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HISTORY

OF THE

INVENTION OF LOGARITHMS, &c.

THE philosophy of Logarithms has been so thoroughly investigated by the many illustrious authors already referred to, that it is unnecessary to attach an algebraic discussion, or analytical theory of Napier's great invention, to his domestic memoirs. I shall attempt, however, to sketch the history of his mathematical studies, especially in reference to those points which appear to have been carelessly or inaccurately recorded. To this shall be added some very curious original matter from our philosopher's unpublished manuscripts, which cannot fail to interest even those who are deeply read in mathematics.

The most popular English history of Logarithms mixes up, in one theoretical view, the Logarithmic properties of numerical progressions, observed for many ages before Napier's time, with "the happy Invention of Logarithms." * But any observations of the kind made by calculators between the time of the sage of Syracuse, and the sage of Scotland, seem to resolve themselves into the celebrated theorem of the former, the history of which has been already given. † A more distinct arithmetical view, of the properties of that theorem, was of necessity obtained through the medium of Arabic or Indian notation, which Archimedes did not possess; but our own philosopher was not led to his invention or discovery by the preparatory labours of others, or at least that aid was afforded him as much by Archimedes as by any one else. This can be easily rendered obvious.

We shall suppose that a mere tyro in modern arithmetic, and one ignorant of geometry, endeavours to make himself master of the theorem in the *Arenarius*. In any geometrical progression from unity, represented by the letters,

A, B, C, D, E, F, G, H, I, K, L,

* Hutton.

† See page 346.

and of which A is unity, he finds from Archimedes, that, "if any two of the terms be multiplied together, the product will also be a term in the same progression; and its place will be at the same distance from the larger of the two *factors* that the lesser *factor* is from unity; and that its distance from unity will be the same, *minus one*, that the *sum* of the distances of the two *factors* from unity is distant from unity." To relieve his attention, our tyro will naturally substitute actual numbers in place of the symbols used by Archimedes. Having mastered the meaning of a *geometrical* progression, he may be supposed to adopt the series most easy to multiply into such a progression, namely,

1, 10, 100, 1000, 10000, 100000, 1000000, 10000000, &c.

where he obtains a proportional increase in the constant ratio of 10, simply by adding an additional cypher to each additional term. He may select the two nearest terms from unity to make his experiment, and will not be long in discovering, that 100 multiplied by 10, gives 1000, the fourth term in the progression, counting unity. His eye will tell him at once that 1000 is at the same distance from the larger *factor* 100, that 10, the lesser *factor*, is from unity. Nor will he have much greater difficulty in ascertaining that the united numbers of the places of the *factors*, counting unity, is equal to 5, and that the product sought is at that number, *minus one*, being the *fourth* term.

So far the theorem is satisfactorily tested. But if the tyro, in repeating his attempts, should select terms at a greater distance from unity and each other, his eye will not so readily assist him to the fact of the respective distances. He would have to count the terms, which might naturally lead him to number them, thus:

1 2 3 4 5 6 7 8

1, 10, 100, 1000, 10000, 100000, 1000000, 10000000.

In this manner, he would soon arrive at the knowledge, that the mere addition of the two upper figures immediately above the two lower terms to be multiplied, will give a sum or figure in the upper line, pointing not to the actual product sought, but to the term immediately beyond it; and he would also easily detect, that the fact of its not pointing immediately to the product, was explained by the *minus one*, which forms a hitch, as it were, in the theorem of Archimedes. Now, supposing the tyro to possess some ingenuity, he will easily get rid of this inconvenience by numbering the distances in the geometrical series differently, and calling 10 not the second term in the series, but the first term after unity, or the first distance from unity; and this would seem the more accurate way of numbering, for 1 cannot be said to be at any distance from itself. He would then arrange them thus:

1 2 3 4 5 6 7

1, 10, 100, 1000, 10000, 100000, 1000000, 10000000.

According to this mode of numbering, he would find that the *sum* of any two figures in the upper line was a number in that same line *directly* over a term in the lower line, which would be the *product* of the two terms respectively below the added figures. After this step it would not be difficult, even for a tyro, to detect all the simpler operations with

the upper progression, affording the same results as the more complicated operations with the lower.

In the case supposed, a rude and limited, and, we may add, useless table of Logarithms, is unconsciously formed; the numbers composing the arithmetical series being truly Logarithms to the terms composing the geometrical. But no step of any value beyond what was demonstrated by Archimedes is thus accomplished. The theorem of the school of Alexandria has been viewed through the facilities of Arabic notation,—a logarithmic adaptation of numerical progressions has been very clearly brought out,—but the *Logarithms* are just as far as ever from being discovered. Yet the very arrangement and base of the common Logarithms is thus exemplified by a tyro's translation of Archimedes's theorem into Arabic numerals!

The fact is, that our system of notation is essentially Logarithmic; and the tyro might have even detected, in the simple algorithm, 1000, the very process he had gone through in testing the theorem of Archimedes. 1000 expresses that 1 has progressed three steps from right to left; the cyphers mark those steps, and therefore may be said to number them. Then the Arabic system is in a decuple progression; *i. e.* each move of the advancing digit increases its value ten times its last value; so 1000 is unit progressed from right to left in this order, 1000, 100, 10, 1. The values of each move are here noted; and the steps themselves may be arranged and numbered, thus:

1	2	3	
1,	10,	100,	1000.

Here we are back again to the Archimedean theorem and Logarithms! It will be observed, that to number the last example is superfluous, for the *cyphers* perform that office. Again, it is equally superfluous to write the whole steps of the progression at full length, for the simple notation 1000 expresses all the steps. It is a short-hand *exemplification* of the most convenient system of Logarithms; the cyphers stand in place of the *arithmetical* progression, 1, 2, 3, &c. as adapted to the *geometrical* progression, 1, 10, 100, 1000, &c. and the whole is based upon the denary scale in use. But if this be true, it must follow that the mere *addition* of the cyphers in the Arabic scale will afford the same result as the *multiplication* of the terms? And such, indeed, is the case; for a thousand multiplied by ten thousand gives ten million: ten million is noted by unit moved to the left seven steps, *i. e.* unit with seven cyphers to the right. A thousand has three cyphers, and ten thousand has four, which added, give seven. Write this out, and we have

1	2	3	4	5	6	7	
1,	10,	100,	1000,	10000,	100000,	1000000,	10000000.

Now, 1000 multiplied by 10000 must give 10000000; for the numbers above the *factors* are 3 and 4, which added, give 7, which number points to the product sought, 10000000.

Thus we find that the Arabic system itself is essentially Logarithmic, and that the properties of the Archimedean theorem may present themselves to a very ordinary cal-

1000
10000

10000000

culator, upon a consideration of the simple notation 1000. I am not aware that the most profound observers of numerical progressions before the time of Napier ever went a single step beyond what we have thus exemplified. They pointed out the effect of the adaptation of an arithmetical to a geometrical series of numbers in relieving the calculation of the terms of the latter series *in a particular case*. They might vary the case by choosing other ratios of progression, and examine their properties more minutely, but none of them (supposing them as numerous as Dr Hutton assumes) ever conceived the possibility of making the principle embrace the WHOLE SYSTEM OF NUMBERS. That was "the reason why tables of such artificial numbers were not sooner formed," and by no means because they were not sooner wanted.

If all numerical operations were performed upon the decuple progression itself, and by means of unit and cyphers, calculators would have an easy time of it, and children might lisp in Logarithms. But where is the advantage of knowing, that to multiply 10000 by 1000 we need only add the cyphers, when we have, for instance, to multiply 4723 by 835? It takes some trouble to discover that the product of those factors is 3943705, a number very little indebted to cyphers for its notation, and which is not to be obtained by reckoning and adding the steps of the factors' digits. In other words, it required no great penetration to discover that this progression, 1, 10, 100, 1000, &c. or this, 1, 2, 4, 8, 16, &c. can have for their logarithms the whole range of natural numbers 1, 2, 3, 4, &c.; but where are the logarithms for the many terms betwixt 1 and 10, 10 and 100, 100 and 1000, &c. or betwixt all the terms of any other geometrical progression? Kepler, in one of his enthusiastic essays on the subject, written not long after Napier's death, exclaims with testy irony against some jealous carping philosophers in Germany:—"Now what is this thing? Of what use are Logarithms? Why to be sure, of the very use that was declared ten years ago by the original inventor, Napier, and which may be conceived in three words. Wherever it happens in common arithmetic, and in the rule of three, that two numbers have to be multiplied together, in that case their Logarithms are to be added; where a number has to divide another, the Logarithm is only to be subtracted from the sum of the Logarithms, so that in the one case the added, and in the other case the remaining Logarithm points out the number sought in either operation. This, I say, is the use of Logarithms. But the featherless chickens of arithmeticians, greedy of facilities, and gaping with their beaks wide open at the mention of this use, as if to gorge every particular gobbet of my precepticles, were not to be satisfied in a work devoted to the fundamental demonstration of the Logarithms."* The use thus characteristically announced by Kepler would have been far beneath the observation of that lofty philosopher, but for its application to the whole system of natural numbers, from unity *in infinitum*; and Kepler himself, in his letter to Napier, draws the mighty distinction which separates the Scotch philosopher from every calculator in the world who had previously considered numerical progressions, when he

* *Joannis Kepleri, Supplementum Chiliadis Logarithmorum.* 1625.

says, "*Vix autem uno tentato exemplo, deprehendi, magna gratulatione, generale factum abs te exercitium illud numerorum, cujus ego particulam exiguam jam a multis annis in usu habebam.*"

We may well believe, that if Kepler, as he tells us himself, did actually observe, and attempt to reduce to practice, logarithmic properties of numbers, without having the least conception of the Logarithms *par excellence*, and also that Stifelius, a most profound arithmetician, examined such properties still more minutely without forming that conception, there was a gulf which totally disunited those speculations from Napier's invention, however Dr Hutton may have been pleased to jumble the ideas together in his history. The fact is, that, from the undeveloped state of the power of Arabic notation at this early period of European science, the speculations referred to had an obvious tendency to check the conception of the Logarithms. The natural system of numbers, 1, 2, 3, 4, &c. composed an arithmetical progression, capable of being Logarithms to various sets of geometrical progressions. How, then, could that system obtain Logarithms adapted to itself throughout its infinite extent? Its nature would require to be changed from an *arithmetical* to a *geometrical* series, without losing any of its terms; and this involved a contradiction, and was clearly impossible! The system of Logarithms is founded upon the correspondence of those different progressions. That system cannot exist as such, unless it be made applicable to the whole range of natural numbers. The whole range of natural numbers are in arithmetical progression, and never can form a geometrical one. How are these facts to be reconciled? Here all the calculators in Europe stopt short except Napier. His mind, of an uncommon cast, enabled him to break in upon this enchanted circle of numbers with perfect success. The general conception he formed was that of two *flowing points*, generating magnitudes by infinitely small degrees, and so regulated in their respective motions, that in the one case, the successive increments would be equal to each other; and in the other case, would differ proportionally from each other in an infinitely small degree. In the latter case, a geometrical progression was conceived, into which, obviously, all the natural numbers 1, 2, 3, 4, &c. might be supposed to enter as terms, having the magnitudes generated in the former case for their arithmetics. Napier knew, indeed, that the infinitely small ratios which he imagined to be generated betwixt the natural numbers, were an approximation merely, and never could equal the determined finite quantity; but he had the sagacity to perceive, that, in such an approximation, the difference or defect would become smaller than any assignable quantity, and therefore would not sensibly affect the calculations to which he meant the system to apply. The two first chapters of the Canon Mirificus contain the developement of this beautiful idea, and no succeeding philosopher, though the most illustrious have tried it, has ever afforded a clearer view of Napier's method than his own statement, which is as follows:—

A conse-
quent.

“ Therefore, the logarithm of the whole sine 10000000 is nothing, or 0 ; and, consequently, the logarithms of numbers greater than the whole sine are less than nothing.

“ For seeing it is manifest by the definition that, the sines decreasing from the whole sine, the logarithms increase from nothing ; therefore, contrariwise, the numbers which yet we call sines, increasing unto the whole sine, that is 10000000, the logarithms must needs decrease to 0, or nothing ; and, by consequent, the logarithms of numbers increasing above the whole sine 10000000, which we call *secants* or *tangents*, and no more *sines*, shall be less than nothing.

“ Therefore we call the logarithms of the sines abounding, because they are always greater than nothing, and set this mark + before them, or else none. But the logarithms which are less than nothing we call defective, or wanting, setting this mark — before them.

“ It was, indeed, left at liberty in the beginning to attribute nothing, or 0, to any sine or quantity for his logarithm ; but it was best to fit it to the whole sine, that the addition or subtraction of that logarithm which is most frequent in all calculations might never after be any trouble to us.

CHAP. II.—OF THE PROPOSITIONS OF LOGARITHMS.

1. Proposi-
tion.

“ The logarithms of proportional numbers and quantities are equally differing.

“ As for example. The logarithms of the proportional sines, namely $c Z$, which is to $e Z$ as $h Z$ is to $k Z$, are respectively the numbers defining AC , AE , AH , AK , as is manifest by the 6th definition. Now AC and AE differ by the difference CE , and AH and AK by the difference HK . But, by the first definition and his corollary, CE and HK are equal ; therefore the logarithms of the foresaid proportional sines are equally differing. And so in all proportionals,” &c.

2. Proposi-
tion.

“ Of the logarithms of three proportionals, the double of the second or mean, made less by the first, is equal to the third.

“ Seeing that by the first proposition the difference of the logarithm of the first and second is equal to the difference of the logarithms of the second and third, that is, the second made less by the first is equal to the third made less by the second ; therefore, the second being added to both sides of the equation twice, the second, or the double of the second made less by the first, shall come forth equal to the third, which was to be proved.

3. Proposi-
tion.

“ Of the logarithms of three proportionals, the double of the second, or middle one, is equal to the sum of the extremes.

“ By the second proposition, the double of the second, made less by the first, is equal to the third. To both the equal sides add the first, and there shall arise the double of the second, equal to the first and third, that is, to the sum of the extremes ; which was to be demonstrated.

4. Proposi-
tion.

“ Of the logarithms of four proportionals, the sum of the second and third, made less by the first, is equal to the fourth.

“ Seeing by the first proposition of the logarithms of four proportionals the second

made less by the first, is equal to the fourth less by the third; add the third to both sides of the equality, and the second and the third made less by the first shall be equal to the fourth, which was propounded.

“Of the logarithms of four proportionals, the sum of the middle ones, that is, of the second and third, is equal to the logarithm of the extremes, that is to say, the first and fourth.” 5. Proposition.

“By the fourth proposition, the second and third made less by the first were equal to the fourth: to both sides of the equality add the first, and the second more by the third shall be made equal to the fourth more by the first, which was to be demonstrated.

“Of the logarithms of four continual proportionals, the triple of either of the middle ones is equal to the sum of the further extreme, and the double of the nearer.” 6. Proposition.

“By the second proposition, the double of the second made less by the first is equal to the third; and by the third proposition the double of this, that is, the fourfold of the second, made less by the double of the first, shall be equal to the sum of his extremes, that is, the fourth more by the second. Now if from both sides of the equality you subtract the second, the triple of the second made less by the double of the first shall be made equal to the fourth. Again, to the sides of this equality add the double of the first, and there shall arise the triple of the second, equal to the fourth, more by the double of the first, which we undertook to prove.

An Admonition.

“Hitherto we have shewed the making and symptoms of Logarithms. Now by what kind of account or method of calculating they may be had, it should be here shewed. But because we do here set down the whole tables, and all his Logarithms with their sines to every minute of the quadrant, therefore passing over the doctrine of making Logarithms till a fitter time, we make haste to the use of them; that the use and profit being first conceived, the rest may please the more being set forth hereafter, or else displease the less, being buried in silence. For I expect the judgment and censure of learned men hereupon, before the rest, rashly published, be exposed to the detraction of the envious.” *

The abstract geometrical mode in which Napier promulgated his system was so perfectly original, as to startle and disturb some of the High Priests of Science in Germany; and although that promulgation was accompanied by a canon, which (to use Dr Hutton's expressions) “is a perfect work on this kind of Logarithms, containing in effect the Logarithms of all numbers, and the logarithmic sines, tangents, and secants, for every minute of the quadrant, together with the description and uses of the tables,” still some of the venerable sages of the 16th century, no less jealous than astonished, shook their gray heads at the auspicious dawn of the 17th, and refused the summons of Kepler to fall down and worship the greatest era of science, as its sun first rose above the remote hills of unlettered Scotland. “When,” says Kepler, “in the year 1621, I travelled into Upper Germany, and discoursed every where with those skilled in the mathematical sciences,

* English translation of the Canon Mirificus. 1616.

concerning the Logarithms of Napier, I discovered that they, of whose minds age had diminished the activity, in proportion as it had increased the experience, were unwilling to admit this description of numbers in place of the usual canon of sines. They said it was degrading to a professor of mathematics to show such childish exultation about any compendious method of numbers; and meanwhile to receive into practice, without even a legitimate demonstration, a form of calculus, which some day or other might betray into errors when least suspected. They complained that Napier's demonstration depended upon the fiction of a peculiar geometrical motion, whose slippery and unstable nature was inadequate to sustain the severe march of reason and demonstration. This (he adds) induced me to attempt to found a legitimate demonstration, not under the nature of lines, or motion and fluxion, or, so to speak, any other sensible quantity, but under that of ratios and abstract quantities," &c. But even Kepler was wrong in this concession, as is admitted in modern science; and the puerility of the objection urged by these venerable bigots might have been retorted by the exulting champion of Logarithms. "Napier's view of the subject (says Professor Playfair) is as simple and profound as any which after two hundred years has yet presented itself to mathematicians. The mode of deducing the results has been simplified; but it can hardly be said that the principle has been more clearly developed." The opinion of the Newtonian age has in like manner been passed upon those commentaries of Kepler, in which he attempted a new demonstration of the Logarithms, and the judgment is, that even he only mystified the system of Napier, while professing to clear it, and at the same time drew his own purest principles from Napier's code. "Whether (says Delambre) these objections were suggested to Kepler or occurred to his own mind, they might have been easily answered. It is true that the consideration of fluents, and fluxions of lines and points in motion, are quite extraneous to the subject; but efface them all, and Napier's calculations are not a whit the less substantial. From two numbers which are in a given proportion, subtract proportional numbers, and the remainder will be proportional. Subtract from 9 and 10, a tenth part of each, there remains 8.1 and 9, and you have $10 : 9 :: 9 : 8.1$, $9 \times 9 = 10 \times 8.1 = 81$. Behold the fundamental theorem of Napier: upon this principle he formed his preparatory tables. Extend these tables sufficiently, and you will there find numbers sensibly equal to all the natural numbers, to the sines, and to every possible numerical quantity. The process is only an approximation. Napier admitted the fact: but where the limit of the error is known, it is always permitted to disregard it: equally admissible is it to adopt a method so eminently commodious: there is nothing puerile in adopting it with exultation: on the contrary, the desire to confine that conception to lines and hyperbolic spaces has something in it of pedantry. All the clearness, simplicity, and generality observable in the theory of Logarithms are the results of processes purely analytical or numerical; and we owe whatever is obscure to extraneous considerations with which the system has been painfully alloyed. I would wish no better proof of the fact than the works of Kepler and Mercator. Who would dream now a days of studying in Euclid the theory of numbers and proportions? These subtleties are more troublesome than useful, and time, which might be more profitably

and judiciously bestowed, is lost in demonstrating such conceptions." Delambre farther remarks of the *Chilias Logarithmorum*; "Kepler lays down 30 propositions; the most part of them appear fit for nothing but to swell the volume; the number was necessary, however, in order to justify a kind of *jeu de mots* in his dedication. The landgrave of Hesse, Philippe, had presented him with 30 pieces of silver, and he evinced his gratitude by dedicating a book to the landgrave containing 30 propositions. The dedication is in Latin verse garnished with Greek words. The book and the dedication are in the taste of the times. Kepler then proceeds to construct his tables, but takes very good care not to employ his 30 propositions; *in fact, he uses no theorem for which he is not indebted to Napier.*"* Such is the opinion of a philosopher, the hero of whose history of science is, nevertheless, Kepler.

But the most illustrious defence of Napier's genesis of Logarithms is to be found in the Life of Sir Isaac Newton. "The notion of flowing quantities *first proposed* by Newton, (says Professor Leslie as if in a day dream,) and from which he framed the terms fluxions and fluents, appears on the whole very clear and satisfactory; nor should the metaphysical objection of introducing ideas of motion into geometry have much weight. Maclaurin was induced, however, by such cavilling, to devote half a volume to an able but superfluous discussion of the question."† Yet the works either of Napier, Kepler, Delambre or Maclaurin might have informed our professor that, whatever its merits or demerits, the notion of flowing quantities was also Napier's, and that the terms said to have been framed by Newton are to be found in the Canon Mirificus. "*Sit punctus A, à quo ducenda sit linea fluxu alterius puncti, qui sit B; fluat, ergo primo momento,*" &c.‡ and from Kepler we learn that the same cavils against which Maclaurin philosophised had been urged against Napier. Maclaurin himself, in the very work referred to by Sir John Leslie, has a chapter "of Logarithms and the Fluxions of logarithmic quantities," in which he observes, "the nature and genesis of Logarithms is proposed by the inventor in a method similar to that which is applied in this doctrine (Fluxions) for explaining the genesis of quantities of all sorts, and is described by him *almost in the same terms.*"§ We must now turn to the passage in Sir Isaac Newton's work, where he announces the method that led him to his great discovery.

"I consider mathematical quantities in this place not as consisting of very small parts, but as described by a continued motion. Lines are described, and therefore generated not by the opposition of parts, but by the continued motion of points; superficies by the motion of lines; solids by the motion of superficies; angles by the rotation of the sides; portions of time by a continual flux; and so in other quantities. These genesis really take place in the nature of things, and are daily seen in the motion of bodies. And after this manner the ancients, by drawing moveable right lines along immoveable right lines,

* Histoire de l'Astronomie Moderne, p. 507 *et infra*.

† Leslie's continuation of Playfair's Dissertation.

‡ Canon Mirificus.

§ Maclaurin's Treatise of Fluxions, Vol. i. p. 158.

taught the genesis of rectangles. Therefore, considering that quantities, which increase in equal times, and by increasing are generated, become greater or less according to the greater or less velocity with which they increase and are generated, I sought a method of determining quantities from the velocities of the motions or increments with which they are generated; and calling these velocities of the motions or increments *Fluxions*, and the generated quantities *Fluents*, I fell by degrees upon the method of Fluxions, which I have made use of here in the quadrature of curves, in the years 1665 and 1666.*

Here Newton seems to have fallen insensibly upon the method of Napier, for I can discover no indications in all his works that he had ever seen the Canon Mirificus, however deeply he entered the theory which that canon created. But the minds of these great men were formed in the same mould, although belonging to very different ages. Constantly bent on conquering where the difficulty seemed greatest, whether it were the mysteries of prophecy or calculation, they attacked their subjects with the same weapons. Had Newton been placed in the situation of Napier, he would have attempted the Apocalypse, and invented the Logarithms. Had Napier possessed the algebraic calculus in the state that Newton took it up from the hands of Girard, Harriot, Cavalerius, Descartes, Roberval, and Wallis, he would have reached the discovery of Fluxions by the very path of Newton; for, as it was, we shall find that he was on the confines of the binomial theorem. But some of the mathematical magnates of the present century, while reviewing the Fluxions of Newton, and the method which led him to attach that nomenclature to his system, make no mention of Napier, † as if there was nothing interesting or worthy of attention in the coincidence. Yet so strong is it, that, when the personal friend of Newton, and the greatest mathematician after Napier that Scotland ever produced, ‡ set his powerful mind to expound the philosophy of Newton's fluxionary method, he wrote a chapter "of the grounds of this method," which serves equally well to illustrate Napier's Logarithms or Newton's Fluxions. Nay, he adopts the very propositions, and nearly the language of Napier. Even Dr Hutton, who has shown himself no friend to our philosopher's fame, observes, "Napier's manner of conceiving the ge-

* Sir Isaac Newton's Treatise of the Quadrature of Curves, &c. translated by John Stewart, A. M. Professor of Mathematics in the Marischal College, Aberdeen, 1745.

† No one should review, even by the slightest sketch, the mathematical sciences, without naming Napier,—far less if that review be in a life of Newton, who was so deeply indebted to the Logarithms. But the remarkable coincidences of the theological studies, and geometrical modes of investigation pursued by these philosophers, render it doubly strange that Sir David Brewster does not once mention Napier in his Life of Newton. How striking, on the other hand, are the observations of Delambre in his History of Astronomy. "*Néper démontre que $\log \sin A > (1 - \sin A)$ et $< (\coséc A - 1)$. Il le prouve par ses Fluxions et ses Fluents.*" Again, "*Képler promet une démonstration légitime; il regarde donc comme insuffisante ou inexacte celle de Néper: il pouvait lui reprocher des longueurs, des inutilités; il lui reproche, en effet, cette idée de fluxions, et de fluents, qu'on a depuis reprochée à Newton. Mais nous verrons que les principaux théorèmes trouvés et démontrés par Néper, n'ont pas été inutiles à la nouvelle démonstration.*"—Tome i. pp. 499, 507.

