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The mathematical principles of mechanical philosophy, and their application to the theory of universal gravitation

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Statics.

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STATICS.

CHAPTER I.

THE COMPOSITION AND EQUILIBRIUM OF FORCES ACTING UPON A MATERIAL PARTICLE.

11. When a single force acts upon a particle, it is clear, from the meaning we attach to the term force, that the particle cannot be at rest.

Experience shews us, however, that two forces may counteract each other's effects in producing motion. In such a case the forces, even though they originate from different causes, are said to be *equal;* since they are measured by their effects. Experience likewise shews us that three or more forces may be in equilibrium with each other, if their directions and magnitudes are properly adjusted. The object of the Science of Statics is to determine the relations which must exist among the forces, in magnitude, direction, and points of application, that they may produce equilibrium when acting on a body.

12. Now any given number of forces acting upon a particle must either be in equilibrium, or else produce an effect on the particle which some single but unknown force would produce. For if the forces be not in equilibrium, the particle will begin to move in some determinate curve line immediately the particle is abandoned to the action of the forces. It is clear, then, that a single force may be found of such a magnitude, that if it act along the tangent at the commencement of

the curve, and in a direction opposite to that in which the motion would take place, this force would prevent the motion, and would consequently be in equilibrium with the other forces which act upon the particle. If, then, we were to remove the original forces, and replace them by a single force, equal in magnitude to that described above