‡ Colin Maclaurin.

neration of the lines of the natural numbers and their Logarithms by the motion of points, is very similar to the manner in which Newton afterwards considered the generation of magnitudes in his doctrine of fluxions; and it is also remarkable, that in Art. 2 of the *Habitudines Logarithmorum et suorum naturalium numerorum invicem*, in the Appendix to the *Constructio Logarithmorum*, Napier speaks of the velocities of the increments or decrements of the Logarithms in the same way as Newton does, namely, of his fluxions, where he shows that those velocities, or fluxions, are inversely as the sines or natural numbers of the Logarithms, which is a necessary consequence of the nature of the generation," &c. And Hutton mentions this more particularly afterwards, when he says, "I shall here set down one more of these relations, as the manner in which it is expressed (by Napier) is *exactly similar* to that of fluxions and fluents, and it is this: Of any two numbers 'as the greater is to the less, so is the velocity of the increment or decrement (*incrementi autdecrementi*) of the Logarithms at the less, to the velocity of the increment or decrement of the Logarithms at the greater,' that is, in our modern notation, as $X : Y :: \dot{y}$ to \dot{x} , where \dot{x} and \dot{y} are the fluxions of the Logarithms of X and Y."*

We thus see that Napier's method was not an accidental idea, indicative of a rude age and country, but one which the loftiest minds were the most apt to adopt. Logarithms mark one great revolution in modern calculation,—Fluxions another; and surely the coincidence is not uninteresting that their immortal authors arrived at these discoveries independently of each other, but by a train of thought identically the same. But Newton, to use the expression of his latest biographer, was "the leader of a mighty phalanx,—the director of combined genius,—the general who won the victory, and therefore wears the laurels." Napier occupies a remote and solitary orbit, whose glory is all his own. He attacked science precisely at the point where the adventure was most uninviting and most laborious; and he did so precisely at the time when the achievement was of the greatest consequence. Men thought that the utmost power of the Indian algorithm was already displayed in the ascending decuple scale; and although some faint idea of Decimal fractions had been obtained, still, until Napier arose, the system of numbers was viewed falsely and in fragments, like the first appearances of the ring of Saturn through the rude telescope of Galileo. The Brahmans themselves never knew the value of the scale whose beautiful notation they transmitted to Europe. Wallis, the successor of Henry Briggs in the Savilian chair, and whose *Arithmetic of Infinites* gave the first impulse to Newton's mind, observes, "there are two very considerable improvements which we have added to the algorism of the Arabs since we received it from them, to wit, *Decimal fractions* and the *Logarithms*." Keill, who succeeded Wallis as Savilian professor, and is distinguished as the opponent of Leibnitz, has also remarked, "The mathematicks formerly received considerable advantages, first by the introduction of the Indian characters, and afterwards by the invention of Decimal fractions; yet it has since reaped at least as much from the invention of Logarithms as from both the other two." In short, there is no doubt that the great frame-work upon which the miraculous powers of modern calcula-

* Hutton's History of Napier's Construction of Logarithms, pp. 42, 48.

tion are reared, consists of three steps, the Arabic numerals, Decimal fractions, and the Logarithms. Now of these, Napier brought the second into operation, and created the last, at a time when other philosophers were engrossed with the fascinations of applicative science, and when physical research was soaring upon unruly wing in dangerous advance of the science of numbers. This view of our philosopher's fame deserves a closer consideration; and we must now glance at the circumstances under which he deliberately undertook to unfold the latent power of the Arabic, or rather Indian, system.

We have reviewed, generally, in the preceding memoirs, the manner in which his great contemporaries of the continent were employed, and the resources they had obtained from their predecessors. The *desideratum* of those times was a philosopher of the intellectual order of Tycho, Kepler, or Galileo, who, possessing also their ardour for the advancement of science, would devote his whole power to conquer the tyranny of Logistic. One or two had made that attempt before Napier's time; and although the fruits of their labours conferred honour even upon Germany, still the results prove that his success was beyond the grasp of their minds. Had our philosopher lived under those cloudless skies where the telescope was first applied; had his lot been cast in some of those countries where the sons of science excited each other in the opening path of physical research; and where, (to use the expressions which, in reference to those countries, Napier addressed to his own monarch,) royalty itself became "the patron and protector of all zealous students, and an allowor and acceptor of their godly exercises;" he, too, might have exerted his powers of calculation in legislating for the stars, or in founding some department of science less abstract and retiring than the path he followed. As it was, however, he turned to the numeral system, where there was so much to do, and where he achieved all that remained to be developed. That he set himself deliberately to the task, we learn from his own accounts, both in the preface to the Canon Mirificus, and in his letter to the Chancellor, already quoted; and the same is repeated in his preface to the *Arithmetica Localis*, where he says, "In the progress of my inquiries by what means the toil and trammels of calculation could be removed, and in the course of occupying every leisure moment I possessed with investigations of compendious methods of computing, I fell (in addition to the Logarithms, Rabdology, the Promptuary of Multiplication, and *other methods*) upon a certain arithmetical table, which, as it performs all the serious operations of vulgar arithmetic upon an abacus or chess-board, deserves to be called an amusement rather than labour." It appears from his manuscripts, to be afterwards considered, that our philosopher, as might be supposed, only accomplished all this by a most systematical progress through the whole theory of numbers, which he was endeavouring to accommodate to the practice of science, in a digest replete with philosophic beauty, when the magnitude of his own success deranged his plan, and the severity of his labours hastened his life to a close. But we cannot appreciate what he did, unless we keep in view both the state in which he found numbers when he determined to create a revolution in the system, and the state in which he left them at his death.

There is reason to believe that Napier took up the subject some time before the year

1594. He tells us himself, that he had invented the Logarithms long before he published them ; and this agrees with the fact already noticed, that Dr Craig, soon after King James's visit to Tycho, wrote some account of the matter to that philosopher. In Napier's manuscripts, however, there does not appear the least trace of his great invention. Now it is impossible to suppose that it would not have been prominent in a work where he enters deeply into the arithmetic of surds, had the conception been then formed. As his treatise on arithmetic and algebra was found among his loose notes in an unfinished state, it seems more than probable that the invention of Logarithms arose out of these deep lucubrations, and would actually have formed a part of a great work on numbers, which he intended to have given the world had his life been spared. We must, therefore, fix the chronology of Napier as an algebraic writer sometime before the close of the sixteenth century, that is, during the rudest period of algebraic science in Europe. *Leonardo of Pisa* composed his work before the invention of printing, and early in the thirteenth century. But this had been lost sight of, and was not known for more than a century after Napier's death, when the manuscript was discovered at Florence. The first printed work on the subject was that of *Lucas de Burgo*, who brought his mercantile travels to the same good account as Leonardo, and from him is generally dated the decided dawn of algebra in Europe. De Burgo's principal work was published about the year 1494, and in 1539 the second printed book upon arithmetic and algebra appeared. This was a work of the great but eccentric *Cardan*, of whom it is affirmed by Scaliger, that he was so devoted to astrology, as to refrain from food, and actually die of starvation, to fulfil his own astrological predictions,—a very equivocal compliment to the mystical science. He died in the year 1575. Germany produced one or two philosophers, who, at the same time that Cardan wrote, gave a more decided impulse to numbers. Hitherto nothing had been added to that recondite science, since Arabic numerals, and the rude and imperfect symbols of Burgo's algebra, except in the theory of equations, which received a great extension from Tartalea and Cardan. The defect which materially clogged and impeded the system was in notation, the mainspring of numerical science, and of which the profoundest minds are most apt to see the philosophy and value. *Michael Stifelius*, a Lutheran clergyman, published at Norimberg, in the year 1544, his *Arithmetica Integra*, a very original Latin treatise on arithmetic and algebra, wherein he viewed numerical quantities, and their combinations, closely and ingeniously, and gave an impulse to algebra by improving its notation. He was the first to introduce the signs + and — for *plus* and *minus*, and also the character \surd , (a contracted R) which denotes the *radix* or root. Besides this he entered systematically into the consideration of arithmetical and geometrical progressions, pointed out the logarithmic properties of a corresponding series of powers and their exponents, which latter term he uses, and approached as nearly to the shrine of the Logarithms as it was possible for any one to do who formed no conception whatever of the great system itself. Another singular coincidence between Napier and Stifelius may

be mentioned. The latter was a zealous Protestant, and had persuaded himself, though by a less cool and philosophical consideration of the subject than Napier's, that the day of judgment was at hand. The anecdote is well told by Sir John Leslie in his Dissertation.—“Captivated, perhaps, by the wonderful properties of numbers, he fancied, as other ingenious persons have since done, that he could interpret the visions of the Apocalypse, and foretel the end of the world. He was so imprudent as to place that awful dissolution very near hand. Early in the morning of the day predicted, in the year 1553, he assembled his trembling flock in a wide open field, where he endeavoured to season their minds for the tremendous change by fervid prayers and pathetic exhortations. The sky was lowering,—the darkness thickened,—a portentous silence prevailed,—and the preacher rolled his thunders with overpowering energy. But the clouds soon passed away, the sun shone forth in his wonted splendour, and all nature smiled. The populace recovered their agitated spirits; and now breathing rage and disdain, they chased the unlucky prophet home with volleys of stones.” Thus it would seem, that, although possessing a mind somewhat similar in its constitution, the German exceeded our own philosopher upon this subject, in the same proportion that he fell short of him in developing the power of the Arabic scale. What he added to the system of numbers was chiefly in algebraic notation; as for his ingenious observations of numerical progressions, they left that subject very much in the state that Archimedes had done. Another German, *Scheubelius*, published a work upon arithmetic and algebra about the same time, and of much the same description. In 1552 appeared the first treatise upon the subject in the English language, written by the unfortunate *Robert Recorde*. His works are curious and original, but only elementary. They are generally in the form of a dialogue between a master and scholar, and under such quaint titles as *the Pathway to Knowledge, the Ground of Arts, the Castle of Knowledge, the Whetstone of Wit*. He is chiefly remarkable in the history of algebra, for having added to its notation the sign of equality; “and to avoid (says he,) the tedious repetition of these woordes, *is equal to*, I will sette, as I doe often in woorke use, a paire of paraleles, or gemowe lines of ane lengthe, thus =, bicause noe 2 thynges can be moare equalle.” England seems to have thought nothing of her solitary algebraist, who, though an able and meritorious man, was suffered to die a prisoner in the Fleet for debt about the year 1558. Sir John Leslie, in his Dissertation, observes of this writer, “he was the first to propose the sign = for equality, but made no other advances; and during a period of most active enterprise, till the close of the century, algebra was not cultivated at all in this country.” In France, the celebrated *Ramus*, under whom Napier may have studied for a time at Paris, wrote on arithmetic and algebra about the year 1560; but his work is also elementary rather than philosophical, and he left the science as he found it. *Raphael Bombelli*, whose algebra was published at Bologna in the year 1572, in Italian, wrote more elaborately and profoundly, but did not add any thing of consequence to the labours of his predecessors. The first that can be said to have done so, between the time of Stifelius and Napier, was

Simon Stevinus of Bruges, who published *La Pratique D'Arithmetique* about the year 1582. He afterwards put forth other works upon arithmetic and algebra, along with a translation of some books of *Diophantus*, in all of which he evinced the highest genius for his subject. Algebraic notation received from his hands another of those impulses by which it has so gradually reached its present perfection; and arithmetic is indebted to him as the first who expressly promulgated the doctrine of *Decimal fractions*. It was most probably about this period that our own philosopher commenced his endeavours to create a decided revolution in the science of numbers. A contemporary of *Napier's* was the great *Vieta*, whose name reflects such lustre upon France. His algebraic work first appeared in 1600, but by this time *Napier's* studies had ripened into the *Logarithms*, which were at least in progress. The most accurate chronology of the time of our own philosopher's preparatory labours, therefore, so far as I have been able to ascertain, seems to be between the publications of *Stevinus* of Germany, and *Vieta* of France. The French philosopher generalized the language of algebra by employing letters to denote known as well as unknown quantities; and he extended the theory of equations. It is not likely that *Napier* ever saw his treatises, which were only first collected into one volume by *Schooten* in 1646. All the other great works that occur in the history of numerical science, are subsequent to the death of *Napier*. Countries the most distinguished in Europe for philosophers, had produced in that recondite path the few we have so briefly noticed; and although their names are illustrious and their labours profound, not one of them struck a blow sufficient to extricate the best wing of the mathematics, which, at the close of the sixteenth century, still remained with its arithmetic undeveloped, and its algebra little beyond the rude and infant state in which it was brought from the East. The name of *Recorde* is barely sufficient to give England a place in that history at all; and as for Scotland, until *Napier* arose, it was only famed for mists that science could not penetrate, and for the *Douglas wars*, whose baronial leaders knew little of the denary system beyond their ten fingers.

It is curious to think how much science had attempted in physical research, and how deeply numbers had been pondered, before it was perceived that the all-powerful simplicity of Arabic notation was as valuable and as manageable in an infinitely *descending* as in an infinitely *ascending* progression. It was only necessary to reverse the notation, and the power of the scale was doubled. How obvious and simple does that expedient now appear, "*Mais ces moyens simples sont le fruit des idées profondes et lumineuses.*"* The decimal fractional division itself was long conceived before that notation was established from which it derives all its value and beauty as a part of the Arabic system. Yet the history of this important chapter of numbers is carelessly recorded where we might have expected accuracy. "*Regiomontanus*," says a successor of *Henry Briggs* in the *Savilian chair*, "introduced that simple, but most valuable, modification of the decimal notation, which consists in fixing the unit's place at any figure, and not neces-

* *Bailly*.

sarily at the right hand, by placing there a comma, all arithmetical operations going on just the same; in a word, the use of decimal fractions.* It may be doubted if this sentence expresses its own meaning at all, but certainly it would leave any one, not previously enlightened on the subject, either in the dark or in error. By "decimal notation," we understand the professor to mean the whole system; and what he characterizes as "a most valuable modification of the decimal notation," to be Decimal fractions. Perhaps they are more properly to be called an extension than a modification of our numeral system; but it may be asked, how is the use of decimal fractions explained by saying that it "consists in fixing the unit's place at any figure, and not necessarily at the right hand, by placing there a comma?" This idea disturbs the simple and beautiful philosophy of the Arabic scale. By the comma of course is meant the *decimal point* or *separatrix*, betwixt the integers and fractions, a comma being sometimes used for that purpose; but the fact, that any number of decimals may follow any number of integers, by no means renders unit's place an arbitrary one. No doubt, in any row of figures, such as, for instance, 465.8329, a point placed between any two of them, say 5 and 8, converts all those to the right of the point into decimal fractions, and also indicates 5 as occupying the unit's place of the integers; but the unit's place is just as necessarily at the right hand as ever; it is at the right hand of the integers of which it is the unit; just as much so as if the fraction were noted in the vulgar form thus, $465\frac{8329}{10000}$; the shifting the point may alter the sum, but it never can change the character of unit's place, which must always be to the right of all the integers, and to the left of all the fractions. The reason is obvious, and the slightest attention to what is sometimes apt to be overlooked, the simple idea of our numeral scale, will clear the matter. The unit's place is said to be at the *right* hand just because its value increases from the bottom of the scale by progressing to the *left*. But Decimal fractions made no alteration in this valuable law; the unit, though accompanied by a tail of decimals as long as a comet's, still becomes 10, or 100, &c. only by progressing from right to left; therefore its place is always *necessarily at the right hand*; not perhaps at the calculator's right hand in any given sum of integers and decimals; but, strictly speaking, at the right extremity of the path, by its fluxion through which the unit acquires value. Again, what had Regiomontanus to do with the *decimal point*? No more than Archimedes. So far as a record of the fact can be traced, Napier of Merchiston, who lived a century after Regiomontanus, was the first to conceive the powerful idea of giving decimal fractions their proper value in the scale, by simply *reversing* the order of the original notation. He was also (according to Delambre) the first to perform calculations with decimal fractions, as in the arithmetic of integers; and he did so in the very work by which he unfolded his method of constructing the Logarithms. This fact is very interesting; and so little observed, that we cannot afford to have it still further obscured in a history of science.

* Historical View of the Progress of the Physical and Mathematical Sciences, &c. by the Rev. Baden Powell, M. A., F. R. S., Savilian Professor of Geometry in the University of Oxford. 1834. P. 116.

The principle of Decimal fractions is susceptible of a very simple illustration. A digit progressing from right to left, or from unit's place *in infinitum*, increases in a tenfold ratio; but, let the unit's place be marked by a point after it, and then suppose the digits to progress from unit's place to the right of the point in the same decreasing ratio of ten; and the Arabic system will thus be doubled simply by reversing the order of its notation. To the left of the unit's place are tens, hundreds, thousands; and to the right are tenths, hundredths, thousandths, &c. The idea, however, though apparently a very natural extension of the original system, is twofold; and only arrived at its present perfection by two distinct steps;—*first*, the decimal division in that direction; and *second*, the converse application of the decimal notation. A *sexagesimal* descending scale was used from the earliest times, having been introduced by Ptolemy in his trigonometrical tables; but the equivalent decimal system is of European growth. Purbach, as we have already noticed in the preceding memoir, first applied the decimal division to the radius of the circle, which he supposed to be divided into 600000 equal parts. Regiomontanus improved and extended this method, by computing the sines of the arcs for every minute of the quadrant to a radius of 1000000. This clearly introduced the *idea* of decimal fractions, but only indirectly; and it is a complete mistake to suppose that he had any thing whatever to do with their present punctuated notation. The decimal descending scale was so far from being practically introduced, even in Napier's time, that his early treatise on arithmetic, though it contains a beautiful exposition of vulgar fractions, makes no allusion whatever to decimals, of which at that time he appears to have had no notion, as a system.

In the *Rabdologia*, however, Napier, after some operations with vulgar fractions, adds what he calls *Admonitio pro Decimali Arithmetica*, the substance of which is as follows: "But should those fractions whose denominators are various be found disagreeable on account of the difficulty of working with them, and should that other kind, whose denominators are always tenth, or hundredth, or thousandth, &c. parts, (which that most learned mathematician *Simon Stevinus* in his decimal arithmetic notes and names in this manner, (1) *firsts*, (2) *seconds*, (3) *thirds*,) be preferred on account of their affording the same practical facilities as integers, then, the vulgar division being completed, and concluded with a period or a comma, you can add to the dividend or remainder one cypher for tenths, two for hundredths, three for thousandths, or more at pleasure; and with these proceed to operate as in the above example, where I have added three cyphers: the quotient being 1993.273, signifies 1993 integers, and 273 thousandth parts, or $\frac{273}{1000}$, or as *Stevinus* has it, 1993, $\overset{1}{2} \overset{11}{7} \overset{111}{3}$," &c. Sir John Leslie, in his Dissertation, alludes to this passage, when he says, "It was our illustrious countryman Napier that brought the notation of decimals to its ultimate simplicity, having proposed in his *Rabdologia* to reject entirely the marks placed over the fractions, and merely to set a point at the end of the units." We must remark, however, that the learned knight's antiquities are here not quite accurate. Napier's work where this occurs was published in 1617; but in the edition

of *Pitiscus's* trigonometry, published in 1612, that author, in reference to the decimal division therein used, takes occasion to observe, "which parts are commonly written thus $\frac{5176381}{1000000}$; but it would be a much more compendious form, and fitter for calculation to write it thus 05176381, which means precisely the same thing; and in like manner the two numbers 09, and $\frac{9}{10}$ have just the same value." This is very nearly the notation proposed by Napier, and now in use; and therefore it would appear that Sir John Leslie's assertion in favour of our philosopher is met by the prior claim of Pitiscus. Dr Hutton, in his history of trigonometrical tables, gives the following version of the matter: "Pitiscus, *Trig.* lib. ii. p. 44, describes, for the first time that I know of, the common notation of decimal fractions as now used; and this same notation was *afterwards* described and used by Baron Neper in *positio* 4 and 5 of his posthumous work on the construction of Logarithms, published by his son in the year 1619." Here it will be observed that Dr Hutton refers to a different work of our philosopher than what is referred to by Professor Leslie, and this is curious, that both commit an error exactly the converse of the other. Hutton refers to Napier's *Constructio*, which was composed long before the work of Pitiscus, and yet he gives the priority of the idea of decimal notation to the latter author. Leslie only refers to the *Rabdologia*, not aware that this was published five years after what has been quoted from Pitiscus. Now, the real facts are as follow: Napier's treatise on the Construction of Logarithms was composed many years before 1614, when he published the Logarithms themselves. He expressly says so in his dedication of the *Rabdologia* to Lord Dunfermline; (*Logarithmorum Canonem, à me longo tempore elaboratum, superioribus annis edendum curavi;*) and in the posthumous work it is mentioned by Robert Napier, that his father had composed that for years (*aliquot annos*) before he invented the word Logarithms. Pitiscus, in his first edition of his trigonometry, printed in 1599, says nothing whatever about decimal fractions, and the work of his wherein he mentions them is dated 1612. With these facts not only within his reach, but actually before him, I can only account for Dr Hutton's merely saying that Napier described the modern decimal notation *after* Pitiscus, by the circumstance, that the English mathematician was upon every occasion unwilling to do justice to the Scotch philosopher. He ought, while recording this interesting portion of scientific antiquities, to have pointed out the certain reasons there were for saying that the exposition of decimals in Napier's posthumous work was written long prior to, and not in consequence of, the remark of Pitiscus in 1612.

But it may be said, why so much about a point in decimal notation? and perhaps it is necessary to assign a reason. The greatest mathematicians are those who are most likely to be interested by such facts in the history of science; because they know best how much has been achieved through the medium of *notation*. Mathematical science obtained one of its most powerful impulses in the simple expedient of the cypher, and it is curious to trace as nearly as possible who was the first to observe, that the same simple notation reversed acted with equal facility in the opposite direction. But it is highly interesting to discover that the first to do so was he who was the first to evolve, from the Arabic

system, the latent power of the Logarithms; and who was also the first to bring into full operation the calculation with decimal fractions, as Napier unquestionably was. Improvements in notation, however simple and gradual, are of vast importance in the history of mathematical science. The Logarithms themselves would inevitably have been *discovered* through the medium of improved notation, had Napier not *invented* them (if I may be allowed the distinction,) even before notation was ripe for their discovery. What transformed numerical computations into analytical science? The simple expedient of using the last letters of the alphabet to denote the object of search before it was found. When this symbolical language became yet more general, the power of algebra increased in proportion, and it was through such successive improvements in notation that Newton reached the binomial theorem. But it is not as a discovery for which the world is indebted to Napier, that we are anxious to record that the first idea of the existing notation of decimal fractions must be traced to him. It was truly at the suggestion of no individual that that notation came to be finally established. Long after Napier and Pitiscus had, independently of each other, proposed the method, mathematicians worked decimals in the vulgar form. Practical experience, however, gradually discovered the most perfect manner of doing so. And it is identically that which was so immediately perceived by Napier. We claim the fact for him, not that we may call him the author of the established notation, but, as affording a valuable illustration of the *power* and *simplicity* of his mind. That which experience has proved to be the perfection of the system is just what occurred to him at the very first, and before the world of science was ripe for the application; and hence it is valuable to show that his idea is of older date than the work of Pitiscus, and was not derived from that source. Had such been the case, Napier would have acknowledged the great Pitiscus as readily as he did Stevinus. We shall now turn to Napier's posthumous work, where the fact is fully recorded, and it will be seen that what in the work of Pitiscus is a mere passing observation, (indicative, however, of a great mathematical mind,) is, in our own philosopher's *prior* lucubrations, a system so brought out as to excite the surprise and admiration of DELAMBRE in modern times.

The circumstances under which the *Logarithmorum Canonis Constructio* appeared before the public after our philosopher's death have been already detailed. This was the work which Kepler in his letter expresses an eager desire to see, and certainly it could not have disappointed his most ardent expectations. No clearer or more elegant treatise upon a more recondite or valuable subject illustrates the history of numbers. The purpose of it is to shew by what methods he conquered the second, and not the least difficulty he had to encounter in his path to the Logarithms; namely, how to calculate the actual numbers to be intercalated betwixt the terms of his progressions, in order to reap the fruits of his first great conception. He begins by defining and giving examples of an arithmetical and geometrical progression as the foundation of his system. He then says, that what is requisite in such a system, is, first, *accuracy*, and second, *facility*,—that the former is obtained by laying a foundation of very large numbers, and the latter by

obtaining those large numbers, through the instrumentality of cyphers. This profound though simple opening leads him at once to the doctrine of decimal fractions, which I shall translate from his own words, and the reader will bear in mind that they were written at a period when not even an elementary treatise on arithmetic that has escaped oblivion existed in Scotland; and when Kepler himself was ignorant both of decimal fractions, and of Logarithms, as practical systems. "The less accurate," says Napier, "take 100000 as the largest sine, but the deeper calculators select 10000000, by means of which number, the difference betwixt all the sines can be better expressed. That is the reason why I have adopted it for the whole sine, and as the *maximum* of the geometrical progression. In computing tables, even very large numbers are to be made still larger by placing a period betwixt the original number, and cyphers added to it. Thus at the commencement of my computation I have changed 10000000 into 10000000.0000000, lest the most minute error might, by frequent multiplication, grow into an enormous one. In numbers so divided, whatever is noted after the period is a fraction, whose denominator is unity with as many cyphers after it as there are figures after the period. Thus 10000000.04 is equivalent to $10000000 \frac{4}{100}$. So 25.805 is the same as $25 \frac{805}{1000}$. 9999998.0005021 is $9999998 \frac{5012}{10000000}$ and so on. From the tables so computed, the fractions placed after the period may be rejected without any sensible error, for in these very large numbers, the error is to be considered insensible and nugatory where it does not exceed unit. Thus when the table is completed, for the numbers 9987643.8213051, which are equivalent to $9987643 \frac{8213051}{10000000}$, there may be taken 9987643 without any sensible error."*

This is the earliest conception of the existing notation of Decimal fractions that can be pointed to, and yet in these few words how completely is it developed! Let us compare it with the very latest treatise on the subject, and that is the article Arithmetic in the Encyclopædia Britannica, now in the course of publication,—Chap. ix. *Decimal Fractions*. "Decimal fractions are such as have 10, or some power of 10, (that is 100, 1000, &c.) for a denominator: such are these $\frac{3}{10}$, $\frac{24}{100}$, $\frac{75}{1000}$, $\frac{462}{100000}$; they are more simply written thus: .3, .24, .075, .00462; the number of figures after the point being always the same as the number of cyphers in the denominators. In decimal fractions, as thus written, the figures next the point to the right indicates so many tenths; the next so many hundredths, and so on. Thus in the fraction .346 the figure 3 expresses 3 tenths, 4 denotes 4 hundredths, and 6, 6 thousandths. The use of cyphers in decimals as well as in integers is to bring the significant figures to their proper places, on which their value depends, as cyphers when placed on the left hand of an integer have no signification, but when placed on the right hand increase the value ten times each; so cyphers when placed on the right hand of a decimal have no signification, but when placed on the left hand, diminish the value ten times each." Thus we see that Napier's first conception and explanation of that

* *Constructio Logarithmorum*, p. 6. How deep, and refined, and far in advance of his times, are the doctrines crowded into this single passage.

system, written many years before it came into universal practice, might be transferred verbatim into a treatise on the subject for the year 1834. It is remarkable that Sir John Leslie, in connecting Napier with the history of Decimal fractions, had not referred to the posthumous work rather than to the *Rabdologia*; for it was in the *Constructio Logarithmorum*, that the ordinary rules of calculation were first displayed working with equal facility upon the descending side of the scale. Delambre (*Astronomie Moderne*, p. 493, *et infra*,) was particularly struck with the fact, and I shall follow so far that illustrious philosopher's profound exposition of the work in question. "Napier," says he, "in his definitions, and even in his calculations, makes use of decimal fractions; but only gives the notation without any rule of calculation. It is the earliest example of them I have met with,—it is a first step, and one of the greatest importance," (*il est de la plus grande importance*.) Delambre then follows Napier through his method of calculating the terms of his geometrical progression, but takes the aid of modern algebraic symbols. It would occupy too much space here to give the process, for which the reader must be referred to Napier's own work, or other reconдите sources. After detailing it, Delambre exclaims, "We here distinctly observe examples of *subtraction* in decimal fractions." Passing through some more of the calculations he again exclaims, "behold manifestly *division* in decimal fractions;" and further on he adds, "I have already remarked that Napier is the first to afford the idea of the calculation of decimal fractions, a little more developed afterwards by Briggs."

Such is the hold that Napier has of Decimal fractions, a part of the system, "which" says Playfair, "completed our arithmetical notation, and formed the *second* of the three steps by which in modern times the science of numbers has been so greatly improved." Of course the *first* step was Arabic numerals, and the *third* was the Logarithms; so when we take into consideration that decimals only came into active operation with the system of Logarithms, and that Napier is the first, who affords examples both of the calculation with decimals, and of their best notation, we may fairly say that his share in the development of the great Arabic system is as two to one. The original algorithm, whose history is lost in distant climes and long past ages, brought as it were the telescope to numbers. When Napier reversed the notation, and caused it to act in the opposite direction, he may be said to have added the microscope; and he did so while creating the last and greatest revolution in the system,—when to ἀριθμοὶ he added that omnipotent word, which nor Greeks nor Brahmans knew, λογαριθμοί.* How proud a contempla-

* Αριθμοὶ signifies numbers, λογαριθμοὶ, the ratios of numbers; or, rather, the number of ratios, λέγων ἀριθμῶς. Napier compounded the word before his system was known, but subsequent to the date of his invention. Dr Minto says, "the term Logarithm was first used by Napier after the publication of the canon in which he uses the term of *numerus artificialis*." (Buchan and Minto's *Life of Napier*, p. 43). This is an extraordinary mistake. In the *Constructio* Napier used the latter phrase, but a profound consideration of his own system led him to frame the term Logarithms before he published his canon; and the first knowledge of the system that the world obtained was through that nomenclature which

tion for Scotland, to observe the most recondite department of science receiving its finest and most powerful expansions in the hand of a Scottish baron of the 16th century.

It is singular, that while Dr Hutton, in the history commented upon, would lead his readers to suppose that the Logarithms had been attained by some natural transition from the observation of numerical progressions, in which many calculators were simultaneously engaged, he has elsewhere recorded another error, the very antipodes of the former, in which he supposes the Logarithms to have been viewed and reached through an algebraic medium which belongs to a period of science whose date is long after Napier. Our author, in his *Mathematical Dictionary*, (*Exponent of a power*,) after stating that "exponents, as now used, are rather of modern invention," and noticing the rude and cumbersome approaches made towards their present notation, finally traces that system to *Descartes* and *Girard*, both of whom, it must be observed, wrote after Napier was dead. He then adds: "The notation of powers and roots by the present mode of exponents, has introduced a new and general arithmetic of exponents or powers; for hence powers are multiplied by only adding their exponents, divided by subtracting the exponents, raised to other powers, or roots of them extracted, by multiplying or dividing the exponent by the index of the power or root. So $a^2 \times a^3 = a^5$, and $a^{\frac{1}{2}} \times a^{\frac{1}{4}} = a^{\frac{3}{4}}$; $a^5 \div a^6 = a^{-1}$, and $a^{\frac{3}{4}} \div a^{\frac{1}{2}} = a^{\frac{1}{4}}$; the 2d power of a^3 is a^6 , and the 3d root of a^6 is a^2 . *This algorithm* of powers led the way to the invention of logarithms*, which are only the indices or exponents of powers: and hence the addition and subtraction of logarithms answer to the multiplication and division of numbers; while the raising of powers, and extracting of roots is effected by multiplying the logarithm by the index of the power, or dividing the logarithm by the index of the root." Thus we have two different accounts of the invention of Logarithms furnished by Dr Hutton. The one is, that many learned calculators, about the close of the sixteenth and the beginning of the seventeenth century, "set themselves" to find the Logarithms through the numerical properties pointed out by Archimedes, and actually laid down all the necessary principles; so that "many persons had thoughts of such a table of numbers;" though, he admits, "the world is indebted for the first publication of Logarithms to John Napier." Dr Hutton's other account, however, is, that

has stood the test of ages, and remains unchanged under every new application, and every refined analysis of the Logarithmic power. The word of itself affords evidence, that, although Napier demonstrated his system by the geometrical means of fluxions and fluents, his consideration of the subject was just as arithmetical as Kepler's. Delambre has well observed of Kepler's method of proportions, "*ce système est celui de Néper—cette origine rend raison de la dénomination logarithmique qui signifie nombre des raisons; mais cette dénomination est de Néper, ainsi que l'idée qui la lui a fournie: λογων αριθμῶν.*"

* Dr Hutton's own explanation of *algorithm* is; "the common rules of computing in any art; as the algorithm of numbers, of algebra, of integers, of fractions, of surds, &c. meaning the common rules for performing the operations of arithmetic, or algebra, or fractions, &c." Now the *arithmetic of powers and exponents* had no existence until after Napier's death.

the *algorithm of powers*, as that was established by Descartes and Girard *after Napier's death*, and towards the middle of the seventeenth century "led the way to the invention of Logarithms!" That we may clear up this matter to the general reader, it is necessary to say a few words of powers and exponents,—a doctrine which derives its whole efficacy from its system of notation.

The product of any number multiplied by itself is called a *power* of that number. Thus 9 is a power of 3, because three times three is nine. The multiplication by the same number may be prolonged to any extent, and all the successive products are called powers of that number. So our arithmetical scale, 10, 100, 1000, &c. is composed of the powers of 10. In this series, however, there is a property inherent in its system of notation, namely, that the number of cyphers of the product mark the number of times that the multiplier, or root, has entered into the operation of producing it. Thus 100 is equal to 10 multiplied by 10; or, to express it algebraically, $10 \times 10 = 100$. So $10 \times 10 \times 10 = 1000$. By a rule in algebra, the philosophy of which it is unnecessary to expound here, a number is considered the first power of itself. So 100 is the 2d power (*square*) of 10; 1000 the 3d power (*cube*); 10000 the 4th power, &c. Another notation, however, to the same effect, is to repeat the root itself with a small number beside it, indicating the order of the power, thus 10^2 , 10^3 , 10^4 , &c. Here is an example of the modern notation of powers and exponents. But it is only the notation in a particular case, and must be generalised before it can acquire the important place it actually holds in the system of numbers. One grand distinction betwixt arithmetic and algebra is, that the former considers and works a question in reference only to a particular case, while the latter affords a general rule for a variety of cases. Hence in algebra the letters of the alphabet are taken as symbols to represent indefinite quantities. The notation of which an example is given above may be considered as applicable to any geometrical progression of numbers, and consequently, is capable of being expressed in the general language of algebra. Thus take any number a for the root, or first power, and its successive powers will be a^2 a^3 a^4 , &c. which signify the same as aa , aaa , $aaaa$, &c. or it may be still further generalised, a being taken for the root, and x for the exponent, thus a^x : this expression is called an exponential quantity, where a may stand for any root, and x for any exponent; and therefore a^x may represent all possible values or numbers from zero to infinity. The universal exponential notation, of which an example is here given, belongs to a vast and fertile field of algebraic analysis, that cannot be said even to have opened in Napier's time. Stifellius, Bombelli, Stevinus, and a few others, made some rude attempts to denote the exponents of powers by indices, or small numbers; but this notation was not immediately appreciated or improved, and even Harriot, whose algebra appeared long after Napier's death, denotes the order of the power, by the defective and cumbrous expedient of repeating the root itself, thus, a , aa , aaa , &c. To the great Descartes is yielded the merit of the exponential notation now in use, and hence it is called the *Cartesian*

notation. Through this it was that the universal arithmetic of powers and exponents became developed. The system was found to be flexible to any extent and in every direction. The *law of continuity* (or that algebraic principle which considers a numerical scale as indefinitely extended in both directions, ascending and descending) introduced inverse powers and negative exponents, as the reciprocals of direct powers and positive exponents,—an extension precisely similar to that which Napier first gave to the arithmetical scale, when he proposed the notation of Decimal fractions. The doctrine of fractions was also applied to exponents; and it was discovered that integral and fractional exponents, whether rational or surds, belonged equally to the same system of notation, and could be worked in the same manner. Thus decimals came to be used as fractional exponents. In short, passing through many illustrious hands, the exponential system obtained an unlimited extension, so that in Newton's it reached the Binomial Theorem, which may be called the bridge that spans the chasm betwixt common algebra and the higher calculus.

One result of this analytical developement, even before it reached the crisis of Newton, was very important, but not so exciting as it would have been had Napier not anticipated the treasure. It is obvious that the exponents of any root compose an arithmetical series, adapted to a geometrical one which is composed of the powers whose values the exponents express; consequently, when it was discovered that an exponent might be a number of any denomination, integral or fractional, negative or positive, rational or surd, it followed directly that every number whatever might be considered as a power of any given number. Thus, for example, 100 is a power of 10, whose exponent is 2, *i. e.* $10^2 = 100$. Now, as the value of a power depends upon its exponent, and as the system is found to be infinitely flexible, it follows that the numbers betwixt (say) 10 and 100 could be viewed as powers of 10, having their values denoted by *fractional* exponents; for, although none of these powers would be *commensurable*, the doctrine of surds afforded a notation expressive of approximations infinitely near the truth. Here, then, is the adaptation of an arithmetical to a geometrical series, including numbers of every possible description, so that the logarithmic principle observable in the Arabic system is no longer confined to the ascending decuple scale, but, from a special arithmetical case, has become an algebraic law of universal application. It would have been impossible to have reached this refined extension of the notation of powers and exponents, without detecting all those operations by means of the exponents which afford the same results as the more complicated operations with the corresponding powers; and by this path Napier's great invention must have been discovered; for the observation is perfectly just, that in whatever terms the method of Logarithms has been stated and explained, its principle *may be reduced* to this, "that all numbers are feigned to be equal to the powers of a certain assumed number."

Had the Logarithms been disclosed through this gradual progress of the notation of powers and exponents, however valuable the discovery, it would probably not have at-

tached an immortal name to any individual. We would have been indebted for it to all those who had improved and advanced the algebraic notation in which it lurked. It would have been insensibly attained, as it were, in the natural and inevitable course of numbers, and would have been due to a system, the very dawn of which had not appeared in our philosopher's lifetime, *unless that dawn be his own work*. Professor Playfair remarks particularly, that Napier could derive no assistance from such analytical considerations, but arrived at the Logarithms by an original path of his own; and for that reason he bestows this eulogy upon him, that, "as there never was any invention for which the state of knowledge had less prepared the way, there never was any where more merit fell to the share of the inventor." Yet the real value of this praise, which is as just as it is high, has been obscured in quarters where we would have least expected confusion on the subject. Professor Powell, whose work, already referred to, is the latest history of mathematics in which Napier is mentioned, has transferred almost *verbatim* to his own text Playfair's account of the Logarithms and eulogy of their author. But, overlooking the real point of that eulogy, the Savilian professor adds rather inconsistently, "Hence, to conceive the fundamental idea, that all numbers might be regarded as some powers of one given number, and to devise the actual means of finding the indices of those powers, must be allowed to have been indications of genius of the highest order." But the fundamental idea here assumed to have been Napier's, belongs to a subsequent development of algebraic analysis, independently of which, for *that* was his great merit, he achieved the Logarithms. It is an idea belonging to that mature state of the exponential system wherein a chapter, "*De quantitibus exponentialibus ac Logarithmis*,"* is made preliminary to an exposition of the infinitesimal analysis. But it is at variance with the history of analytical science to suppose that Napier could generalize like Euler, which, however, he must have done if he really reached the Logarithms by the contemplation in question. Delambre, while he views the Canon Mirificus through the modern analysis, is most careful to avoid giving the impression that Napier did so, or had the aid such a view implies; "C'est par anticipation (says he) que j'écris na , n^2a , n^3a , &c. on n'avait encore aucune idée des exposans;" and wherever that philosopher uses such expressions in reviewing our philosopher's work, he reminds the reader that Napier did not look through any such medium as this *translation* of his thoughts might seem to imply; "ce calcul est *la traduction* de ses raisonnemens." Herschel observes, that Wallis's *Arithmetica Infinitorum*, published in 1655, is the first work "in which we find that full reliance on what is called the law of continuity in analytical expressions, which has since led to so many brilliant generalizations;" but that "the notation of exponents was invented by Descartes." Now unquestionably the fundamental idea attributed to Napier is only co-existent with a knowledge of the full meaning and utility of exponential notation; and we would put it, therefore, to the *manes* of Dr Hutton, and the *cathedra* of Professor Powell, whether, before the close of the six-

* *Introductio in Analysin Infinitorum. Auctore Leonhardo Eulero, 1797, cap. 6.*

teenth century, Napier can be supposed to have generalized in this form $a^x = y$?* Did he select a base for his system? or consider a base in Logarithms at all? or can he be supposed to have known that e (a transcendental number begotten upon the Canon Mirificus by the Binomial Theorem,)[†] was really the base of his own original and parent system of Logarithms? If he could know nothing of all this, then it is only confounding the history of his invention to say, that the algorithm of powers led to it, or that the foundation of his conception was the analytical idea, that all numbers might be regarded as some powers of one given number.

But we verily believe, that, had Napier lived twenty years longer, he would have reaped in rapid succession many of those laurels which the path of analytical science yielded so gradually to many philosophers between him and Newton. In his letter to King James, he tells that monarch that he could bring him gifts as rare as Tycho's. He verified that hint with the Canon of Logarithms. In his dedication of the Logarithms, he tells Prince Charles, that, if he received them in good part, it would "encourage me, that am now almost spent with sickness, shortly to attempt other matters perhaps greater than these." Had he been spared, this promise, too, would have been realized. There was before him the whole of that wonderful field of analytical inquiry, from which, by anticipation, he had already snatched one of its most precious disclosures. We must now turn to his manuscript "Booke of arithmeticke and algebra," which affords the most convincing proofs, that with an innate algebraic power equal to Newton's, but without one

* The equation $a^x = y$ contains various relations. x is the *exponent* of a ; y is the *power* of a ; a is the *root* of y ; x is the *logarithm* of y ; y is the number of which x is the logarithm. Thus it is obvious that the *exponent* of a and the *logarithm* of y mean the same quantity. In this equation a is also termed the *base*, and so x is the *logarithm* of y to the base a . Of this algebraic generalization the algorithm $10^2 = 100$ is a particular arithmetical case. 2 is here the *exponent* of the operation of raising 10 to its second power, 100; 2 is therefore the logarithm of 100, and 10 is the *base* of that logarithm. These are modern refinements in analytical science of which Napier knew nothing. He had not the algorithm.

† The letter e in modern algebra is taken to represent the base of Napier's first system of logarithms, which is the fundamental and parent system of all logarithms. That base is equal to the number 2.7182818. Now, until the binomial theorem, and the modern doctrines connected with it, afforded new and comparatively easy methods of computing logarithms, the number e was unknown. A treatise on arithmetic and algebra, published by the Society for the Diffusion of Useful Knowledge, details the algebraic process which produces the number e , and the author adds, "the student will find $e = 2.7182818$; this quantity then is known; the discovery of it does not at present appear to have brought us nearer our object, but we shall find it a necessary instrument in arriving at it; it is the base of a system called the *Napierean*, from Napier, a celebrated mathematician of the seventeenth century, who invented logarithms, and *calculated them to this base*." But this is a complete mistake. Napier did not calculate his system to a base at all; it might as well be said that he computed his tables through the *expansion of a^x* , or by means of a *rapidly convergent series*. Napier was so far in advance of science that men forget when he lived. Delambre most justly observes, that the easiest methods of computing logarithms were discovered after the greatest difficulty and toil had been accomplished.

of the many powerful aids which the English philosopher obtained from the algebra of his day, Napier, ere he found the Logarithms, had launched himself in the very path most likely to have led him even to the Binomial Theorem.

This very interesting fragment has hitherto been secluded in the family charter-chest. Unfortunately it is written in Latin, and would occupy upwards of 130 quarto pages, so is not suited for an appendix to his memoirs in its original state. I shall endeavour to give such an account of it as will afford some insight into the nature of the preparatory study, and mental discipline, through which our philosopher passed to the complete accomplishment of his greatest design. The reader must not fail to keep in view the circumstances of its ancient date, and the local disadvantages under which the treatise was written. We have already noticed the works upon the same subject that were published before he can be supposed to have written anything; and, considering how few they were, how slowly books were then spread abroad, and that literary communication between Scotland and the Continent was then so slight, as to leave Kepler in ignorance of Napier's death two years after that event, we must not suppose that our philosopher had at his command even those scanty sources of information the Continent could afford on the abstruse subjects to which he was attached. This is not to excuse defects or rudeness in his treatise on numbers, but to enhance the surprise that he should then have written as he did, and that even his unpublished papers should be so worthy to meet the eye of modern mathematicians. Any one who now takes the trouble to peruse the Canon Mirificus, and his other published works, (and this is rarely done even by men of science), will be struck, not merely with the invention, but with the power, simplicity, and elegance that characterise all his treatises, and the air that pervades them of having been written a century after his time. The very same may be said of the manuscript we are about to consider. Whole chapters of it might be literally translated and transferred to the most careful and recondite treatise on numbers of the present day. Yet it is the oldest treatise of the kind composed in Britain. Recorde's works are rudely elementary compared to this of Napier's, which is a beautiful treatise on the philosophy of numbers, free not merely from the puerile *facetia* * of the old English writer, but, what is remarkable, from every vestige of mysticism or superstition. It must have been composed before he had formed an idea of the Logarithms, because although the arithmetical part is entire, and brought to a close, there is not the slightest allusion to his great invention, nor to the system of Decimal fractions. I presume, therefore,

* "Master. Exclude number, and answer this question; how many years old are you? Scholar. Mum.—Master. How many days in a week? how many weeks in a year? what lands hath your father? Scholar. Mum.—Master. So that if number want, you answer all by mummies. How many miles to London? Scholar. A poak full of plums.—Master. If number be lacking it maketh men dumb, so that to most questions they must answer mum," &c. "What call you the science you desire so greatly? Scholar. Some call it *arsemetrick*, and some *augrime*.—Master. Both names are corruptly written, *arsemetrick* for *arithmetic*, as the Greeks call it, and *augrime* for *algorisme*, as the Arabians sound it," &c.—*Recorde's Arithmetick*.

it was written before he had seen the work of Stevinus, which he quotes in the *Rabdologia*. Unquestionably it is the oldest philosophical treatise on numbers composed in Scotland.

The general plan and division of his subject is of itself sufficient to show the profound and comprehensive view he had taken of numerical science. He terms his subject, generally, LOGISTIC (*logistica*), which he defines "the art of computing well," and his principal division of it is into four books, of which the *first* (he says) regards the computation of quantities common to every species of logistic; the *second* relates to Arithmetic, which he defines, "the Logistic of *discrete quantities by discrete numbers*;" the *third*, he calls Geometrical Logistic, and defines it "the Logistic of *concrete quantities, by concrete numbers*;" the *fourth* is Algebra, which he defines, "the science of solving questions of magnitude and multitude" (*quanti et quoti*.) The classification of his system, minute, clear, and philosophical, affords a striking illustration of what Robert Napier declared to be the acknowledged characteristics of his father's mind, namely, the power with which he could condense, and the simplicity with which he could expound. The first book consists of eight chapters, and commences in this simple manner.

"*Logistic* is the art of computing well. *Computation* is the action or operation which, from several given quantities and their properties, finds what is sought. These are given either by *vocal nomination*, or in *written notation*. Hence in all Logistic, first comes *nomination* and *notation*, and then follows *computation*. *Computation* is either *simple* or *compound*. That is simple computation which, from two given quantities, finds a third by a single or uniform operation. Simple computation is either *primitive* or *derivative*. That is primitive computation which computes one quantity with another only once; and which, from any two of a whole, a part and a remainder given, finds the third. Primitive computation is either *Addition* or *Subtraction*. Addition is that primitive computation in which several quantities are added, and a whole is produced: *ex. gr.* let 3 and 4 be added, and there will be produced 7 for the whole: so let 2, 3, and 4 be added, and there will be produced 9. Subtraction is that primitive computation in which the *subtrahend* is taken from the *minuend*, (*subtrahendum a minuendo*), and a *remainder* is produced. Thus, let 4 be taken from 9, and 5 remains. 4 is called the subtrahend, 9 the minuend, and 5 the remainder. Subtraction is either of *equal* quantities and nothing remains, or of *unequal* quantities. The subtraction of unequal quantities, is either of the *less* from the *greater*, and the remainder is a quantity greater than nothing (*major nihilo*), or it is of the *greater* quantity from the *less*, and the remainder will be less than nothing (*minus nihilo*). Thus, subtract 5 from 5, and there remains nothing: subtract 3 from 5 and there remains 2 more than nothing: but subtract 7 from 5 and there remains 2 less than nothing, or nothing diminished by 2. Hence the origin of *defective* quantities, namely, by the subtraction of the greater from the less, and of these I shall speak in their proper place. From the premises, it is clear that Addition and Subtraction are related; and thus the one is the proof (*examen*) of the other. Thus, as a proof whether

2 is the remainder of 3 from 5, add 2 and 3, and 5 is restored. On the other hand, as a proof whether 2 and 3 added make 5, subtract 3 from 5 and 2 is restored; or otherwise subtract 2 from 5 and 3 is restored. There is, besides, another proof of subtraction in itself, namely, by subtracting the remainder from the minuend, so as turn the first subtrahend into the remainder; thus, as the proof whether 2 be the remainder of 3 from 5, subtract 2 from 5 and 3 is restored. And so it is, that any two of a *whole*, a *part*, and a *remainder* being given, you have the third by Addition and Subtraction."

Having disposed of his general view of primitive computation in this first chapter, Napier passes in the second to derivative (*ortæ*), which he defines, "the computation of quantity with quantity more than once." He considers it as derived either from Addition and Subtraction, by a repetition of those primitive operations (*ortæ ex primis*), which gives *Multiplication* and *Division* (*Partitio*;) or as derived from these again (*ortæ ex primo ortis*), which gives *radical-Multiplication* and *radical-Partition*; in other words, involution and evolution. Nothing can be more elegant and symmetrical than the manner in which he brings out the *genealogy* of those great operations whose prolific field was all before him. We have seen, in the first chapter, that his leading division is into *simple* and *compound* computation. He regards all simple computation as having to do with three quantities, of which any two are given, and the third is to be found from them; and he also shows how intimately all simple computations are related to each other; the different species of the same kind being the mutual proofs of each other, and the different kinds naturally arising each out of the more primitive. He shows how Addition and Subtraction test each other. Multiplication he views as *continued Addition*, and defines it thus elegantly; "Multiplication is the continued addition of either of the two given quantities, as often as there are units in the other; the product is the *multiple*; thus 3 multiplied by 5 is the same as 3 five times added, or 5 three times added; being 15." The three quantities in this operation he calls the *multiplier*, the *multiplicand*, and the *multiple*. Division he views as "the continued subtraction of the *partient* from the *partitor* until nothing remain, and the number of subtractions is the *quotient*." He then shows that Division may be perfect or imperfect, and points out how "*fractions* derive their origin both from the partition of the less by the greater, and the *imperfect* partition of the greater by the less;" and he concludes, as in the previous chapter, by showing that Multiplication and *perfect* Division mutually prove each other. The third chapter contains the third class of simple computation, namely, that which is derived from Multiplication and Division by a *repetition* of those operations. Here the three quantities considered are thus defined; "the *radicate* * is that quantity

* What Napier calls *radicatum* is now called *power*. It forms another of the several coincidences between Napier and Sir Isaac Newton, that the latter also wrote a Latin work upon arithmetic and algebra, entitled *Arithmetica Universalis*, being the substance of his lectures delivered at Cambridge. In that work I find Newton, like Napier, uses the words *index* and *radix*; but the third quantity he calls "*dimensio, vel potestas, vel dignitas*." Napier's *radicatum* will bear the most hypercritical scrutiny; it regards the quantity as rooted, or composed of roots, which are to be decomposed, or evolved, in

which returns to unit by repeated partition by some other quantity; the number of partitions is the *index*, the dividing number is the *root*." These three quantities he considers subject to three operations; 1. *radical-Multiplication*, which he defines, the continued multiplication of the given *root*, as often as there are units in the *index*, to produce the *radicate* sought;" and he shows that the remultiplication may be infinite; it may be "*duplication*, which is the multiplication of two equal quantities together, or the given quantity placed twice, (*bis posite*); *triplication* is the given quantity thrice placed, &c. in these cases the radicate becomes the *duplicate*, or *triplicate*, or *quadruplicate*, &c. the index is *two*, or *three*, or *four*, &c. the root is *bipartient*, or *tripartient*, or *quadripartient*," &c. The example he affords is by placing 2 for the root, and 2 for the index, and then he raises 2 to the seventh power, as "in the following table, where the prior series (*prior series*) are *indices*, and the latter *radicates*."

“	I	.	II	.	III	.	IIII	.	V	.	VI	.	VII	.	&c.*		
	1	.	2	.	4	.	8	.	16	.	32	.	64	.	128	.	&c.”

He next considers, (in the same chapter) 2. *radical-Partition*, which he defines, "the continued partition of the *radicate* by the *root* down to unit, and the number of partitions is the *index* sought. In the fourth chapter, he takes up the important case of, 3. *Extracting the root* itself. He defines this process, "finding that third quantity which, the *index* being given, raises the given *radicate* by radical-Multiplication, or resolves it by radical-Partition." He then lays down that the extraction may be *perfect* or *imperfect*; "perfect where there is no remainder,—imperfect where a remainder is left irresoluble; thus, if the *tripartient-root* is to be extracted from the radicate 9, the nearest number is 2, which by radical-Multiplication raises 8, and not 9; it is, therefore, called an imperfect extraction, as 1 remains unextracted; whatever numbers so remain are termed irresoluble (*irresolubiles*;) the number obtained by the imperfect extraction is called the *lesser term*, to which, by adding unit, the *greater term* is obtained; between which terms the true and perfect root lies hid." Our philosopher then proposes a very

order to produce the *index*, which again denotes the quality of the *radix*. *Radicatum* being thus expressive, I have translated it *radicate* instead of power.

* It is curious to find, in this example, the inventor of Logarithms framing a logarithmic table *unconscious* of that property of the particular arrangement. The reader will at once perceive the *Archimedean theorem* in the numeral arrangement quoted; the upper series being truly logarithms to the lower. But Napier gives it without any reference to that particular property. Had this been his first step to the Canon Mirificus, that work would have presented a very different aspect. We would probably have heard nothing of his *fluxions* and *fluents*; but every thing about the arithmetic of *indices*; he would have *selected a base* for his system, and that a simple one; the tables computed under those circumstances would probably have been of the kind called *antilogarithms*. (See *Dr Hutton's History of the Construction of Logarithms, and Dodson's Preface*.) The example shows that Napier had not the exponential system in a state to reach the logarithms by that path, though he unconsciously affords a rude table of powers and exponents as well as of logarithms; had he simply repeated the *root* instead of giving the *radicate*, and then reduced his *indices* to small numerals thus, $2^1, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7$, he would also have afforded a specimen of the *Cartesian notation*.

curious notation of his own for these imperfect roots, which shall be afterwards noticed more particularly. His next proposition is, that "in radical-computations, some indices are *even* and some *odd*; some again are *prime*, *i. e.* only divisible by unit, others *composite*, *i. e.* perfectly divisible by some other number." After giving examples, he adds, "hence a compendious method of radical-Multiplication and Extraction where the indices are composite, for it is easier to multiply, or extract, by means of the component parts of the index separately, than by the composites themselves," &c. He closes this chapter, as the former ones, by showing that each of the three operations of radical computation is proved by the other two. This concludes his general view of *simple* computations,—their relations to and dependencies upon each other. I may here observe that he never leaves a term without a definition, or a proposition without examples.

The remaining four chapters of the first book are devoted to the general view of "*compound* computation, or *Rules*." This he defines "the computation which, by several and divers modes of operation, produces the quantity sought from several given quantities." The fifth chapter accordingly treats of compound computation, embracing rules of *proportion* and *disproportion*. It contains a remarkable example of his practical powers, and of his unremitting attempts to create compendious rules where he found them wanting. I shall translate it, therefore, nearly at length, as he seems to have laid some stress upon his own peculiar method; and it may be doubted if any thing better is to be met with on the subject even now.

"Rules of Proportion are those which solely by means of simple proportionate computations, such as Multiplication and Partition, discover from several given quantities the quantity sought; as, if it be asked, how many miles he may go in 6 hours who goes 4 miles in 3 hours? or,—if 6 oxen be nourished for 4 days upon 3 measures of hay, how many oxen may be nourished in 2 days upon 5 measures? or,—20 shillings Scotch are 1 pound, 2 pounds are 3 marks, 5 marks are worth 1 crown; how many shillings, then, are 9 crowns worth? Questions of proportion have no introduction through Addition and Subtraction; for Multiplication and Partition are proportional computations as a consequence of their definitions. Two things are considered in such computations,—*position*, and *working*. Position is regulated by four precepts. *First*, that a line be drawn, and a place prepared under it for the quantity sought, along with its collaterals, as follows, in terms of the three examples given above.

$$1. \frac{6 \text{ hours, } 4 \text{ miles.}}{3 \text{ hours, how many miles.}} \quad 2. \frac{6 \text{ oxen } 5 \text{ meas. } 4 \text{ days.}}{\text{how many ox. } 3 \text{ meas. } 2 \text{ days.}} \quad 3. \frac{20 \text{ shil. } 2 \text{ pnd. } 5 \text{ mr. } 9 \text{ cr.}}{\text{how many shil. } 1 \text{ pnd. } 3 \text{ mr. } 1 \text{ cr.}}$$

Second, that two quantities, of which the one decreases as the other increases, be placed as collaterals on the same side of the line. As, in the first example, by how much the first hours abound, namely, 3, so much fewer will be the miles sought; in the second example, as the number of oxen increase, the number of days in which they may be nourished on the same measure decrease; hence, 3 hours and the miles sought,—6 hours and 4 miles,—again, 6 oxen and 4 days,—the oxen sought and 2 days,—are respectively

placed on the same side of their lines. *Third*, that two quantities increasing or decreasing together, must be placed on the opposite sides of the line; thus, as the 3 hours increase, so must the 4 miles, *et contra*," &c. "*Fourth*, that two cognominate quantities be always separated by the line; as in the first example, 3 hours to 6 hours, and 4 miles to the miles sought," &c. "These precepts of *position* being attended to, the following single general precept of *working* will suffice for the solution of every question of this kind."—"Multiply the upper quantities together, also the lower together; then divide the multiple of the upper quantities by the multiple of the lower; and the quotient will solve the question by giving the quantity sought. Thus, in the first example, 6 and 4 multiplied make 24, which divide by 3, and that will give 8, the number of miles sought;—or, in the second example, 6, 5, and 4 multiplied make 120, then multiply 3 and 2, which make 6, by which divide 120, and that will give 20 and solve the second question;—or, in the third question, multiply 20, 2, 5, and 9 together, which make 1800; then multiply 1, 3, and 1, which make 3, by which divide 1800, and that will give 600, the number of shillings which are equal in value to nine crowns. In this manner, *I bring every species of rules of proportion under one general method and operation.* The authors treat of infinite species and forms of the doctrine of proportion, such as the rules of three or the golden, of simple, double, five-quantities, six-quantities, direct, inverse, &c. but they have not touched the triple rule, or any of its manifold forms, all of which you have here in this brief form."*

"So much for Rules of Proportion; the Rules of Disproportion follow; but as these, besides the proportional computations, embrace additions and subtractions and other computations disturbing proportion, mixed up together, therefore I dismiss all these, as what may be sufficiently comprehended under algebra. As the rules of alligation, society, falsehood, simple proportion, double proportion, and many others, form the greatest part of all arithmetical rules, so of geometrical rules do propositions, problems, theorems, &c. which, confused both from their variety and number, disturb the memory. These therefore I leave, to be presently treated of under algebra."

Having disposed of quantities "*in genere*," Napier takes up the division "*suarum specierum*." His first division of the species is into abundant and defective quantities, (*abundantes et defectivæ*), to which the sixth chapter is devoted. Upon this chapter our philosopher lays much stress, and I shall give it entire.

"Abundant quantities are those which are greater than nothing (*majores nihilo*), and carry the idea of increase along with them. These have either no symbol prefixed, or this one +, which is the copulative (*copula*) of increase. Thus, if you are not in debt, and your wealth be estimated at 100 crowns, these may either be noted 100 crowns, or

* *Recorde's Arithmetic* confirms this remark. There I find, the golden rule direct and inverse, the double rule of proportion, the rule of proportion composed of five numbers, the rule of fellowship, the rule of alligation, the rule of falsehood; but nothing similar to Napier's. He made every rule *golden* that he touched; witness his *trigonometrical rules*.

+ 100 crowns; and are read a hundred crowns of increase; always signifying wealth and gain. The computations of such quantities are to be learnt both from what has been said and what is to follow. Defective quantities are those which are less than nothing (*minores nihilo,*) and carry the idea of diminution along with them. These are always preceded by this symbol —, which is the copulative of diminution. Thus, in the estimation of his wealth whose debts exceed his goods by 100 crowns, justly his funds are thus prenoted, — 100 crowns, and are read, a hundred crowns of decrease; signifying always loss and defect.* I have already shown that defective quantities have their origin in

* *Abundant* and *defective* terms are now used in a totally different sense. A number is sometimes considered as composed of *aliquot* parts, *i. e.* of other numbers, any one of which, being repeated a certain number of times, makes up the whole number precisely; thus 1, 2, and 3, are the aliquot parts of 6. Now when the aliquot parts of a number, added together, make up a sum greater than that number, they are the aliquot parts of an *abundant* number; if less, of a *defective* number; if precisely the number, as in the example given, it is a *perfect* number. The terms now in use to express Napier's idea are *negative* and *positive*. Sir Isaac Newton, in his Algebra, says, "*Quantitates vel affirmativæ sunt, seu majores nihilo; vel negativæ seu nihilo minores. Sic in rebus humanis possessiones dici possunt bona affirmativa, debita vero bona negativa;*" the very example which Napier gives. Dr Horsley, Newton's commentator, observes at this passage; "*Albertus Girardus, ni fallor, omnium primus, (quem summum interea mathematicum agnosco,) durâ quâdam verborum figurâ, Diophanto et Vieta prorsus ignotâ, quam vellem Cartesius et nostrates minus avidè arripuissent, nihilo minores, dixit.*" This shows how neglected Napier's great work is by the learned. Horsley, of course, could not know, that in Napier's *unpublished manuscript* there was a chapter upon this distinction, but he might have read in the *Canon Mirificus*, c. i. p. 5, "*Logarithmos sinuum, qui semper majores nihilo sunt, abundantes vocamus, et hoc signo +, aut nullo prenotamus; logarithmos autem minores nihilo defectivos vocamus, prenotantes eis hoc signum —.*" This was published *fifteen years* before the work of Girard, to which Horsley alludes. Dr Hutton, in his History of Algebra, has fallen into the same mistake; "Girard was the first who gave the whimsical name of *quantities less than nothing* to the negative ones." Here is another indication that Hutton analyzed Napier's works, and presumed to attack his character, without reading the original proofs as he ought to have done. Even Leslie and Playfair had not read the *Canon Mirificus*. The former says, "Girard was possessed of *fancy* as well as invention; and his fondness for philological speculation led him to frame *new terms*, and to adopt certain modes of expression which are not always strictly logical; though he stated well the contrast of the signs *plus* and *minus*, in reference to mere geometrical position, he *first introduced* the very inaccurate phrases of *greater* and *less than nothing*." Playfair says, "Girard is the author of the figurative expression, which gives the negative quantities the name of *quantities less than nothing*; a phrase that has been severely censured by those *who forget* that there are correct ideas which correct language can hardly be made to express." It is, indeed, *unphilosophical* fastidiousness to call the phrase "very inaccurate." Napier fortified it by a better *nomenclature*, in the terms *abundant* and *defective*, than those now in use,—*positive* and *negative*, which are said to convey erroneous impressions. Again, his *exemplification* of the idea is that which is invariably adopted now, though not from him. Surely Euler was never rummaging in the Merchiston charter-chest? Yet his illustration is identically Napier's; "In algebra, simple quantities are numbers considered with regard to the signs which precede or affect them. Farther, we call those *positive quantities*, before which the sign + is found; and those are called *negative quantities* which are affected by the sign —. The manner in which we generally calculate a person's property is an apt illustration of what has just been said; for we denote what

subtracting the greater from the less. Abundant and defective quantities come under the operation of Addition; where the signs are alike, by prefixing their common sign to their aggregate sum; thus, $+3$ and $+2$ make $+5$; but if their signs are unlike, they are added by prefixing the sign of the greater quantity to the difference between them; thus, $+6$ and -4 make $+2$. In Subtraction they are worked by changing the sign of the *subtrahend*, and adding it to one or other of the given quantities according to the foregoing rules; thus, in subtracting $+5$ from $+8$, change $+5$ to -5 , then, as before, add -5 to $+8$, which gives $+3$, the remainder sought; so to subtract $+8$ from -5 , change $+8$ into -8 , which added to -5 gives -13 , the remainder sought; so -5 from $+8$ gives $+13$; and $+5$ from -8 gives -13 ; and -5 from -8 gives -3 ; and $+8$ from $+5$ gives -3 ; and -8 from $+5$ gives $+13$; and -8 from -5 gives $+3$. Abundant and defective quantities are multiplied and divided, where the signs are alike, by prefixing to the *multiple* or the *quotient* the sign of *plus*, (pluris;) or, if unlike, the sign of *minus*, (minutionis;) thus, $+3$ multiplied by $+2$, or -3 multiplied by -2 produce the multiple $+6$; and if $+6$ be divided by $+3$; or -6 by -3 , the quotient $+2$ is produced. But if $+3$ be multiplied by -2 , or -3 by $+2$, the multiple will be -6 ; and if $+6$ be divided by -3 , or -6 by $+3$, the quotient will be -2 ."

"*Roots*, both abundant and defective, having an *even index*, when radically multiplied produce an abundant *radicate*; thus, multiply the root $+2$ to the index 4, and there will be given, *first*, $+2$; *second*, $+4$; *third*, $+8$; *fourth*, $+16$; in like manner -2 gives, *first*, -2 ; *second*, $+4$; *third*, -8 ; *fourth*, $+16$, as before. Hence it follows, that an *abundant radicate*, whose index is even, has two roots, one *abundant*, and the other *defective*, and that a *defective radicate* has *no root*; for in the above example both the abundant $+2$, and the defective -2 , are the quadripartient (*fourth*) roots of the abundant radicate $+16$; therefore there are none remaining, either abundant or defective, which can be the

a man really possesses, by positive numbers, using or understanding the sign $+$; whereas his debts are represented by negative numbers, or by using the sign $-$: Thus, when it is said of any one that he has 100 *crowns*, but owes 50, this means that his real possession amounts to $100-50$; or, which is the same thing, $+100-50$, *i.e.* 50. Since negative numbers may be considered as debts, because positive numbers represent real possessions, we may say that negative numbers are *less than nothing*; thus, when a man has nothing of his own, and owes 50 crowns, it is certain that he has 50 crowns *less than nothing*; for if any one were to make him a present of 50 crowns to pay his debts, he would still be only at the point of nothing, though really *richer* than before. In the same manner, therefore, as positive numbers are incontestably greater than nothing, negative numbers are *less than nothing*." *Maclaurin*, too, defends the phrase, but illustrates the idea more poetically; "the depression of a star below the horizon may be equal to the elevation of a star above it; but those positions are opposite, and the distance of the stars is greater than if one of them was at the horizon so as to have no elevation above it, or depression below it; it is on account of this contrariety that a *negative quantity* is said to be *less than nothing*; because it is opposite to the positive, and diminishes it when joined to it, whereas the addition of 0 has no effect; but a negative is to be considered no less as a *real quantity* than the positive." The opinion of *Leslie*, who calls the phrase *inaccurate*, and of *Hutton*, who calls it *whimsical*, must go down before the opinions of *Napier*, *Newton*, *Maclaurin*, *Euler*, and *Playfair*.

quadripartient root of the defective radicate — 16. Abundant roots, having *uneven indices*, when radically multiplied yield abundant radicates; and defective roots, defective radicates; thus, the abundant root + 2, when radically multiplied to the uneven index 5, yields + 32; namely, *first*, + 2; *second*, + 4; *third*, + 8; *fourth*, + 16; *fifth*, + 32, the abundant radicate; so the defective root — 2, with the index 5 radically multiplied, yields — 32, namely, *first*, — 2; *second*, + 4; *third*, — 8; *fourth*, + 16; *fifth*, — 32, the defective radicate of the said root. In like manner hence it follows that a radicate with an uneven index has only one root, an abundant radicate an abundant root, and a defective radicate a defective root; as in the former example, the abundant radicate + 32 with the index 5, will have the abundant root + 2; so the defective radicate — 32, with the same index, will have the defective radicate — 2. It is unnecessary to repeat here the rules of proportion, as they are compounded of multiplications and partitions, and may be learned from what is premised."

In a subsequent part of his manuscript, when treating of the notation of irrational roots, we shall find Napier referring to this chapter as the foundation of a *great algebraical secret, not previously revealed by anyone*. This shall be considered more particularly, when we come to notice the chapter where these expressions occur. It must be observed, however, that he here lays down the general rules of the arithmetic of *plus* and *minus*, and connects the chapter with his system, in a manner not surpassed, if equalled, in the treatises of Newton, Maclaurin, and Euler.

Our philosopher, in the next place, passes to his second special division of quantities, namely, into *integral* and *fractional*. "Those quantities," says he in chapter seventh, "are called *integral*, which have no denominator, or whose denominator is unit; and those *fractional*, whose denominators are various. The *denominator* is the quantity placed *under* the line, and indicates the number of parts into which unity is divided; the *numerator* is the quantity placed *above* the line, and denotes how many of those parts are taken. For instance, this quantity $3ab$ is an integral quantity; so is $\frac{3ab}{1}$, which is the same thing written in the form of a fraction; again, $\frac{3ab}{2bc}$ and $\frac{3a}{2}$ and $\frac{3a}{2a}$ or $\frac{3}{2}$, which is the same thing, are fractions or broken quantities, whose higher terms are the numerators, and whose lower terms are the denominators." Our philosopher then reminds his reader that it had been previously observed how broken quantities are also produced greater than unit, namely, by the *imperfect* division of the *greater* quantity by the less. "Thus, 9 divided by 2 yields $4\frac{1}{2}$, or, if you prefer it, $\frac{9}{2}$ greater than unit. Hence every numerator sustains the part of the quantity divided; and its denominator, of the quantity that divides it; as in the former example, $\frac{3ab}{2bc}$ signifies that $3ab$ is divided by $2bc$; so $\frac{3a}{2a}$ has the same value as $3a$ divided by $2a$; or, more briefly, 3 divided by 2; or, finally, it has the same value as three parts of unity divided into 2; so $\frac{3}{4}$ are three-fourths of unity, or three divided by four, which is the same thing.* Every quantity having a

* EULER might have written all this; indeed he has written something very like it; he traces from the same source as Napier does the "particular species of numbers called *fractions*, or *broken numbers*;"

numerator and denominator is considered and worked as a fraction, and hence, in order to compute with integers as if they were fractions, 1 is placed beneath them as their denominator. The computation with fractions is facilitated by contracting and abbreviating their terms (*termini*). This is done by dividing the terms in their increased form by their greatest common divisor.* The greatest common divisor is that than which a greater cannot be found capable of *perfectly* dividing each term; and it is found, first by dividing the greater term by the less; then by always dividing the preceding divisor by its remainder until nothing remain; that last divisor, the quotients being neglected, is the greatest common divisor sought; thus, the greatest common divisor of the terms 55 and 15 is found in this manner: divide 55 by 15, there remain 10; divide 15 by 10, there remain 5; divide 10 by 5, and nothing is left over; 5, therefore, is the greatest common divisor, measuring 15 by 3, and 55 by 11. If, however, you arrive at unit for the divisor, then the terms are inabbreviable, and prime, or prime to one another; thus, let the terms be $5a$ and $3a$, divide $5a$ by $3a$, and there remain $2a$; then divide $3a$ by $2a$ and $1a$ remains, by which divide $2a$ and there is no remainder. Hence $5a$ and $3a$ have not a greater divisor than unity, or $1a$, by which if those terms be divided, they become to each other the prime numbers 5 and 3, as more fully shall be laid down in its proper place. But this must be specially looked to in the partition of incommensurable quantities that it will go on eternally without end, as will plainly appear in its proper place; thus of the number 10 and its *bipartient* root, or, as it is called, *square root*, no common measure will be found in eternity; much less that greatest divisor; as in its proper place.

he explains the example $\frac{7}{3}$, and then says, "so, in general, when the number a is to be divided by the number b , we represent the quotient by $\frac{a}{b}$, and call this form of expression a fraction; we cannot, therefore, give a better idea of a fraction $\frac{a}{b}$, than by saying that it expresses the quotient resulting from the division of the upper number by the lower; we must remember also that in all fractions the lower number is called the *denominator*, and that above the line the *numerator*." He turns and views his subject precisely as Napier does; "the nature of fractions is frequently considered in another way, which may throw additional light on the subject. If, for example, we consider the fraction $\frac{3}{4}$, it is evident, that it is three times greater than $\frac{1}{4}$. Now, this fraction $\frac{1}{4}$ means, that if we divide 1 into 4 equal parts, this will be the value of one of those parts; it is obvious, then, that by taking 3 of those parts, we shall have the value of the fraction $\frac{3}{4}$."—*Hewlett's translation of Euler's Algebra*, 1822.

* "In order to reduce a given fraction to its least terms, it is required to find a number by which both the numerator and denominator may be divided. Such a number is called a *common divisor*; and as long as we can find a common divisor to the numerator and the denominator, it is certain that the fraction may be reduced to a lower form; but, on the contrary, when we see that, except no unity, other common divisor can be found, this shows that the fraction is already in its simplest form." This property of fractions preserving an invariable value, whether we divide or multiply the numerator and denominator by the same number, is of the greatest importance, and is the principal foundation of the doctrine of fractions. For example we can seldom add together two fractions, or subtract the one from the other before we have by means of this property reduced them to other forms."—"All whole numbers may also be represented by fractions; for example 6 is the same as $\frac{6}{1}$ because 6 divided by 1 makes 6," &c.—*Euler's Algebra*.

Having obtained the greatest common divisor, and divided by it each term, the new terms arise; and this operation is termed abbreviation."

We now come to the eighth and last chapter of Napier's first book, and he treats his subject so very like Euler, that we are almost surprised to find him at the addition and subtraction of fractions in his eighth chapter, when Euler is only at the same subject in his ninth. But then we must always recollect what his son said of our philosopher, *ex optimorum hominum sententia, inter alia præclara hoc eximii eminibat, res difficillimas methodo certâ et facili, quam paucissimis expedire.* "Fractions," says he, "of the same denomination are subject to the operations of addition and subtraction. If their denominators are diverse they may be reduced to the same. This is done by dividing each denominator by the greatest common divisor, the quotients being noted, then by multiplying both the terms of the first into the quotient of the latter denominator, you have the first new fraction; * and multiplying both terms of the latter by the quotient of the former denominator, gives the latter new fraction of the same denomination: thus, to reduce the fractions $\frac{2}{3}$ and $\frac{7}{9}$ to the same denomination; of denominators 3 and 9, the greatest common divisor is 3, by which divide them and you have 1 for the first, and 3 for the latter; then multiply each term of $\frac{2}{3}$ by the last quotient 3, and $\frac{6}{9}$ is produced as the first new fraction; in like manner multiplying $\frac{7}{9}$ by unit, which is the first quotient, gives the fraction $\frac{7}{9}$ of the same denomination as $\frac{6}{9}$. Being so reduced, these fractions are *added or subtracted* by adding or subtracting the numerators; the sum, or remainder, being taken as numerator, and the common denominator retained," &c. In the same minute and lucid manner, and always preserving the perfect symmetry of his arrangement, our philosopher proceeds to lay down rules for multiplication, partition, extraction of roots, radical multiplication, and radical partition of fractions. This closes the first book, being his exposition of the principles and rules "common to every species of logistic."

It is particularly striking to observe that his manner of treating the subject is not surpassed, if equalled, in modern times. With few resources beyond his own mind, living in a rude age, and in a country whose barbarian darkness in science he was the first to break, Napier surveys the vast field of computation, and not only reduces its complicated elements to a lucid order far before his times, but displays in the task

* We must here observe, that Napier at once gives the most simple and perfect method of adding and subtracting fractions, and that Euler, although he indicates his knowledge of the rule, only details more imperfect ones. It is a striking fact, of which any one may easily satisfy himself, that this perfect rule of Napier's is not taught in the elementary books. A note to Euler's algebra says, "the rule for reducing fractions to a common denominator may be concisely expressed thus: Multiply each numerator into every denominator except its own, for a new numerator, and all the denominators together for the common denominator." This is also the rule *Maclaurin* gives. Now it happens to be the *worst* rule, and Napier's is the *best*. Napier's exposition of fractions, throughout all his manuscript, is perfect.

a philosophical power, and a grasp of mind superior to that of EULER.* That philosopher's *Elements of Algebra*, written in the eighteenth century, are perhaps the severest test we could adopt of the excellence of Napier's unpublished fragments of the sixteenth century. There is, indeed, a remarkable similarity between the treatises, and it is manifest that the illustrious German viewed his subject nearly with the same mental eye that Napier did. Still his treatise is less methodically arranged, less symmetrical, less classic than Napier's, the characteristics of which may be expressed in the words (written so lately as 1830) of Sir John Leslie; "Nothing is more wanted for the purpose of education than a classical treatise on algebra, which, avoiding all vague terms and hasty analogies, should unfold the principles with simplicity and rigid accuracy, and follow the train of induction with close and philosophical circumspection." Our philosopher's exposition fulfils this rule in every particular, and many of his sentences are actually to be found in our most distinguished modern treatises on algebra, as if they had been translated from him. For instance, in the English translation of Euler, I find it said, "this rule for the division of fractions is often expressed in a manner that is more easily remembered, as follows: *invert the terms of the divisor*, so that the denominator may be in the place of the numerator, and the latter be written under the line; then multiply the fraction, which is the dividend, by this inverted fraction, and the product will be the quotient sought: thus, $\frac{3}{4}$ divided by $\frac{1}{2}$ is the same as $\frac{3}{4}$ multiplied by $\frac{2}{1}$, which makes $\frac{6}{4}$, or $1\frac{1}{2}$." Turning to Napier, to see how he treated this rule two centuries earlier, I find the very same; "*partiuntur autem (fractæ) invertendo terminos divisoris, et inversos per partiendum multiplicando omnimode ut superior in multiplicatione: ut sint $\frac{5}{10}$ partiendæ per $\frac{3}{4}$, hujus divisoris inverte terminos, et fient $\frac{4}{3}$, quæ per $\frac{5}{10}$ multiplicatæ fient primò per abbreviationem $\frac{1}{10}$ $\frac{4}{1}$, deinde $\frac{1}{5}$ $\frac{4}{1}$, deinde per multiplicationem superiorum invicem, et inferiorem invicem fient $\frac{2}{3}$ quotus optatus, et superiores multiplicationis examen.*" Again, Maclaurin's† expressions, "when unit is the greatest common measure of the numbers and quantities, then the fraction is al-

* "The algebra of Euler is in various respects a most remarkable production. That illustrious analyst, when totally deprived of sight in his advanced age, dictated it in the German language to a young domestic whom he trained for an amanuensis. He was obliged, therefore, to be plain, distinct, and perspicuous; and these qualities he combined with richness of invention."—*Leslie*. Euler seems to have resembled Napier in his moral character also. "Sweetness of disposition, moderation in his passions, simplicity of manners, were his leading features. Nor did the equability and calmness of his temper indicate a defect of energy, but the serenity of a soul that overlooked the frivolous provocation, the petulant caprices, and jarring humours of ordinary mortals. Possessing a mind of such wonderful comprehension, and dispositions so admirably formed to virtue and to happiness, Euler found no difficulty in being a Christian. Accordingly his faith was unfeigned, and his love was that of a pure and undefiled heart."—*Account of Euler prefixed to the Translation of his Algebra*.

† "In our own language, Maclaurin's *Elements of Algebra*, though a posthumous work, is perhaps the ablest on the whole, and the most complete."—*Leslie*.

ready in its lowest terms; and numbers whose greatest common measure is unit are said to be *prime* to one another," might stand as a translation of Napier's, "*verum si ad unitatem partitorem perveneris, inabbreviables, discreti tamen sunt termini, aut se invicem habentes ut discreti.*" Again, the author of the article Arithmetic in the latest edition of the Encyclopædia Britannica, observes, after going through the rules of the multiplication of fractions, "hence we infer that fractions of fractions, or compound fractions, such as $\frac{2}{7}$ of $\frac{5}{8}$, are reduced to simple ones by multiplication; the same method is followed when the compound fraction is expressed in three parts or more." Napier, after going through the rules of the multiplication of fractions, in like manner adds, "*hac multiplicatione fractiones fractionum, imo et fractiones fractionum iterum atque iterum fractarum, ad simplices fractiones reducuntur: ut duæ quintæ trium quartarum sic notatæ $\frac{2}{5}$ ex $\frac{3}{4}$ per præmissam fiant primo $\frac{1}{5}$ $\frac{5}{2}$ per abbreviationem,*" &c. Sir John Leslie, in explaining Lord Brounker's fractions, observes, "when the original fraction is expressed by rational numbers, its decomposition must always terminate; but, if the numerator and denominator be mutually incommensurable, the process of evolving their elements will never draw to a conclusion." Napier notices the property in these words, "*verum hic summopere cavendum est a partitione incommensurabilium quantitatum, cujus nullus in æternum, erit finis, ut suo loco perspicuum evadet.*" Maclaurin gives the rule to reduce an improper fraction to a mixed quantity thus: "Divide the numerator of the fraction by the denominator, and the quotient shall give the integral part; the remainder set over the denominator shall be the fractional part." Napier gives it thus: "*fit autem restitutio hæc partiendo numeratorem per denominatorem, et emerget in quotiente integra quantitas, et relinquæ erunt numerator, et divisor erit denominator fractionis illæ mixtæ et adjunctæ.*" In short, it appears that our philosopher, before he, or any one else, had conceived the system of Decimal fractions, so thoroughly commanded the difficult doctrine of vulgar fractions, that his exposition of them may be placed side by side with the best treatises on the subject now. Profoundly conscious of the unlimitable play of numbers, his mind penetrated the unexplored field of the Arabic system in every direction. His first, and leading idea throughout, is to show how the prominent operations upon quantity and number, gradually unfold; and how the vast fabric produces itself, growth after growth, every rule the parent of another, and the whole intimately related in all its parts, as one endless family of numbers. This is peculiarly interesting from the person for whom the immortality was yet in store to compress with such effect that very expansion. He shows how Multiplication and Division rise out of the parent operations Addition and Subtraction, and how the involving of radicates, and the evolving of roots, rise in their turn out of Multiplication and Division. He afterwards, by his invention of Logarithms, provided the means of obtaining all the third quantities, hitherto sought in the complicated rules, from the more simple operations of their respective primitives. He explored the prolific system in all its channels, and then condensed it to a greater power.

Having given the *genealogy* of numbers, in the next place with what genius he seizes *unit*, breaks it into a new and infinite scale, and reduces to order and beauty all the great operations of arithmetic upon its fractions. The subsequent computation of his Logarithms, however, brought out a new system of fractions in *Decimals*. No sooner had he found these, than he at once took the view that now prevails; he regarded the great Arabic scale as acting reciprocally, in opposite directions, from right to left, and from left to right; and, rejecting in this case the notation of broken numbers, he *proposed the point* to distinguish the reciprocal play of the decuple progression. But the treatise we are considering shows that his mind had been long previously matured for such fearless and prolific views of computation. His arithmetic of *plus* and *minus* is a most interesting chapter, and full of genius. Before viewing an infinite scale in the fragments of *unit*, he takes *zero*, and considers that unpromising symbol as the focus of a reciprocal scale of *integers* extending infinitely above and below the thus dignified cypher. Destined to accomplish the Logarithms out of their natural course of discovery, he dared to conceive a scale below nothing, and to say *quantitates minores nihilo!* He showed, in this conception, how the primitive operations of Addition and Subtraction, with their distinguishing signs, gave out another infinite scale in opposite directions from zero; and in this profound exposition of $+$ and $-$, he is followed as closely by Euler as if the German philosopher had written with Napier's manuscript before him. In some particulars, however, the modern treatise is superior to the ancient fragment. In the *first* place, it possesses that perfect system of algebraic notation which, between the dates of Napier's work and Euler's, had been successively moulded in the hands of Vieta, Girard, Wallis, Harriot, Descartes, and Newton. In the *next* place, Euler has a chapter upon *Decimal fractions*, and three chapters upon *Logarithms*, so that his system is complete and Napier's is not. We shall find, however, that the important subject of *notation* was not left untouched by our philosopher; and as for the systems he omits, what made him throw aside and leave unfinished this beautiful institute of numbers, but that he paused to create those very systems, that he *did create them*,—and died.

In the second book, Napier comes, as he says, to *particulars*. Through these I must follow him less closely, but shall endeavour to select what is curious and interesting. In the first chapter he proposes a third division of computation, and I shall translate the most of it, as it contains his definitions, and also a beautiful statement of the Indian notation, before that had been enriched by its European, or we may say *Neperean* stores. "In the third place, computation is either of *verinomial*, or *fictinomial*, otherwise *hypothetical* quantities; and hence logistic is either of *verinomes*, which are treated of in this second book, and also in the third; or of *fictinomes*, otherwise *algebraics*, concerning which the fourth book treats. *Verinomes* are quantities defined by the actual terms in which their multitude or magnitude is expounded; and they are either *discrete*, *i. e.* named in discrete number; or *concrete*, *i. e.* named in concrete number. Hence *verinomial logistic* is either of discrete quantity, and called *Arithmetic*, of which this book treats, or

of concrete, called Geometric (*geometrica*), of which in the third book. Arithmetic, therefore, is the logistic of discrete quantities by discrete numbers. A discrete number is that which is measured by its single individual number. A discrete number is either whole or broken. Hence arithmetic is of *integers* and *fractions*. An integer is that which is measured by its own individual unity. Every idiom supplies its own vocal nomination of integers; as, in Latin, *unum, duo, tria, quatuor*, &c. But the written names of integers, or their *notation*, are these nine significant figures, 1, 2, 3, 4, 5, 6, 7, 8, 9. These signify various numbers, according to their change of place. Besides these nine figures, there is the circle 0, which has no signification wherever it is placed, but is destined to supply the vacancies. The series of places is considered from right to left, in the first of which the figure is named by its own value as above; in the second place, by its tenfold value; the third, a hundredfold; and so on *in infinitum*, always progressing by a tenfold increase." After giving examples, our philosopher proposes, for the sake of facility in reading great numbers, to point them off in threes; thus, 4.734.986.205.048.205, which he reads in Latin, *quatuor millies mille millena millia millium . septingenta triginta quatuor millies millena millia millium . non-genta octoginta sex millena millia millium . ducenta quinque millia millium . quadraginta octo millia . ducenta et quinque*.

In the second chapter, he passes from *nomination* and *notation* to *computation*, and displays the operations of Addition and Subtraction, taking his first example from the book of Genesis. The third and fourth chapters are devoted respectively to Multiplication and Division, and he shows the most perfect command of all these operations. He gives the well-known multiplication table. The fifth chapter is entitled, "Miscellaneous short methods of Multiplication and Division." In this occurs a distinct genesis and notation of Decimal fractions in Arithmetic, and perhaps the earliest on record. Our philosopher observes, that to divide any number by a divisor composed of unit and cyphers is easily effected by striking off so many figures from the right of the partiend, as the divisor contains cyphers; and he directs the figures so struck off to be placed above a line as the numerator of a fraction having the divisor for denominator; and the fraction thus formed to be adjoined to the remainder of the partiend in order to form the quotient. The example he gives is, 865091372, to be divided by 100, and, according to the above rule, $8650913\frac{72}{100}$ is the quotient. Napier goes through this operation apparently unconscious of the important nature of the fraction thus obtained. Had he proposed simply to point off the figures deducted, so as to separate the right extremity, or unit's place, of the remaining integers from the broken numbers, he would have obtained his quotient by the most compendious rule possible, and at the same time have given his own notation of Decimals, and that now in use. But the system was comparatively valueless in Arithmetic until the Logarithms appeared, and it is obvious from the above example that Napier's manuscript must be referred to a very early date. Clearly he had not seen the work of Stevinus, which he afterwards mentions in *Rabdologia*, and had formed no conception

of his system of Logarithms, which, indeed, may be called the parent of the system of Decimal fractions.

In chapter sixth our philosopher, with a fearless composure becoming the conqueror and king of numbers, enters the formidable field of involution and evolution. This, as we have seen, he terms radical multiplication, partition, and extraction. Euler himself had not a more thorough command of the relative quantities, *root*, *power*, and *exponent*, than Napier had of *radix*, *radicatum*, and *index*. His opening statement of involution is less perplexing than that of the illustrious German, whose statement might leave the student at a loss to know why the *square* of a number is called the *second* power, seeing Euler at the same time informs him that a power of a number derives its dignity from "the number of times it is multiplied by itself," and that "we obtain a *cube* by multiplying a number *twice* by itself." Napier creates no such perplexity at the outset, for he commences by saying that the first step in the process of involution is to "multiply *unit* by the root, which multiplication returns the root itself; *secondly*, multiply that by the root and the *duplicate* [*i. e.* square or 2d power] is raised, and so on, according to the quality of the index; thus if 235 is to be multiplied to the index 4, [*i. e.* raised to the 4th power] first multiply unit by the root 235, which gives 235; multiply that again by the root, and 55225, the duplicate, is obtained," &c. "Hence," he adds, "radical multiplication repeated any number of times from unity is the same thing as to multiply together so many equal roots; thus, 235 four times multiplied from unity is the same thing as 235.235.235.235 multiplied into each other;" a law which now would be thus generally and shortly expressed $a^4 = a \times a \times a \times a$. Napier, indeed, had not arrived (and be it remembered that he is writing before *Vieta*, *Harriot*, and *Oughtred*, and when "algebra was not cultivated at all in this country,") at that powerful notation without the aid of which it was impossible for him to take some more recent views of the exponential or potential system. He did not possess the algebraic refinement of working *known* quantities by means of other symbols than the significant digits, or of expressing powers by small letters instead of numerals and initial signs. He did not, for instance, consider *aaaa* as (to use his own term) the *quadruplicate* of any number *a*; far less did he consider the same quantity in this form, a^4 . While he had not the *literal* notation of *powers*, neither had he the *numeral notation* of *indices*; for although, in explaining their genesis, he named the indices, one, two, three, &c. and even noted them 1, 2, 3, &c., yet he did not systematically attach them to the root for the expression of the power. To have done so would have been to have established the *Cartesian* notation, whose epoch is 1637. But in each definition he shows his thorough command of the subject, and how capable he was of reaping every laurel in that great field of analytical inquiry which *notation* opened to his successors. For instance, the exponent of a power is thus defined in modern science: "Exponent of a power in arithmetic and algebra denotes the number or quantity expressing the degree or elevation of the power, or which shows how often a given power is to be divided by its root before it be brought down to unity or 1; it is otherwise called the

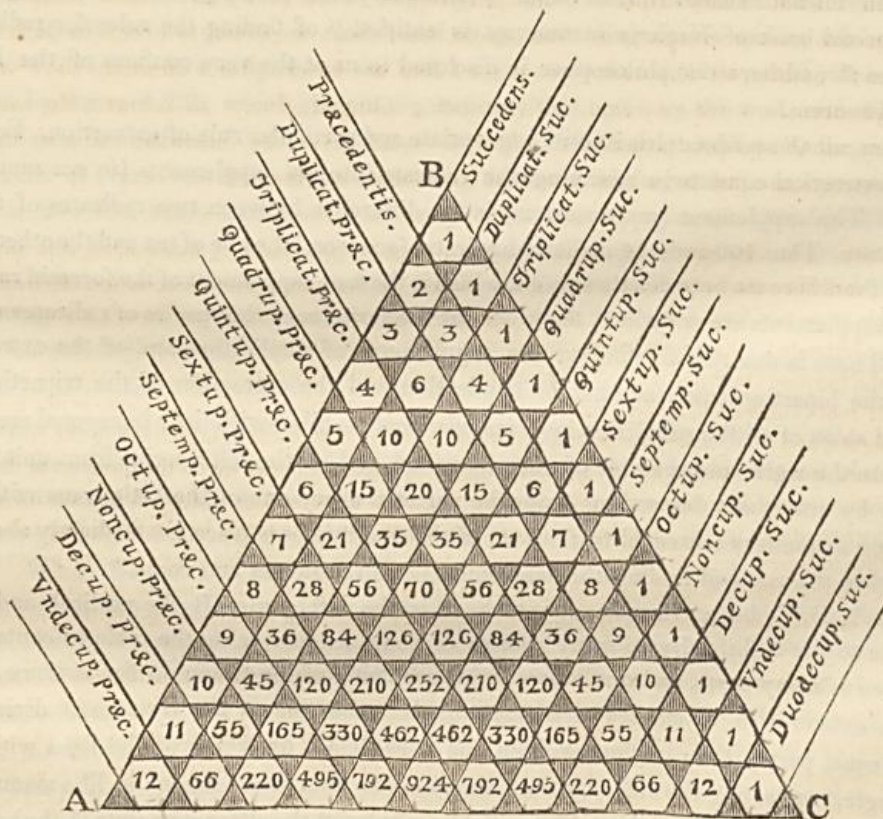
index. Exponents, as now used, are rather of modern invention," &c.—(*Hutton's Math. Dict.*) Now, although Napier had not the *algorithm* which opened the arithmetic of exponents, (and which Dr Hutton so unaccountably says, "led the way to the invention of Logarithms,") his view of that important quantity is precisely what is here stated. He says, "the number of the index, or quality of the root, is obtained as well in descending from the radicate to unity by partition, as in ascending from unity to the radicate by multiplication, for in either case the *number* of the operations is the index and quality of the root."

We must now turn to his chapter of the extraction of roots; a subject of which it has been observed, that among all the questions which the developement of our ideas of number places in review before us, there is none which, independently of the importance of the solution, has a greater tendency to excite the curiosity of every mind born for calculation; it is comparatively easy to raise roots to powers, but when we demand the roots back again it is not so easy to obtain them. (*Bertrand.*) Accordingly, the seventh chapter of the second book of Napier's manuscript is entitled "of finding the rules for radical extraction;" and here our philosopher is disclosed to us at the very confines of the Binomial Theorem.

"Every root," says he, "has its own appropriate and particular rule of extraction. Each rule of extraction consists in resolving the radicate into its supplements (*in sua supplementa.*) The supplement (*supplementum*) is the difference between two radicats of the same species. Thus 100 and 144 are both duplicates [squares,] the one of ten and the other of 12; and the difference between them is 44, which is the true supplement of the foresaid radicats. Supplements are as various, therefore, as the varieties of the species of radicats and roots. There is one rule for finding the supplements of duplication and of the extraction of the bipartient root, another of triplication and the extraction of the tripartient root, and so on of all the rest. But *my triangular table*,—filled with little hexagonal areas, having, on the right side, a series of units inscribed, and on the left a series from unit increasing by unity, and descending from the vertex; every one of the little areas within containing a number each equal to the sum of the two numbers placed immediately above it,—teaches the rules of finding the supplements of all radicats and roots."

"Let A, B, C, be a triangle, of which A is the left angle, B the vertical, and C the angle to the right. By so many species of roots as you wish the table to contain, into twice as many parts, and one more, divide each side of the triangle; for instance, in order to extend it to 12 species of extractions, let each side of the triangle be divided into 25 equal parts; then beginning from the base A C, draw 12 parallel lines within the triangle, connecting the sides by the points in them alternately taken: in like manner, begin from the side A B, and draw 12 parallel lines betwixt the alternate points of the base, and the side B C, extending the lines beyond the side B C, about the space of an inch; exactly in the same manner draw the lines betwixt the side B A and the base, extending

them an inch beyond B A ; and you will have the triangle filled with little hexagonal areas. Of these, the 12 to the right, and next the line B C, must each have unit inscribed within it ; those on the left must have the numbers 1, 2, 3, 4, &c. as far as 13, (exclusive) successively inscribed in each, descending in their order from the vertex B to the angle A ; then each interior hexagonal remaining vacant must have inscribed the sum of the two numbers immediately above it ; thus, under 2 and 1, must be written 3, under 3 and 3, 6 ; under 3 and 1, 4, and so on down to the heel of the table. Lastly, the table must be titled, on the left side above the second hexagonal (2,) let there be written *præcedentis*, above the third hexagonal, (3,) write *duplicatum præcedentis*, and so on as far as *duodecuplicatum*. On the right hand of the table write above the first hexagonal, *succedens*, above the second, *duplicatum succedens* ; above the third, *triplicatum succedens*, and so on down to *tredecuplicatum* ; as you have here in the diagram of the table itself written below."



"To every supplement two parts of the root correspond, the one part consisting of one

The above diagram is a fac-simile from the manuscript.

or more left hand figures, already found, and which is called *præcedens*; the other consisting of a single figure immediately on the right, which is to be sought for, and this is called *succedens*. The supplement and these parts of the root mutually compose each other, and are built up together, as will afterwards appear." *

In the rest of this chapter our philosopher lays down rules for reading the table by means of the titles annexed, and refers generally to its use in the extraction of roots. In the two following chapters, namely, the eighth and ninth, he shows its application more particularly, and affords a long and profound exposition of the difficult doctrine of evolution.

The remarkable similarity between Euler's Elements and Napier's is even observable in the tables that illustrate the respective works; and if Euler's arrangement had been as purely and philosophically symmetrical as Napier's, (in which circumstance, however, it is far inferior,) his work would almost have seemed a modern translation of the ancient manuscript. If our philosopher were to be any where completely thrown out in the comparison, that might have been expected to occur in Euler's chapter of the Binomial Theorem; yet there I find the latter, after examining the "important question how we may find, without being obliged always to perform the same calculation, all the powers either of $a + b$, or $a - b$," gives the following table as that which discovers the law by which binomial coefficients are formed.

* Dr Wallis, in his Algebra, 1685, reviews Oughtred's *Clavis Mathematicæ*, first published in 1631, (fourteen years after Napier's death,) and in the chapter of the nature and composition of powers, gives a table of powers from Oughtred's work, of which I find the counterpart in Napier's manuscript, but further extended. Napier gives it immediately after his arithmetical triangle, and uses it precisely for the purpose Oughtred did. "From hence," says Wallis. "we may take, without more ado, the nearest root (quadratick, cubick, &c. respectively,) of any number whose root requires not more than one figure, and the respective power of any such root. But because in extracting the root of great numbers, it will be necessary to seek out the root by piece-meal, (as we do the quotient in division,) he doth afterwards consider the root as consisting of two parts, $A + E$, (which he calls a *binomial* root,) whereof one part is supposed to be *already known*, (or to be found by the preceding table,) and the other *unknown*, to be found by the following table, which he calls his *latter table of powers*." This latter table is Napier's binomial table; but under the notation of *Vieta*, whose symbolical method, called *specious arithmetic*, was unknown to Napier, and forms an important step in the progress of notation.

There is an old-fashioned, but excellent work, entitled, "A New System of Arithmetick, Theoretical and Practical, by Alexander Malcolm, teacher of the mathematics at Aberdeen, 1730," containing a full exposition of the Binomial Theorem, wherein I find a remark that illustrates our philosopher's explanation of his diagram. "These expressions of powers of a *Binomial* root shew us how the difference betwixt any two similar powers is composed of the various powers and multiples of any one of the roots, and the difference betwixt the roots," &c.

fore he had seen it. I think it is equally certain that he had never seen the *Arithmetica Integra* of Stifelius.* While he praises the former author in *Rabdologia*, I cannot find that he any where mentions the latter, whose very curious work, however, must have excited his warmest admiration had he met with it. The celebrated *Blaise Pascal*, one of the most profound minds ever created, has in more modern times obtained the highest praise for his *Arithmetical Triangle*, which, as the reader will easily perceive from the following diagram of it, is just Napier's table under a less beautiful form.

1									
1	2								
1	3	3							
1	4	6	4						
1	5	10	10	5					
1	6	15	20	15	6				
1	7	21	35	35	21	7			
1	8	28	56	70	56	28	8		
1	9	36	84	126	126	84	36	9	
1	10	45	120	210	252	210	120	45	10
1	11	55	165	330	462	462	330	165	55
1	12	66	220	462	792	924	792	462	220
1	13	78	286	636	1287	1716	1287	636	286
1	14	91	364	900	1848	2709	1848	900	364
1	15	105	462	1287	2709	4353	2709	1287	462
1	16	120	588	1638	3542	6352	3542	1638	588
1	17	136	742	2184	4862	9237	4862	2184	742
1	18	153	936	2982	6720	12870	6720	2982	936
1	19	171	1170	4074	9690	19449	9690	4074	1170
1	20	190	1456	5635	13800	28695	13800	5635	1456
1	21	210	1800	7770	19449	40740	19449	7770	1800
1	22	231	2200	10626	27090	54284	27090	10626	2200
1	23	253	2660	14348	36960	75287	36960	14348	2660
1	24	276	3200	19134	50000	102606	50000	19134	3200
1	25	300	3828	25480	67200	138000	67200	25480	3828
1	26	325	4556	33642	89952	184344	89952	33642	4556
1	27	351	5394	43878	119784	252000	119784	43878	5394
1	28	378	6352	56640	159840	336000	159840	56640	6352
1	29	406	7440	72378	210000	453600	210000	72378	7440
1	30	435	8668	90420	276000	604800	276000	90420	8668
1	31	465	10036	111000	359040	806400	359040	111000	10036
1	32	496	11564	134280	462000	1080000	462000	134280	11564
1	33	528	13272	160440	590000	1411200	590000	160440	13272
1	34	561	15180	190680	748000	1824000	748000	190680	15180
1	35	595	17308	235200	942000	2352000	942000	235200	17308
1	36	630	19666	294480	1180000	3052800	1180000	294480	19666
1	37	666	22374	369120	1470000	3960000	1470000	369120	22374
1	38	703	25452	460680	1818000	5083200	1818000	460680	25452
1	39	741	28920	570600	2324000	6552000	2324000	570600	28920
1	40	780	32808	701400	2916000	8424000	2916000	701400	32808
1	41	820	37136	855600	3614000	10752000	3614000	855600	37136
1	42	861	41924	1035600	4440000	13680000	4440000	1035600	41924
1	43	903	47192	1243800	5418000	17328000	5418000	1243800	47192
1	44	946	52960	1482000	6584000	21840000	6584000	1482000	52960
1	45	990	59248	1752000	7974000	27360000	7974000	1752000	59248
1	46	1035	66076	2156400	9624000	34176000	9624000	2156400	66076
1	47	1081	73464	2600400	11580000	42576000	11580000	2600400	73464
1	48	1128	81432	3100400	13896000	52944000	13896000	3100400	81432
1	49	1176	90000	3664800	16620000	65760000	16620000	3664800	90000
1	50	1225	99200	4300400	20820000	81600000	20820000	4300400	99200

Montucla, in his *History of Mathematics*, refers to it in these words: "Quelques questions sur les jeux l'engagèrent (Pascal) à approfondir les combinaisons, et ses meditations sur ce sujet donnerent lieu à l'invention de son *triangle arithmetique*, au moyen duquel il résoud divers problêmes sur cet objet. Il écrivit sur cette matière un traité qui paroît avoir été achevé vers 1653, quoique imprimé seulement en 1665. Les usages de ce triangle arithmétique sont nombreux, et c'est une invention *vraiment originale et singulièrement ingénieuse*." The properties of this triangle are so intimately connected with the Binomial Theorem that Bernoulli, on that account, claims for Pascal the merit of being its first inventor. In his annotations upon a work of Mr Stone, upon the infinitesimal analysis, where the latter speaks of "that marvellous theorem," Bernoulli notes, "*Pour l'elevation d'un binome à une puissance quelconque. Nous avons trouvé ce merveilleux theoreme aussi-bien que Mr Newton, d'une maniere plus simple que la sienne. Feu M. Pascal a été le premier qui l'a inventée.*" (*Johan. Bernoulli Opera*, iv. p. 173.) Baron Maseres (*Scriptores Logarithmici*, Vol. iv.) republished Pascal's works on Arithmetic and Algebra, and says, "These works are so full of genius and invention, that I thought I should do a service to the mathematicians of Great Britain, by republishing them in

* Dr Minto acutely observes, "Not only Napier's manner of conceiving the generation of the Logarithms, but his having computed that species of Logarithms which has been described, before the common Logarithms occurred to him, is a *convincing proof* of his not taking the Logarithms from the remark of Stifelius."

this collection. Some of them, and more especially his *Arithmetical Triangle*, have a considerable connection with Logarithms, by affording a good demonstration of Sir Isaac Newton's Binomial Theorem in the case of integral and affirmative powers, which is of great use in the construction of Logarithms." Very probably the invention was original in Pascal's hands, and the application to games of chance seems entirely his own. It is a curious fact, that Napier's friend, Henry Briggs, to whom the manuscript we are considering is addressed, did also, in his *Trigonometria Britannica*, give a table of the same description; and Dr Hutton, when noticing this work in his History of the Construction of Logarithms, has accordingly claimed the Binomial Theorem for Briggs. He says, after giving some account of the table and its properties, "this is the first mention I have seen made of this law of the coefficients of the powers of a binomial, commonly called Sir Isaac Newton's Binomial Theorem, although it is very evident that Sir Isaac was not the first inventor of it; the part of it properly belonging to him seems to be only the extending of it to fractional indices, which was, indeed, an immediate effect of the general method of denoting all roots like powers with fractional exponents, the theorem being not at all altered," &c. Briggs' table, which he called *Abacus Παρχρηστος*, is in this form, only carried further on.

ABACVS ΠΑΡΧΡΗΣΤΟΣ.							
H	G	F	E	D	C	B	A
—(8)	—(7)	+(6)	+(5)	—(4)	—(3)	+(2)	(1)
1	1	1	1	1	1	1	1
9	8	7	6	5	4	3	2
45	36	28	21	15	10	6	3
165	120	84	56	35	20	10	4
495	330	210	126	70	35	15	5

Notwithstanding the many long and delightful discussions that must have passed between Henry Briggs and the Baron of Merchiston upon their favourite topics, there seems no ground for alleging that the former had borrowed his idea from his illustrious friend. We have elsewhere ventured to call him a satellite of Napier's, and fairly enough, as his memory is chiefly logarithmic, and his persevering pilgrimages to the old tower in Scotland is an ample justification of the epithet. But Briggs has evinced in his two logarithmic works a mind capable of great mathematical conceptions.* In reference to the arithmetical triangle, he appears to have been the first to point out a

* The kind assistance of an Oxford friend enabled me to ascertain, with tolerable certainty, that there are no traces among Briggs' papers, preserved at that university, of a correspondence between him and Merchiston; probably he found the Baron a better host than a correspondent. Among Briggs' papers in the British Museum, there is one entitled *Imitatio Nepeirea, sive applicatio omnium fere regularum, suis Logarithmis pertinentium, ad Logarithmos*, supposed to have been written immediately after the publication of the *Canon Mirificus*.

particular law of that configuration which brought him as close to the Binomial theorem as the notation of his day rendered possible. The passage is remarkable, and as his work is rarely to be met with, I shall give it here. "*Numerus quilibet est ad suam Diagonalem, ascendendo versus sinistram, ut verticalis primi ad Marginalem secundi. Numeri in Columna A sunt ad suos Diagonales in B ascendendo, ut 2. ad Marginalem secundi. Hinc sequitur numeros margini dextro adjacentes, reliquosque deinceps proximos, posse inveniri et continuari quo usque visum fuerit; licet totus Abacus a Capite non sit adscriptus.*" I have looked anxiously, but in vain, through Napier's manuscript to discover some expressions indicative of his observation of this important law of proportion actually existing in the table he had formed. There is, however, no question that his triangle is what would be now called a *table of coefficients of the powers of a binomial*, which he framed for its most important application, that of extracting roots. In doing this he was certainly at the confines of the Binomial Theorem. Had he only recorded the observation of Briggs, it must have been admitted that he had actually *stated* the leading principle of that elegant theorem, which is engraved upon the tomb of Newton as one of the greatest of his discoveries. The observation, which leaves that laurel with Briggs, (and which Napier may have seen, though he did not state it,) amounts to this, that, by a certain law of proportion existing betwixt the figures of the diagram, which law he points out, all the terms of the binomial quantity could be successively deduced, or raised, from the second term (the coefficients of the first and second terms being always known,) without the necessity of finding the intermediate and preceding powers. The *application* of this law (which Briggs verbally stated) is that algebraic generalization of the principle of Napier's triangle which supersedes the necessity of actually composing the whole table in order to obtain the terms and successive powers of a binomial root; and upon the strength of Briggs' observation of that law Dr Hutton claims the Binomial Theorem for him, certainly with better reason than Bernoulli does for Pascal. But the value of it is really dependent upon a play of symbols not known in the time either of Napier or Briggs. What was necessary in order to make the property, which the latter unquestionably pointed out, a valuable extension of the arithmetical triangle, was to have the means of stating it in this form, $1 \times \frac{m-0}{1}, \times \frac{m-1}{2}, \times \frac{m-2}{3},$ &c. being Sir Isaac Newton's genesis of the binomial powers in question. So far, indeed, the Prince of Mathematicians only made the *algebraic application* of the principle of the figurate table in the case of *integral* quantities, to which alone the triangle is applicable. But Dr Hutton, probably for the sake of planting so fine a laurel upon the brow of Briggs, seems inclined to slur over a most important extension of the Binomial Theorem, when he says, "Sir Isaac was not the first inventor of it, the part of it properly belonging to him seems to be *only the extending of it to fractional indices*, which was, indeed, an immediate effect of the general method of denoting all roots like powers with fractional exponents." True it was an improved notation that led Newton to consider the theorem as he did, and moreover, to expand it into an infinite algebraic series, which, without that notation, it were impossible to have done; but in this it was, that, to use the phrase

of his last biographer, Newton must be acknowledged as "the General who won the victory, and therefore wears the laurels." In his hands the binomial table of Stifelius, Napier, Briggs, and Pascal (each one of whom appears to have invented it) was expanded into the Binomial Theorem *par excellence*. What he did beyond his predecessors is somewhat analogous to Napier's merit when he generalized the logarithmic principle (previously observed by Archimedes, Stifelius, and others,) into a system of universal application and omnipotent power. In that comparison, however, the important distinction must be kept in view, that Newton's generalization of the table of coefficients was forced upon the attention of such a mind by the then ripened doctrine, and notation, of powers and exponents, the very medium through which, in like manner, he must have detected the Logarithms. Napier, on the other hand, instead of using that means to extend the principle of Archimedes into a system of *common* Logarithms, and before such means was in existence, took a totally different path of his own construction, and tore the veil from a *transcendental* system of Logarithms, thus disclosed, as it were, before its time.

Although the Binomial Theorem is "so very closely connected with the subject of Logarithms as to be the foundation of the best methods of computing them," (*Maseres*,) and although our philosopher approached the confines of it in his beautiful diagram, (a form perfectly original,) these circumstances must not be supposed to connect with his great invention. In that path he could do nothing without algebraic notation, which in his day was totally inadequate for such refined purposes. The analytical language may be said to have first dawned in the works of *Vieta*, which only commenced to be spread abroad, and to give an impulse to science after Napier's career was closed. It is of consequence, then, to see if, in the manuscript we are considering, there be any indications that Napier felt the trammels of a rude notation, and struggled to remove them. As his system of numbers was never finished, and is only now first noticed to the world, of course what he did in this manuscript can form no link in the progress of science, and can be only referred to in further illustration of the mind that invented the Logarithms. But it will be acknowledged, by all lovers of science, to be a very striking and interesting circumstance, if, as we shall immediately show to have been the case, Napier not only determined to become the liberator of the numerical scale, but had turned his powerful mind to algebraic notation, with the same premeditated intention of reforming that. I am not aware that any writer before his time had made the systematic attempt now to be noticed. Immediate necessity, and accidental ingenuity, added very sparingly to the abbreviated language of algebra during the period between its introduction into Europe and when Napier commenced a work of extreme beauty and high conceptions, which, had he published it, even in its uncompleted state, must have given a decided impulse to science, and Britain a distinguished place in the history of Algebra, independently of the Logarithms.

In his consideration of radical partition, and extraction of roots, Napier did not fail to observe, most profoundly and successfully, a species and property of numbers exceedingly curious, and of high importance in the science. The quantities alluded to are the

roots of those numbers whose roots cannot be *numerically* expressed ; and for this reason, that a root is that quantity which is contained in another quantity any number of times *exactly, i. e.* without a remainder less than the root itself ; and there are some numbers that contain no number whatever any number of times without a remainder. An ordinary mind might be apt to conceive that such quantities had no roots, according to the definition of that term. Mathematicians have decided otherwise. The roots lurk in those quantities, though they cannot be extracted ; they may be hunted into a corner, but they cannot be caught ; or, to use Napier's expressions with regard to them, they may be *named*, but they cannot be *numbered*. Having decided that such latent quantities have a real existence, mathematicians, of course, will not suffer them to remain in idleness, or unsubjected to the dominion of science. They have been called irrational quantities or surds, and hence the *arithmetic of surds* has become a special and important department of numbers. No man before or since his day, knew better how to hunt a surd than John Napier. He was thoroughly master of their whole philosophy, and the manuscript before me contains, perhaps a more beautiful and complete exposition of their arithmetic than has ever been published. Consequently, this very curious property had not escaped him, that a surd root, though it cannot be expressed in finite number, lies between two other numbers that can be so expressed, and whose terms can be brought closer and closer to each other by *infinite* approximations, without, however, being capable of catching the latent surd. To give an easy example,—the square root of 9 is 3, because 3 times 3 is 9 ; but what is the square root of 10 ? In other words, what is the number which, multiplied by itself, makes 10 ? Not 3 times 3, that is too little ; nor 4 times 4, that being too much. But the doctrine of fractions enables us to express numbers betwixt 3 and 4, and, consequently, nearer to each other than these. The approximations, however, are still found to be terms, the one too great, and the other too small, to express the surd sought ; and the curious property is, that the fractional terms may be brought closer and closer together by an endless approximation, and still the surd shall be latent between them. Thus the actual existence of the quantity is ascertained, but it can only be expressed by two separate finite terms indicating its position, or by some special symbol invented to represent it. Now it was to the notation of these surds that Napier, in that department, first turned his attention, as such quantities seemed peculiarly dependent upon a symbolical notation. The notation he proposed was never published ; and I shall premise the translation with some notices of the state of irrational expressions after his day, and, indeed, as it exists now.

Dr Wallis, the great contemporary of Newton, in his Algebra already quoted, after explaining the nature of a surd root, adds, “ in such case we must either content ourselves with an approximation instead of the accurate value, or else with such note of radicality as shall intimate what is supposed to be, but cannot accurately be expressed in numbers. As $\sqrt{2}$, or $\sqrt{q}2$, the square root of the number 2. $\sqrt{c}3$, the cubick root of the number 3. Which supposed roots, thus designed, cannot in numbers be accurately expressed, there being no effable number, integer or fraction, which, being mul-

multiplied into itself, can make 2; or, being cubically multiplied, can make 3." Euler, in his Algebra, says, "there is a sort of numbers which cannot be assigned by fractions, and which are, nevertheless, determinate quantities; as, for instance, the square root of 12; and we call this new species of numbers *irrational numbers*; they occur wherever we endeavour to find the square root of a number which is not a square; thus, 2 not being a perfect square, the square root of 2, or the number which, multiplied by itself, would produce 2, is an irrational quantity; these numbers are also called *surd quantities*, or *incommensurables*. These irrational quantities, though they cannot be expressed by fractions, are, nevertheless, magnitudes of which we may form an accurate idea; for, however concealed the square root of 12, for example, may appear, we are not ignorant that it must be a number which, when multiplied by itself, would exactly produce 12; and this property is sufficient to give us an idea of the number, since it is in our power to approximate towards its value continually. As we are, therefore, sufficiently acquainted with the nature of the irrational numbers under our present consideration, a particular sign has been agreed on to express the square roots of all numbers that are not perfect squares; which sign is written thus $\sqrt{\quad}$, and is read the *square root*." A great improvement, however, in this notation became established between the time of Wallis and Euler, and that was to express the number of the root, or the order of the power, by a numeral index placed within the radical sign, instead of the cumbrous repetition of initial letters. Besides this improvement there is a more modern alternative notation of surd roots. Euler, in his chapter "of the method of representing irrational numbers by fractional exponents," shows "that $a^{\frac{1}{2}}$ is the same as \sqrt{a} ," and so on; and then he adds, "we might therefore entirely reject the radical signs at present made use of, and employ in their stead the fractional exponents which we have explained; but as we have been long accustomed to those signs, and meet with them in most books of algebra, it might be wrong to banish them entirely from calculation; there is, however, sufficient reason also to employ, as is frequently done, the other method of notation, because it manifestly corresponds with the nature of the thing." It must also be observed, that, notwithstanding "the rule, that we must adhere to one notation for one thing," the radical notation in question has not been exclusively devoted to the same species of quantity. Euler, in his chapter "of roots, with relation to powers in general," and, speaking of *rational* roots, takes occasion to exhibit the different roots of the number a , with their respective values.

$$\left. \begin{array}{l} \sqrt{a} \\ \sqrt[3]{a} \\ \sqrt[4]{a} \\ \sqrt[5]{a} \\ \sqrt[6]{a} \end{array} \right\} \text{ is the } \left\{ \begin{array}{l} 2\text{d} \\ 3\text{d} \\ 4\text{th} \\ 5\text{th} \\ 6\text{th} \end{array} \right\} \text{ root of } \left\{ \begin{array}{l} a \\ a \\ a \\ a \\ a \\ a \text{ and so on,} \end{array} \right.$$

being the same radical signs that are taken to express surds. Thus it appears that even at present the notation of such irrational quantities is not of a very determined character; but, in the first place, possesses an alternate mode of expression; and, in the second place, a set of radical signs, shared in common with an opposite species of quantity. We may now turn to Napier's consideration of this subject in which we shall find, as usual, the most unequivocal proofs of his original and penetrating genius.

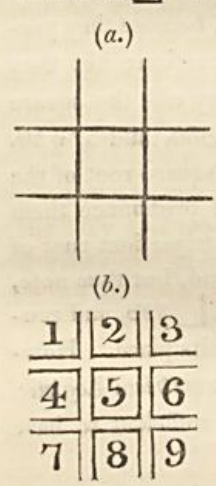
In the fourth chapter of his first book, our philosopher, after explaining the genesis of a surd root, and of the approximating terms between which it lurks, (*see supra*, p. 467,) adds, "but geometricians, studious of greater accuracy, choose rather to prefix the sign of the index to the radicate itself, than to include the root between two terms; thus they note the tripartient root of nine in this manner, $\sqrt[3]{c}9$, which they pronounce the *cube* root of nine. I, however, note it thus, $\lfloor 9$, and call it the *tripartient* root of nine; these signs I shall discuss more fully in their place." In the ninth chapter of his second book, entitled, "Of the method of amending imperfect extractions," our philosopher enters minutely into the subject of the approximating fractional terms, and teaches how to express an irrational root with the least sensible error. "So that," to take the result of one of his examples, "without any sensible error, especially in practical science (*in mechanicis*,) the bipartient root of 164860 may be called $406\frac{24}{813}$, or $406\frac{24}{812}$." He afterwards observes, "these methods, as they do not make imperfect roots perfect, but merely render them less imperfect, are more pleasing to practical men (*mechanicis*) than to mathematicians, as I have noticed in *C. iv. Lib. i.* Geometricians, therefore, prefix the appropriate sign of the root to such radicates as have no roots in numbers. Hence, from the radicates with these signs prefixed, arises the first species of geometrical numbers, called *uninomes*. As in the above example of the duplicates 164860 and 50, they neither extract the bipartient roots, because they possess none precisely in numbers, nor do they amend the imperfect extraction; but they prefix to the number the sign of the root to be extracted, which they call the square root (*quadratam*,) thus, $\sqrt{Q} 164860$, and $\sqrt{Q} 50$, or thus, $\sqrt{q} 164860$, and $\sqrt{q} 50$, which they pronounce the square root of the number 164860, and the square root of the number 50. I, however, note them thus, $\lfloor 164860$, and $\lfloor 50$; and pronounce them the bipartient root of 164860, and the bipartient root of 50. So the tripartient root of the number 998 they neither extract, as it is not in numbers, nor amend, but thus note, $\sqrt[3]{c} 998$, and pronounce, the cube root of 998. I, however, note it thus, $\lfloor 998$, and pronounce it, the tripartient root of 998, as I shall discuss more amply in its place. However, these are called *uninomia*, or *medialia*, and are the foundation of *Geometrical Logistic*. They shall be treated of, therefore, in the following book; here it is sufficient to have pointed out their origin."

In order to connect this subject, I shall pass immediately to the third book here referred to, reserving in the meantime what remains to be noticed of the first. It is entitled, *Liber tertius de Logistica Geometrica*,* *Cap. i.* Unfortunately it is a fragment, being

* I am not aware of a department of science known under that term now. Probably the best explanation of it is that afforded by the fragment itself.

all that his son Robert could find among his papers upon the subject, as he notes at the end of his transcript. It is, however, so original and full of genius that no apology need be offered for giving our readers a literal translation of the whole of it.

“ In the preceding book I have taught Arithmetic; here in order follows Geometrical Logistic. The computation of concrete quantities by concrete numbers is called Geometrical Logistic. Thus 3^3 , if it relate to three lines, each a finger-breadth (*digitales*) thus, — — — — — , is a *discrete* number. When, however, it refers to a concrete and continuous line of three finger-breadths, such as this — — — — — , it is called a *concrete* number; but this *improperly*, and subject to reason. The roots of numbers which cannot be measured by any number, integral or fractional, are properly, and in themselves, called concrete numbers. Thus the bipartient or square root of seven is greater than two, less than three, and with no fraction in the universal elements of broken numbers is it equal or commensurable; it is therefore properly called a concrete number. So the tripartient, or cube root of the number 10 is not a discrete number, nor commensurable with number, but is concrete; and so are an infinity of other roots of numbers, commonly called surds and irrational numbers (*surdos et irracionales.*) These concretes arise out of the extraction of roots from numbers in which those roots are not seated; as I have already noticed, *C. iv. Sect. 8, Lib. i.* and *C. ix. Sect. 7, Lib. ii.* Hence, from the variety of roots arise various notations and nominations of concretes. As the bipartient root of seven, which is usually called the *square* root of seven (*quadratam,*) I note in this manner $\sqrt{\quad}$ 7, and pronounce the *bipartient* root of seven. So the *cube* root 10, I pronounce the *tripartient* root of ten, and write it thus $\sqrt[3]{\quad}$ 10. So the *quadrupartient* of 11, I note thus $\sqrt[4]{\quad}$ 11. So the *quintupartient* of any number, thus, $\sqrt[5]{\quad}$; the *sextupartient* thus $\sqrt[6]{\quad}$. This single scheme, (*a.*) divided into compartments, (*b.*) with the indices numbered, (to assist the memory,) supplies us with this variety of radical characters. As in the preceding examples, $\sqrt{\quad}$ $\sqrt[3]{\quad}$ $\sqrt[4]{\quad}$ $\sqrt[5]{\quad}$ $\sqrt[6]{\quad}$ prefixed to the numbers, denote the bipartient, tripartient, quadrupartient, quintupartient, sextupartient roots; so $\sqrt[7]{\quad}$ is the septupartient, $\sqrt[8]{\quad}$ the *octupartient*; $\sqrt[9]{\quad}$ the *nocupartient*; $\sqrt[10]{\quad}$ the decupartient; $\sqrt[11]{\quad}$ the undecupartient; $\sqrt[12]{\quad}$ the duodecupartient; $\sqrt[13]{\quad}$ the tredecupartient; $\sqrt[14]{\quad}$, or $\sqrt[14]{\quad}$ the quadrudcupartient; $\sqrt[15]{\quad}$ the quindecupartient; $\sqrt[16]{\quad}$ the sedecupartient; $\sqrt[17]{\quad}$ the septemdecupartient; $\sqrt[18]{\quad}$ the octodecupartient; $\sqrt[19]{\quad}$ the novemdecupartient; $\sqrt[20]{\quad}$ the vigecupartient; $\sqrt[21]{\quad}$ 21^{ent.} $\sqrt[22]{\quad}$ 22^{ent.} $\sqrt[23]{\quad}$ 23^{ent.} $\sqrt[24]{\quad}$, or $\sqrt[24]{\quad}$ 24^{ent.} *et cetera.* Also $\sqrt[30]{\quad}$ 30^{ent.} $\sqrt[40]{\quad}$ 40^{ent.} $\sqrt[50]{\quad}$ 50^{ent.} $\sqrt[60]{\quad}$ 60^{ent.} $\sqrt[70]{\quad}$ or $\sqrt[70]{\quad}$ 70^{ent.} $\sqrt[80]{\quad}$ 80^{ent.} $\sqrt[90]{\quad}$ 90^{ent.} $\sqrt[100]{\quad}$ 100^{ent.}, and so on *in infinitum* upon the principle of figurate arithmetic. * “ Geometrical numbers, which rather name quantity than number it, are on that account commonly called nomials (*nomina.*) Of



* Napier's notation is written about this size in the manuscript, apparently for the sake of distinctness in teaching; but it would appear that he meant it to be much smaller in practice, as it sometimes is written of a diminutive size, and even attached to fractions, thus $\sqrt[29]{\frac{29}{4}}$, and $\sqrt[29]{\frac{29}{4}}$.

nomials some are *uninomials*, others *plurinomials*. A uninome is the same as a single concrete number, proper or improper. Hence it follows that a uninome is either a single simple number, or any root of a single simple number. Thus 10 is a simple number, and, by geometricians, in frequent use as a uninome. So $\sqrt{\quad} 10$, $\sqrt{\quad} 12$, $\sqrt{\quad} 26$, and such like, are roots of numbers, and, when taken by themselves, are truly uninomial radicates.

“ Now, since it is the case that a uninomial *radicate* may be the *root* either of an *abundant* or *defective* number, and its index may be either *even* or *odd*, from this fourfold cause it follows, that some uninomes are abundant, some defective, some both abundant and defective, which are called double, and, finally, some are neither abundant nor defective, which are called imaginary (*mugacia*.) I have already (Lib. i. C. vi.) laid the foundation of this *great algebraic secrete*; and although never, that I know of, *hitherto revealed by any one*, how much it will enrich this art and the rest of the mathematics, shall afterwards be manifest. *

“ In abundant and defective uninomes, it is not of much consequence whether the appropriate sign be prefixed or interposed; it is better, however, to prefix it. But in double and imaginary uninomes, the appropriate sign must be always interposed. An example of the first case is $\sqrt{\quad} 10$, or (which by C. vi. Lib. i. is the same thing) $\sqrt{\quad} + 10$, an abundant uninome. An example of the second case is $\sqrt{\quad} - 10$, a defective uninome. An example of the third case is $\sqrt{\quad} 10$ or $\sqrt{\quad} + 10$, (being, as above, the same,) which

* This certainly has no connection with the Logarithms, and most probably refers to some of those profound views in algebra, and the theory of equations, which compose the triumphs of subsequent philosophers. Unfortunately, the algebraic part of the manuscript is not entire; but from what has been preserved, it is quite obvious that Napier was capable of any thing in that science, so far as the existing notation made it possible for him to advance. Without attempting to say what Napier here particularly contemplated, (which I leave for the learned,) some interesting illustrations of what he actually lays down may be derived from the history of algebra. It must be kept in mind, that what he calls *abundant* and *defective* quantities are now known under the terms *positive* and *negative*; (*supra*, p. 469, &c.) as for *imaginary* quantities, I am not aware that any one before the date of this MS., or for long after it, was so bold or profound as to give them their important place in calculation. Accordingly Playfair, speaking of *Girard*, in the passage already quoted as to quantities less than nothing, (*supra*, p. 469,) whose *Invention Nouvelle en Algèbre* was printed in 1629, says, “ the same mathematician conceived the notion of imaginary roots.” Dr Hutton observes, “ Albert Girard gives names to the three kinds of roots of equations, calling them greater than nothing, less than nothing, and *enveloppée*, as $\sqrt{-bc}$; but this was soon after called *imaginary* or *impossible*, as appears by Wallis' Algebra, p. 264, &c.” Yet we find that Napier considered, and was expounding, such quantities, in their philosophy, nomenclature, and notation. So much is this the case, that a great part of Euler's 13th chapter “ of impossible or imaginary quantities” may stand, as usual, for a translation of Napier's discussion of the same subject. The passages are too long to quote; but any one who takes an interest in the history of algebra, or the genius of Napier, will be struck with the similarity betwixt that chapter of Euler and what we have quoted at p. 471 from our philosopher's manuscript, and also above. Euler even adds the same warning against confounding the radical signs: “ We must not,” he says “ confound the signs + and —, which are before the radical sign $\sqrt{\quad}$, with the sign which comes after.”

signifies both an abundant quantity multiplied into itself, and yielding + 10, and also a defective quantity multiplied into itself, and yielding + 10; or, for the sake of a more lucid example $\sqrt{}9$, or $\sqrt{}+9$ is as much + 3 as - 3, according to what I have already demonstrated, *Lib. i. C. vi.* An example of the last case is $\sqrt{}-9$, which is merely imaginary, and signifies nothing that either abounds or is deficient, for *defective nine* has no bipartient root, as is made plain in *Lib. i. C. vi. s. b.*

“ In imaginary quantities special care must be taken that the sign *minus* —, to be *interposed*, be not *prefixed*. Thus, if for $\sqrt{}-9$, which is the bipartient root of *minus nine*, (*minuti novenarii*,) and infers an absurd and impossible quantity, there be taken $-\sqrt{}9$, which signifies a quantity *less by the square root of nine*, a great mistake will be committed; for the bipartient root of nine, here abundant, namely, $\sqrt{}9$, is double; that is, + 3 and - 3; and therefore, a quantity *minus* these geminals + 3 and - 3 will be geminal; so that whoever for $\sqrt{}-9$ writes $-\sqrt{}9$, puts forth a quantity of a geminal, or double signification, instead of a quantity absurd, impossible, imaginary, and of no signification (*absurdo, impossibili, nugaci et nihil significante.*) Take care, then, of such prevalent confusion.

“ In all other uninomes (significant that is) it is the same thing whether the copulative sign be placed between the radical sign (*signum radicale*,) and the number, or prefixed to both; nor does it change the value of those uninomes to place the sign + before them or in the middle. Thus $\sqrt{}9$, and $\sqrt{}+9$, and $+\sqrt{}9$, and $+\sqrt{}+9$, are all precisely the same, namely, as much + 3 as - 3. So $\sqrt{}27$, or $\sqrt{}+27$, or $+\sqrt{}27$, or $+\sqrt{}+27$, have the same value as + 3 only. So $\sqrt{}-27$, or $+\sqrt{}-27$, or $-\sqrt{}27$, or $-\sqrt{}+27$, have the same value as - 3 only. So in imaginary quantities, $\sqrt{}-9$ and $+\sqrt{}-9$ signify the same, as they both imply the same impossibility. But *take care* not to write in their stead $-\sqrt{}9$ or $-\sqrt{}+9$, as in the preceding section I have admonished.

“ So much for the affections of uninomes in themselves. The next consideration is the manner in which they stand affected to each other. Two uninomes (*uninomia bina*) are either commensurable with each other, or incommensurable. Those are commensurable which are to each other as discrete or absolute numbers. Hence every absolute number is commensurate with every absolute number. Moreover, two uninomes radicated alike, [*consimiliter radicata*, *i. e.* raised to the same power, or whose indices are alike,] of which the one simple number, when divided by the other, yields a number possessing such a root as the radical sign indicates, are said to be commensurable with each other in the ratio that the root indicates. Thus 5 and 7 are commensurable, because they are absolute or rational numbers. So, of the two uninomes radicated alike $\sqrt{}8$ and $\sqrt{}2$, if the simple number 8 be divided by the simple number 2, the quotient is 4. Now the number four has a root whose sign is $\sqrt{}$, that is to say, bipartient, and it is the number 2. Therefore $\sqrt{}8$ and $\sqrt{}2$ are commensurable with each other in the ratio of the root, which is as two to one. Consequently, all other uninomes which cannot be reduced to this are incommensurable. Thus $\sqrt{}12$ and $\sqrt{}3$, because they are differently ra-

dicated, are incommensurable. So $\sqrt{6}$, et $\sqrt{2}$, (although radicated alike,) are incommensurable, because 6 divided by 2 produces 3, which wants the root whose sign is $\sqrt{\quad}$, that is, the bipartient. But 12 and $\sqrt{4}$ are commensurable, because when reduced they are equivalent to 12 and 2, &c."

"I could find no more of his geometrical pairt amongst all his fragments." *

I now return to the concluding chapters of Napier's second book, of which it is only possible here to give a hurried and imperfect view. Having in the 6th, 7th, 8th, and 9th chapters disposed in the most brilliant manner of involution and evolution, our philosopher, never losing sight of perfect symmetry in his arrangement, again takes up, in chapter 10, the rules of proportion of integers. Referring to those already given in his first book, he now expounds several particular and compendious rules of proportion, of which one example may be selected, as being characteristic of the constant war he waged against the tyranny of derivative computation.

"There is," says he, "another compendious method without the omission of figures. Let all the given numbers of the question be arranged in their proper places above and below the line, as I have expounded in the general method proposed in *C. v. Lib. i.* Then let each of two numbers, one above the line, as if numerator, and the other below the line, as if denominator, be divided by the greatest common divisor until each of the numerators shall be to each of the denominators in the first or least ratio to each other, all the last quotients being noted. Finally, let the multiple of all the upper quotients be divided by the multiple of the lower quotient; this quotient will be the answer sought, and solve the question. Thus, if 4 builders construct a wall 6 feet high and 48 ells long in 42 days, it is demanded, in how many days will 5 builders erect a wall 9 feet high, and 50 ells long? Let all the numbers be arranged according to the rule laid down in *C. v. Lib. i.* and they will stand as on the margin. Then abbreviate the upper number 4, and the lower number 6, by 2, the greatest divisor, which gives $\frac{2}{3}$ in this form $\frac{2 \cdot 9 \cdot 50 \cdot 42}{5 \cdot 3 \cdot 48}$. Then divide 2 above, and 48 below, by the common divisor 2, which gives 1, and 24, in this form $\frac{1 \cdot 9 \cdot 50 \cdot 42}{5 \cdot 3 \cdot 24}$. Then divide 9 and 3 by 3, which gives 3 above and 1 below, in this form $\frac{1 \cdot 3 \cdot 50 \cdot 42}{5 \cdot 1 \cdot 24}$. Then divide 50 and 5 by 5, which gives 10 above and 1 below, in this form $\frac{1 \cdot 3 \cdot 10 \cdot 42}{1 \cdot 1 \cdot 24}$. Then divide 10 and 24 by the greatest common divisor 2, which gives 5 above and 12 below, in this form $\frac{1 \cdot 3 \cdot 5 \cdot 42}{1 \cdot 1 \cdot 12}$. Finally, divide 42 and 12 by the greatest common divisor 6, which gives 7 above and 2 below in this form, $\frac{1 \cdot 3 \cdot 5 \cdot 7}{1 \cdot 1 \cdot 2}$. So you now have the familiar and tractable numbers 1.3.5.7. and 1.1.2. to be multiplied together, instead of the given numbers, which were somewhat bigger. Let, then, 1.3.5.7 be multiplied into each other, which gives 105; let the same be done with the lower numbers 1.1.2. which gives 2, by which divide 105, and the

Ædific.	Pedes.	Ulnæ.	Dies.
4	9	50	42
5	6	48	quot dies.

* Note by Robert Napier, addressed to Henry Briggs.

quotient will come forth $52\frac{1}{2}$, being the number of the days, satisfying the question without great and laborious multiplications and divisions."

The five remaining chapters of this book, namely, the 11th, 12th, 13th, 14th, and 15th, are devoted to the arithmetic of fractions, the general rules of which have been already given. He carries them minutely through all the operations of addition, subtraction, multiplication, involution, evolution, and rules of proportion. It would occupy too much space to give any thing like a satisfactory abstract of these operations, in which the elegant and profound character of the work is completely sustained. This must again be observed, however, that his division of the subject of fractions clearly intimates, that at this time Napier had not considered decimal fractions as a distinct department. He says, "of fractions, some are called *vulgares*, others *physicæ*." He defines vulgar fractions as those "whose denominators are various and free; as, one-half, two-thirds, four-elevenths, &c." He then explains that "the denominator is that which *names* into how many equal parts unity is distributed; the numerator is that which *numbers* how many of these parts are taken; the numerator is pronounced in *cardinal* number, the denominator in *ordinal*, the numerator *above* the line, the denominator *below*." He then refers to the fractions of fractions, and his own method of noting them; "there are some improper fractions, he says, which are not expressly a part or parts of unity, but are the parts of fractions; and these are called fractions of fractions. *I note them by interposing the particle ex; others note them by omitting the line or lines of the posterior fractions.* Thus, two-fifth parts of three-fourth parts I note $\frac{2}{5}$ ex $\frac{3}{4}$; others note them thus, $\frac{2}{5} \frac{3}{4}$," &c.*

Napier defines physical fractions, "the part or parts of a whole, divided by some appointed and commonly received divisor, which its authors put in the place of denominator. Thus it hath pleased *our mint-masters* to divide the pound of money, not into what number of parts you will, but into 20 parts, and to put shillings in the place of denominator; so the Apothecaries divide the pound weight into 12 parts, which they name ounces, an ounce into 8 drachms, a drachm into 3 scruples, &c.; Chronologists divide the year into 12 months, the months into 30 days or thereabouts, the day into 24 hours, &c.; Astronomers divide the degree into 60 prime scruples or minutes, the primes into 60 seconds, the seconds into 60 thirds, &c." But Napier nowhere, in all his minute exposition of fractions in this work, refers to the system of decimals. The chapter of physical fractions closes his book of arithmetic, the last sentence of which must not be omitted,—“And now to God, the Father Almighty, and in all His Numbers, infinite, immense, and perfect, be ascribed all the praise, honour, and glory, for ever and ever. Amen. *Finis.*”

Napier, in the first chapter of his Arithmetic, refers to Geometrical Logistic as the subject of his third book, (the fragment already given,) and to Algebra, as treated of in his fourth book. It would appear, however, that although he has also left a manuscript treatise on Algebra, it is an earlier production than what we have been considering.

* "Compound fractions are fractions of fractions, and consist of several fractions connected together by the word *of*; as $\frac{2}{3}$ of $\frac{3}{4}$, or $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{3}{4}$."—*Hutton's Math. Dict.*

This is manifest from several circumstances. 1. It is entitled, "The Algebra of John Napier, Baron of Merchistoun," but not *liber quartus*, in correspondence with the other books. 2. Arithmetic is referred to in it; but there is no reference to his own book of Arithmetic, as unquestionably (according to his practice throughout the rest of the manuscript,) there would have been had that existed at the time. 3. This treatise is itself divided into two books; and while there is a systematic reference to its component parts, there is none whatever to the treatise we have considered. 4. Napier adopts in his Algebra the radical *nomination* and *notation*, which in the other treatise he had superseded by a superior system of his own; and there is here no reference to his peculiar notation of surds. There can be little doubt, therefore, that, although what we have reviewed was written before he had conceived the Logarithms, this treatise is a still earlier production. From the circumstance, however, that Robert Napier has paged the two books of Algebra continuously with the rest, it is probably that they are so much of what the philosopher intended to compose the fourth book, to which he alludes. Yet it is singular that there is no appearance of crude or youthful composition in this his earliest work. It is stamped with the same characteristics of simple exposition, profound views, and symmetrical arrangement as all his other productions. Our limits will not enable us to do it justice; but some extracts from the first chapter, which he entitles, "of the definitions, the divisions of the parts, and the vocabulary of the art," will afford an interesting specimen, and also evidence that his Algebra was written prior to his arithmetic.

"Algebra is the science which treats of solving questions of magnitude and multitude. It is twofold; the one part regards *nominate* quantities, the other *positive*. Nominate quantities are named from numbers, rational or irrational. Rational numbers are either absolute numbers, or fractions, of which arithmetic also treats. Irrational numbers are roots of those rational numbers which have no roots in numbers; and these, *as they are quantities*, also belong to geometry. The *positive* part of algebra is that which explicates quantities and numbers through the medium of fictitious suppositions, and of which I shall treat in the second book. In this first book I shall teach the first part of algebra, concerning nominate numbers and quantities. There are three species of nominates; *uninomia*, *plurinomia*, and *universalia*. *Uninomia* are either a single simple number, or any root of a single simple number. But the roots of numbers are various; therefore, for the sake of art and learning, they are expressed by characters prefixed, called radical signs (*signa radicalia*;) and noted thus:—

√ Q,	-	<i>radix quadrata.</i>
√ C,	-	<i>radix cubica.</i>
√ QQ,	-	<i>radix quadrati quadrata.</i>
√ Ss,	-	<i>radix supersolida.</i>
√ QC,	-	<i>radix quadrati cubica.</i>
√ SSs,	-	<i>radix secunda supersolida.</i>
√ QQQ,	-	<i>radix quad. quad. quadrata.</i>
√ CC,	-	<i>radix cubi cubica. et sic de cæteris in infinitum.</i>

Our philosopher then minutely expounds the various compositions and combinations of these radical signs and quantities, with their relations to each other. In the second chapter he commences their arithmetical operations with "addition of uninomes;" and thus, in seventeen chapters, which compose this first book, he gives the most beautiful treatise on the arithmetic of surds perhaps ever written. His leading arrangement is always *genealogical*. He shows how *uninomes* are born of the extraction of roots that have no roots of numbers, of which his first part treats,—how, from the addition and subtraction of uninomes that are incommensurable arise *plurinomes*, of which his second part treats,—and how, from the extraction of the obscure roots of plurinomes arise *universals*, of which the third and last part treats; and then he adds, "so in like manner from universals arise universals of universals, and from these again others *ad infinitum universalissima*, the art of which, should it require to be practised, which rarely happens, may easily be gathered from what has been laid down."

Napier's second book, entitled "of positive or cossic algebra," commences, like the last, with definitions, divisions, and a vocabulary. He defines the positive part of algebra to be that which "discloses, by means of feigned suppositions, a true quantity and true number sought." He defines *suppositions* or *positions*, "certain fictitious symbols attached to unity, which, in the place and on the part of quantities and numbers *unknown*, we add, subtract, multiply, and divide. Positions and the symbols of positions are as various and dissimilar as the unknown quantities which the question embraces. Their figures and names are, *ex. gr.* 1 R, which is pronounced *one first position*; 1 a, pronounced *one a*, or *one second position*; 1 b, *one b*, or *one third position*, and so on through the rest of the alphabet." These symbols compose what our philosopher calls "things first in order." He then proceeds to deduce the successive orders [*i. e.* powers] *in infinitum* by the involution of these symbols, and illustrates his exposition by the following table:

Numeri ordinum.	Characteres et exempla ordinum primæ positionis.	Characteres et exempla ordinum secundæ positionis.	Characteres et exempla ordinum tertiæ positionis.	&c.
0				
1	1 R	3	1 a	2
2	1 Q	9	1 a Q	4
3	1 C	27	1 a C	8
4	1 QQ	81	1 a QQ	16
5	1 Ss	243	1 a Ss	32
6	1 QC	729	1 a QC	64
7	1 SSSs	2187	1 a SSSs	128
8	1 QQQ	6561	1 a QQQ	256
9	1 CC	19683	1 a CC	512
10	1 Q Ss	59049	1 a Q Ss	1024
11	1 SSSs	177147	1 a SSSs	2048
12	1 QQC	531441	1 a QQC	4096
13	1 SSSSs	1594323	1 a SSSSs	8192

"In this table," he says, "I have supposed, for example's sake, that 1 R is equivalent to 3, 1 a to 2, and 1 b to 4; which being given, the values of the successive orders follow, necessarily, as noted."

The symbolical language and applications of algebra have undergone so great a revolution since Napier wrote, that to give a sufficiently illustrated analysis of the whole of this part of his work would occupy more space than we can afford. It is rich in definitions, and he leaves no step in his progress unexplained. He uses figures for the *known* quantities, the universal literal system having been introduced at a later period. We see from the preceding table and nomenclature, that the *unknown* quantities he classed in *positions*, and called *positives*, or *things*, which last term is not strange in the history of algebra, the science having been called by the Italian authors *Regola de la Cosa*, or *Rule of the thing*, which is also the derivation of the term *cossic*. From the second chapter to the eighth inclusive, Napier proceeds, in his usual minute and symmetrical manner, through the whole arithmetic of the *cossic* art. In chapters ninth and tenth he enters upon the *theory of equations*, one of the most important and complicated departments of analytical science, and in which he is far before the algebra of the period when he composed this treatise. How little was it ever suspected that the algebraic triumphs of Vieta, Harriot, and Girard, whose principal works were not known to the world for many years after the date of this manuscript, were some of them actually in the possession of this retired and unpretending Scottish baron, though laid aside among his papers, and never known publicly till now! Professor Playfair, in his Dissertation, sketches the history of the slow progress of this branch of algebra, and shows that the genesis of equations first received a decided explication in the works of Harriot, not published till the year 1631. He adds, "Their slow progress arose from this, that they worked with an instrument, the use of which they did not fully comprehend, and employed a language which expressed more than they were prepared to understand; a language which, under the notion first of *negative*, and then of *imaginary* quantities, seemed to involve such mysteries as the accuracy of mathematical science must necessarily refuse to admit." But early and rude as was the period in the history of algebra to which we must refer the composition of Napier's manuscript, we find him treating these mysterious quantities as if he had a perfect command of them, and looking forward with exultation to his future applications of such *great algebraic secrets*. Nothing can be more interesting in the whole history of his studies, than his opening chapters of that redoubtable subject Equations. They prove beyond question that he was among the very first to understand that recondite subject; which he did so thoroughly as to compose a treatise, the fragment of which may be compared with any of the greatest that have succeeded him, from Harriot to Euler. Now this is very striking. The internal evidence is irresistible that Napier composed his algebra before his arithmetic, and geometrical logistic. The progress of his studies appears to have been in this order. Having mastered algebra he conceived the noble project of composing four books embracing every department of numerical science; he returned accordingly to the simplest elements, and with an extensive prospect and command of the vast field before him, he had digested his subject, and "sett it orderlie down," nearly as far as his original books of al-

gebra, and had even commenced a systematic reformation of the symbolical language of algebra, when the invention of Logarithms interrupted his original plan. This great invention, however, had, it seems, occurred to him before the year 1594, when Tycho got a hint of it from Napier's friend Craig; and, indeed, from his own expressions, we must date his conception of the Logarithms many years before their publication. His treatise on Numbers, then, in which he betrays no idea either of Logarithms or Decimal fractions, must be referred to a very early period, and it is impossible that when he wrote it he could know any thing of the writings of Vieta. "Most of Vieta's algebraic works," says Dr Hutton, "were written about or before the year 1600, but some of them were not published till after his death, which happened in 1603." And this is most material to observe that, "the two books *de æquationum recognitione et emendatione*, which contain Vieta's chief improvements in algebra, were not published till the year 1615;" indeed, his scattered works were only first collected into a volume thirty years after our philosopher's death.

But the historians of science are agreed that, although some important conquests were achieved in that department by Tartalea, Cardan, and a few others, the general theory of equations was only first opened by Vieta, who paved the way for Harriot and Descartes. Montucla says, "The different transformations which may be adopted to give an equation a more commodious form, are, at least for the most part, the invention of M. Vieta, who taught the method in his book entitled *De Emendatione Æquationum*. We there learn how to perform all the operations of arithmetic, addition, subtraction, multiplication, and division, upon the roots of equations. By means of that he causes the second term of an equation to disappear; an operation which at once *resolves* quadratic equations, and *prepares* the cubic. It is thus, too, he causes the fractions to vanish which embarrass an equation, that he delivers it from irrationality when any of the terms are embarrassed thereby; all these things have been adopted by the modern analysts, and form what they call the preparation of equations; after these preliminaries M. Vieta passes to the resolution of equations of all degrees." This is just the object of the two chapters on equations with which, unfortunately, our philosopher's manuscript concludes. The first of them, being the 9th of the 2d book of his algebra, is entitled "of Equations and their Roots," and the one following is "of the general Preparation of Equations." No more is extant; but in these chapters he refers to succeeding ones, as if already composed, and expressly mentions that, after laying down all the rules of *preparation*, he means to give the methods of *resolution*. Though none of these valuable lucubrations were ever published, and only a fragment has been saved, yet in the history of his own mind, and in estimating the honour he confers upon his country, the fact is most interesting. Euler, of whom those most capable to judge have said, "that he was indisputably the greatest analyst that has ever appeared," concludes the work, by which I have all along tested Napier's, with the theory of equations. So does our philosopher his, and here again the same comparison may be safely challenged. Even in the state in which he left his work,

among his loose papers, and not remodelled and fitted to the first books, as he obviously intended to have done, all that remains of his doctrine of equations is richer in tuition, more systematically arranged, and appears to lay the foundation for a more masterly examination of the subject than the corresponding chapters of Euler's finished work. This pretension is so high that, in order to justify it, I have given the two last chapters of the manuscript entire in the Appendix, and have translated them for the benefit of those who might not take the trouble to read algebra in Latin. In the translation, I have adhered as literally as possible to the original. Some of his terms, and of course his symbolical language, differ from that now in use; but he is so precise and explanatory that, with the aid of the vocabulary already quoted, it is easy for any one acquainted with the history of algebra to follow him. The learned will there find that he is not only anticipating Vieta in what Montucla refers to that philosopher, and from whose merit, of course, Napier's unpublished work cannot detract, but that he is evidently stretching beyond the triumphs of Vieta to those of Girard and Harriot. It is impossible to read his opening chapters of equations, and not admit that they indicate a maturity in the subject for which Vieta is held only to have paved the way. Girard is considered the first in whose work, published long after Napier's death, the refined and difficult doctrine of imaginary quantities and roots assumes a place in science. Our philosopher clearly has this doctrine, and apparently a great command of the subject. The reduction of equations he calls *expositio*, and the root *exponens*. He states how various are those roots; that they are *valida* when prenoted with the sign +, and *invalida* with the sign —; in other words, positive and negative roots. He also defines the nature of an *impossible* equation, with the view of preparing the way for his doctrine of imaginary roots; a doctrine which it is obvious he had profoundly considered; indeed, he lays the foundation for it, as a great algebraic secret not then known, in his chapter of abundant and defective quantities which has been quoted. He also refers to roots of every description, capable of being expressed by number or quantity, or both, or neither; clearly embracing all roots, rational and irrational, real, and imaginary; and then he expressly adds, that "these with their examples shall be amply discussed in chapters 11, 12 and 13,"—the chapters which ought immediately to follow that with which the manuscript abruptly concludes. The terms he so frequently and fearlessly uses of *quantities less than nothing*, and *impossible or imaginary quantities*, all of which have been referred to Girard as their originator, indicate that command of the subject which was not to be daunted by the difficulty of naming such quantities, and that he was prepared to show how the phrases were justified in science. It is also very interesting to observe, that although he does not adopt, as Vieta did, letters for the known quantities, his notation is in some material circumstances beyond that philosopher's. Mr Babbage, in his History of Notation, observes, "it is a curious circumstance that the symbol which now represents equality was first used to denote subtraction, in which sense it was employed by *Albert Girard*, and that a word signifying equality was always used instead *until the time of Harriot*." This sentence, it must

be observed, overlooks the claim of *Recorde*, who, if he did not succeed in establishing the sign of equality, unquestionably proposed it, as I have elsewhere noticed. Napier, however, adopts it, and, with his usual precision, defines it in these words; "betwixt the parts of an equation that are equal to each other a double line is interposed, which is the sign of equation (*signum æquationis*); thus, $1 R = 7$, which is pronounced, *one thing equal to seven.*" To Vieta is ascribed the *vinculum* in algebraic notation, which Girard changed to the *parenthesis*. This, as is well known to algebraists, is used to denote the compound of binomial surds yielding what are termed *roots universal*. The English algebraists, chiefly, use the vinculum, which is drawn above the compound thus, $\sqrt{a+b}$. Napier explains and uses this notation, with the simple variation of drawing the line under the compound. In the 12th chapter of his arithmetic of surds he lays down; "to extract the square root of this quantity $\sqrt{Q} 48 + \sqrt{Q} 28$, prefix to this binomial (*huic binomio*) the following radical sign \sqrt{Q} , with a period after it thus, $\sqrt{Q} \sqrt{Q} 48 + \sqrt{Q} 28$," &c. and in the 17th chapter of the same book he gives this example, after explaining the notation, "the square root is extracted from this quantity $5 + \sqrt{c} 2 - \sqrt{Q} 3 - \sqrt{Q} 2$, by prefixing the sign of the root universal with a line drawn in this manner, $\sqrt{Q} 5 + \sqrt{c} 2 - \sqrt{Q} 3 - \sqrt{Q} 2$," &c. Accordingly, this vinculum will be found frequently used in his equations, and sometimes a vinculum within a vinculum. Yet even later than Vieta that convenient notation was not in constant use. Oughtred adopts the *u* after $\sqrt{\quad}$ to denote universal, instead of what is called "the vinculum of Vieta." I can nowhere find in Napier the sign \times of multiplication, which Oughtred introduced. In the preparation of equations our philosopher is far in advance of the date of his manuscript. "Harriot," says *Bossut*, "was the first who thought of placing all the terms of an equation on one side, and thus distinctly saw, what Vieta had only pointed out in a confused manner, that in every equation the coefficient of the second term is the sum of the roots taken with contrary signs," &c. But it will be observed that Napier had this mode of preparation, and made much of it. "If," says he, "you transpose all the terms of one side of an equation to the opposite side, the whole will be made equal to nothing, and this is called *an equation to nothing*," &c.

What I have thus imperfectly abstracted from this most interesting relic will enable the world to see that the Inventor of Logarithms was not a mere calculator who had made a lucky hit in a path where others were close behind him; but that had he only published his treatise on Logistic, without having invented the Logarithms, he would have taken the place of Vieta,—have anticipated the triumphs of Harriot—and, at a still earlier period, have placed Britain in the very highest ranks of those countries from which analytical science has received its greatest impulses.

It appears to me unquestionable that Napier composed his Arithmetic, and consequently his Algebra, before conceiving any of those mechanical inventions in aid of calculation, of which his own account has been given in the preceding memoirs. His

Rabdologia and his *Promptuarium* would otherwise have been frequently and prominently referred to.* In the former of these inventions, so well known under the name of *Neper's Bones*, the philosopher's object was to reduce the labour of multiplication and division to the less laborious operations of addition and subtraction,—to make the *primitives* do the work of the *derivatives*. From the moment he commanded the genealogy of numbers, this seems to have been his constant endeavour. “Napier,” said poor Pinkerton, “was not a great inventor, he was only a useful abbreviator of a particular branch of the mathematics.” But it was the *power* of his mind that impelled him to this. The finest geniuses are they who have felt most intensely the trammels of calculation. Many a man passes for a great mathematician, because he is a huge computer. Hutton and Maseres were *great calculators* rather than *great mathematicians*. When their pages were full of figures and symbols they were happy; and they took up the subject of Logarithms, *con amore*, from the very love of that labour to which the Logarithms were opposed. Archimedes and Napier were *anti-calculators*. But Napier alone, of all philosophers in all ages, made it the grand object of his life to obtain the power of calculation without its prolixity. At whatever period, therefore, our philosopher composed his minor works, they must be regarded with great interest, from the evidence they afford, that, with this object constantly before him, he left no department of numerical science not enriched by his most original genius. They compose a chapter, and no mean one, of his universal system of numbers.

Mr Herschel, in his *History of Mathematics*, has said, “Napier, struck with the difficulties which encumbered arithmetical computation of any length, and which various circumstances had about that time concurred to place in a very prominent light, after bestowing much fruitless labour on the invention of mechanical contrivances for multiplication and division, rejected this plan, and struck on the happy idea of Logarithms.” Yet the great Wolff has devoted an elaborate chapter of his *Elementa Matheseos* to the “*Lamellas Neperianas, quarum ope multiplicationem ac divisionem facilius absolvere licet quam per abacum Pythagoricum.*” The great Leibnitz did not disdain such mechanical inventions, and has referred pointedly to Napier's while praising his own in competition with the machine invented by Pascal. † It is interesting to regard our philosopher as

* The only reference to his minor inventions which occurs in the manuscript tends to confirm this remark. In his chapter entitled “Miscellaneous short methods of Multiplication and Division” this note is marked as an interpolation to a passage regarding short methods of multiplication, “*sive omnium facillime per ossa Rabdologiæ nostræ,*” clearly implying that he had not the method when he wrote his *Arithmetic*. Had Napier lived to finish his treatise on *Logistic*, it would have been the most splendid work of the kind in existence. His *Mechanical Arithmetic, Logarithms, and Decimal Fractions*, with all his improvements in notation would have been added to his system; and how much of that system would have been his own!

† “J'ai encore eu le bonheur de produire une machine arithmétique infiniment différente de celle de *M. Pascal*, puisque la mienne fait les grands multiplications et divisions en un moment, et sans additions ou soustractions auxiliaires; au lieu que celle de *M. Pascal*, dont on parloit comme d'une

the father of this school too,—a school whose labours are fruitless, just because the Logarithms have superseded their utility, unless, perhaps, we except that Leviathan of an abacus, so fearfully constructed “that the machine can itself correct the errors which it may commit, and that the results of its calculations, when absolutely free from error, can be printed off without the aid of human hands, or the operation of human intelligence” (*Brewster*); and this Mr Babbage is inventing chiefly for the purpose of computing Logarithms. I am inclined to doubt the theory that Napier rejected Rabdologia, and then set himself to seek the Logarithms. From his letter to the Chancellor, a very different idea may be gathered. He appears to say that he contrived such artifices for the special benefit of those who might distrust the *artificial numbers*. There occurs in the work, “*Tabulato anno Domini 1615*,” an example, however, that may possibly have been added when he was preparing for the press this profound and elegant little volume, which we are sure Mr Herschel had never looked at when he slighted its contents. Independently of other merits, it is hallowed by the fact, of containing perhaps the earliest chapter upon decimal fractions ever composed in Britain, and under the perfect notation which Napier was the first to adopt.

It is singular, that, after having proceeded so far in the path of numbers, our philosopher achieved his greatest conquest, which lay directly in that very path, and not far before him, by a different and an eccentric route, belonging to an opposite branch of science. The Logarithms should have been the offspring of his Arithmetic and his Algebra. He made them the offspring of his Geometry and his Arithmetic. Instead of prosecuting the arithmetic of powers and exponents, he turned to the geometry of his fluxions and his fluents,—terms unknown till then,—a method strange and startling to the philosophers of his times,—distrusted in another age when once again it reappeared in the hands of Newton,—yet successively productive of the Logarithms and the Calculus. The fact is, that Napier was as fearless and as powerful in geometry as he was in logistic, which accounts for the method he adopted. Who but himself, with the whole system of arithmetic and algebra brought under his control, would, in aid of calculation, have set to work with a flowing point! His fluxionary method was characteristic of the same unfettered genius that commanded the scale on either side of zero, and could even see that quantities, “*impossibiles et nihil significantes*,” though revolting in language, were precious in calculation. The application of *arithmetic* to geometry created the science of trigonometry. Napier made the application anew, and revolutionized that science, not merely in its *tables*, but in its *rules*. As a geometrician, therefore, he may almost be said to have been more successful than as an arithmetician, for the Logarithms themselves were

chose merveilleuse, et non pas sans raison, n'étoit proprement que pour les additions et soustractions, qu'on pouvoit combiner avec les batons de *Neper*, comme a fait depuis Mr Moreland.”—*Leibnitii Opera*, Tome vi. p. 248.

a geometrical conquest. "As a geometrician," says Playfair, "Napier has left behind him a *noble monument* in the two trigonometrical theorems which are known by his name, and which appear first to have been communicated in writing to Cavalieri, who has mentioned them with great eulogy; they are theorems not a little difficult, and of much use, as being particularly adapted to logarithmic calculation." *

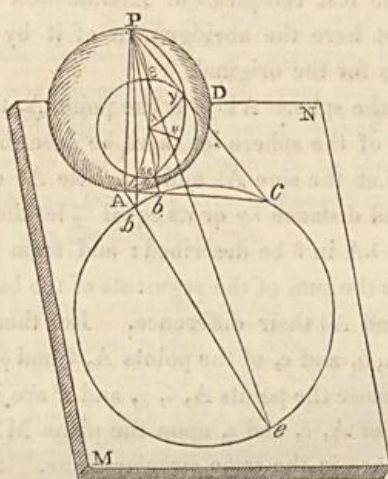
The rules alluded to, generally termed *Napier's Analogies*, are well known to mathematicians. One of his demonstrations is characterized by peculiar elegance and originality. In the optical illustration, we may observe an indication of those habits and acquisitions which led him to revive the lost catoptrics of Archimedes, whose history is given in the memoirs. I shall adopt here the abridgement of it by Dr Minto, referring the reader to the *Canon Mirificus* for the original.

"Let a plane MN touch the sphere ADP at the point A, the extremity of its diameter PA. Upon the surface of the sphere let there be described the triangle $A\lambda\gamma$ acute in γ , or $A\lambda\epsilon$ obtuse in ϵ . Let the sine $A\lambda$ and the base $A\gamma$ or $A\epsilon$ be produced to the point P. With the pole λ and distance $\lambda\gamma$ or its equal $\lambda\epsilon$ let the small circle of the sphere $\epsilon\gamma\delta$ intersecting λP in ϵ and λA in δ be described: and from λ let the arc $\lambda\mu$ be drawn perpendicular to $A\epsilon\gamma$. $A\gamma$ is the sum of the segments of the base and $A\epsilon$ their difference. $A\epsilon$ is the sum of the sides and $A\delta$ their difference. Let there be supposed a luminous point in P: The shadows, $A, b,$ and $c,$ of the points A, ϵ and γ , upon the plane MN, are in the same straight line, because the points A, $\epsilon, \gamma,$ and P are in the same circular plane: also the shadow $A, d,$ and $e,$ of A, $\delta,$ and ϵ , upon the plane MN, are in the same straight line, because A, $\delta, \epsilon,$ and P, are in the same circular plane. Since PA is perpendicular to the plane MN, the plane triangles PAc, PAb, PAe, and PAD, are rectangular in A: therefore, to the radius PA, the straight lines Ac, Ab, Ae, and Ad, are the tangents of the angles APc or AP γ , APb or AP ϵ , APe or AP δ , and APd or AP δ respectively.

* Professor Powell has also said (*Historical View, &c.* p. 194,) that Napier, before he published his trigonometrical theorems, "communicated them in manuscript to Cavalieri, who mentions them with high commendation." There is, however, a strange mistake here. Napier never corresponded with Cavalieri. That great philosopher was the first Italian commentator upon the Logarithms, but he was only born in the year 1598, as Professor Playfair himself tells us, and as Professor Powell of course repeats. Consequently, when Napier had his rules in manuscript, Cavalieri was an infant, or, at least, a child. Besides, Bonaventura Cavalieri was a Papist! a friar of the order of the Jesuati of St Jerome! And the old Scotch baron, who, God bless him, never communicated the scrape of a pen to any philosopher, would not have sent his theorems to one who was a jesuitical friar. Playfair quotes Wallis as his authority; but the passage has been misunderstood. Wallis (*Opera Math.* Tom. ii. p. 875,) says, "*Proportiones sequentes duas Cavalierius acceptas refert Nepero; nec immerito eas dicit alte indaginis.*" But it is manifest that this means no more than that the Italian philosopher acknowledged that science was indebted to Napier for those rules, which, he adds, evince a lofty genius. The same is also apparent in Cavalieri's great work on Logarithms, of which the editions are dated 1632 and 1643.

This philosopher has the honour of being the first who established the Logarithms in Italy.

But these angles, being at the circumference of the sphere, have for their measures the halves of the arcs intercepted by their sides: therefore Ac , Ab , Ae , and Ad , are the tangents of the halves of $A\gamma$, $A\epsilon$, $A\delta$, and $A\delta$ respectively. Now, by optics, the shadow of any circle, described on the surface of the sphere, produced by rays from a luminous point situated in any point of that surface excepting the circumference of the circle, forms a circle on the plane perpendicular to the diameter at whose extremity the luminous point is placed: therefore the points c , b , e , and d , are in the circumference of a circle: therefore $Ac \times Ab = Ae \times Ad$. Q. E. D."



But it is not merely by his *Analogies* that our philosopher is distinguished in trigonometry. The same object that he constantly pursued in numbers, he struggled to attain in his geometrical path. He determined to enable the student, with the least retentive memory, to carry as it were the whole science of trigonometry in his head, and he actually succeeded. There is not a modern work upon the subject in which *Napier's rule of the circular parts* is not the relief of study, and the theme of praise. If we turn to the most distinguished elementary works we find it said, "the rule of the *Circular Parts* invented by Napier, is of great use in spherical trigonometry, by reducing *all the theorems employed in the solution of right-angled triangles to two*. These two are not new propositions, but are merely enunciations which, by help of a particular arrangement and classification of the parts of a triangle, include all the six propositions with their corollaries; they are perhaps the happiest example of artificial memory that is known." (*Playfair's Elements*). If we turn to the most distinguished philosophical treatises, we find, "these forms are not easily remembered, and, therefore, an artificial memory has been supplied to the student and computist, by rules known by the title of *Napier's Rules for Circular Parts*; and in the whole compass of mathematical science, there cannot be found, perhaps, rules which more completely attain that which is the proper object of rules, facility and

brevity of computation."—(Woodhouse, of Cambridge.) If we turn to the great historians of science the same eulogy is to be met with. Wallis expounds the rule, and adds, "this, Napier excogitated for the relief of memory, and Cavallerius, Ursinus, Vlaccus, and our own Gellibrand, Oughtred, Norwood, Ward and Wing, have applied it to various cases." Montucla observes, "it would appear that Napier's views always tended to the simplification of practice; among his inventions, one for the resolution of spherical rectangular triangles is especially remarkable, and in the judgment of all acquainted with it extremely ingenious and convenient; indeed those versant in spherical trigonometry know that sixteen cases in spherical rectangular triangles may be proposed, and of these there are ten or twelve so difficult that authors who have written on the subject have been obliged to construct a table to consult for the relief of memory; Napier's rule reduces all these cases to a single rule, composed of two parts, whose elegant form is particularly apt to impress itself profoundly on the memory; hence the English trigonometrists generally adopt it, and I cannot conceal my surprise at scarcely finding a trace of it in various French and Continental treatises upon trigonometry, published since that epoch; M. Wolff, however, has felt the merit of it, and taught it in his *Elementa Matheseos Universalis*."

No wonder, then, that, with such geometrical powers of invention, our philosopher reached the Logarithms through that path. But it would, indeed, have been wonderful, if, after having done so, he had not, with all his command of numbers, have immediately perceived that the *transcendental* system he had created was not fitted for ordinary calculation, and if he could not have supplied the desideratum. There were various practical inconveniences in his system which it was impossible he could fail to perceive. Above all it was inconvenient and unsuitable, for common operations, to have a system of Logarithms whose fundamental progression was not accommodated to the root, or base, of the arithmetical scale in use. This fact could escape no calculator the moment he attempted to work with the new-born power, and to doubt the fact which Napier asserts, and which Briggs never upon any occasion hesitated to admit, namely, that he (the object of whose life was to increase the power, by simplifying the means of calculation,) had himself observed and provided against that inconvenience, is just as absurd as we have seen that it is unjust. He had only to return from his *geometrical* flight,—which, however, had brought out the lofty system that is the parent of all others,—to his simplest *arithmetical* considerations, in order, as he says himself, "to set out such Logarithms as shall make those numbers to fall upon decimal numbers, such as 100,000,000, 200,000,000, 300,000,000, &c. which are easy to be added or abated to or from any other number." It was the practical inconvenience, and not the algorithm of powers and exponents, that led to this change; a change which itself first opened the doctrine of fractional exponents.*

* 1000 equals 10 raised to the third power; 10,000 equals 10, to the fourth power; 3 and 4 respectively are the logarithms of those numbers; and, taken as powers and exponents, are written thus, 10^3 10^4 . But what is the logarithm of 2000? which, in the modern view of the subject, is the same as

There never was before, or has been since, or can be again, such a destiny in numbers. What could have compensated his country for the suppression of his system of algebra but that he forsook it to invent the Logarithms? Who would not have advised him to turn neither to the right hand nor to the left from that analytical career in which he had triumphed so far? A step or two in notation,—and he had systematically commenced to clear that path,—would have opened to him the arithmetic of exponents,—the Logarithms,—the Binomial Theorem! all of which, from a mind such as his analytical treatise displays, we may safely say nothing but a rude notation veiled. But he was not satisfied with the powerful machinery of integers and fractions, abundant, defective, and imaginary quantities, uninomes and plurinomes, and all the play of radical and cossic signs that he had reduced to obedience. The stars were becoming too many for Tycho and Kepler,—so he determined to attack the numeral scale through another medium. Then what a result—what an episode in his analytical labours—what a corollary to his great design! Having surveyed, and mastered, and nearly digested the whole field of Logistic, so that his unfinished manuscript may compete with Euler's finished production,—having conquered computation and attacked notation, the ARCHIMEDES OF THE NORTH paused, not to rest, but to seek another path of conquest. In that very departure from his algebraic career he brought out, as it were by a single blow, two great sections of the Arabic scale, which had been latent till then, and caused an important end of the exponential system to become the means of developing that very doctrine. Then how thoroughly was the object of his constant study fulfilled in the Logarithms! “By their means it is that numbers almost infinite, and such as are otherwise impracticable, are managed with ease and expedition. By their assistance the mariner steers his vessel—the geometrician investigates the nature of the higher curves—the astronomer determines the places of the stars—the philosopher accounts for other phenomena of nature—and lastly, the usurer computes the interest of his money.”—(Keill.) But in what age, or in what department of science, can we limit the impulse which the crowning success of Napier's ruling propensity created? “The quadrature of the hyperbola,” says another elegant and distinguished philosopher, “was now no longer a matter of mere speculative curiosity. Practical utility was become deeply interested in the investigation by a discovery which the beginning of the seventeenth century produced, but which we deferred speaking of that we might connect it with its *proper link* in the great chain.”—“The invention of Logarithms was a most invaluable present to the calculator, but its influence extended still wider. Gregory St Vincent in 1647 had demonstrated the grand property of the hyperbola which connects its area with the logarithmic function; and Mercator, pursuing this subject at length in his *Logarithmotechnia* (1667,) distinctly reduced the construction of logarithmic tables

to ask, what power of 10 is 2000? It must be represented by three integers and a decimal fraction thus, 3.301. It is obvious, therefore, how the common logarithms are connected with the doctrine of *fractional exponents*. So, reversing Dr Hutton's dictum, we may say, that the *invention of Logarithms led to the algorithm of powers and exponents*,—the very path that would have led to them.

to the quadrature of hyperbolic spaces. The unsuccessful attempts of Wallis now came under his contemplation, and what that geometer could not accomplish, Mercator effected by the simple but happy idea of continuing the division of the numerator by the denominator to infinity, as in the decimal arithmetic, and applying the method of Wallis to the series of positive powers which results. The first general quadrature of the hyperbola was thus obtained at the same time that the regular developement of a function in series was now distinctly exhibited."—"Such were the grounds upon which Newton was to raise the mighty fabric of his mathematical discoveries. Previous to the publication of Mercator's series, the perusal of Wallis' work, as himself relates, had led him to consider how the general or indefinite values had afforded that writer his quadrature of the whole circle. This was a work of comparatively greater facility than that undertaken by Wallis, and his undertaking was accordingly successful. It immediately struck him that the same method of interpolation might be applied to the ordinates as to the areas, and, by pursuing this idea, he arrived at his *Binomial Theorem*, which proved the key to the whole doctrine of series."—(*Herschel*.)

I have done my best to illustrate the domestic history—the Christian character—the philosophical power of Napier; and, however rudely the task may have been performed, the world has now a better basis for his eulogy than, perhaps, England's historian was aware of when he called him "the person to whom the title of a GREAT MAN is more justly due than to any other whom his country ever produced."



THE HISTORY OF THE UNITED STATES

The history of the United States is a story of growth and expansion. From a small collection of colonies on the eastern coast, it grew into a vast nation that stretched across the continent. The early years were marked by struggle and conflict, as the colonies fought for their independence from British rule. The American Revolution was a turning point in the nation's history, leading to the signing of the Declaration of Independence in 1776. The new nation then faced the challenge of building a government that would unite the diverse states and provide for the common good. The Constitution was drafted in 1787 and put into effect in 1789, establishing a system of checks and balances that has endured to this day. The years following the Revolution were a period of rapid growth and development. The nation expanded westward, settling new lands and building a strong economy. The Industrial Revolution brought about significant changes in the way people lived and worked, leading to the rise of cities and the growth of a middle class. The American Civil War, fought from 1861 to 1865, was a defining moment in the nation's history, as it resolved the issue of slavery and preserved the Union. The war led to the passage of the Emancipation Proclamation and the Reconstruction Amendments to the Constitution, which guaranteed the rights of all citizens. The Reconstruction era was a period of great challenge and struggle, as the nation sought to rebuild and reunite. The Civil War also led to the rise of the Gilded Age, a period of rapid economic growth and industrialization. The late 19th and early 20th centuries were marked by the rise of big business and the growth of a powerful industrial sector. The Progressive Era, which began in the late 19th century and continued into the early 20th century, was a period of reform and social change. Progressives sought to address the problems of the Gilded Age, such as corruption and the exploitation of workers. The Progressive Era led to the passage of many important laws, including the Pure Food and Drug Act and the Antitrust Act. The Progressive Era also led to the rise of the labor movement and the passage of laws that protected workers' rights. The Progressive Era was a period of great achievement and progress, but it was also a period of great challenge and struggle. The Progressive Era led to the rise of the New Deal, a series of programs and policies that were implemented in response to the Great Depression. The New Deal was a period of great reform and social change, leading to the passage of many important laws, including the Social Security Act and the National Labor Relations Act. The New Deal was a period of great achievement and progress, but it was also a period of great challenge and struggle. The New Deal led to the rise of the Cold War, a period of tension and conflict between the United States and the Soviet Union. The Cold War was a defining moment in the nation's history, leading to the development of nuclear weapons and the space race. The Cold War also led to the passage of many important laws, including the Civil Rights Act and the Voting Rights Act. The Cold War was a period of great achievement and progress, but it was also a period of great challenge and struggle. The Cold War led to the end of the Cold War and the beginning of a new era of peace and cooperation. The end of the Cold War led to the passage of many important laws, including the North American Free Trade Agreement and the Personal Responsibility and Work Opportunity Reconciliation Act. The end of the Cold War was a period of great achievement and progress, but it was also a period of great challenge and struggle. The end of the Cold War led to the rise of the 21st century, a period of rapid technological advancement and global interconnectedness. The 21st century has been a period of great achievement and progress, but it has also been a period of great challenge and struggle. The 21st century has led to the passage of many important laws, including the Affordable Care Act and the Dodd-Frank Wall Street Reform and Consumer Protection Act. The 21st century is a period of great achievement and progress, but it is also a period of great challenge and struggle. The 21st century will continue to shape the future of the United States and the world.

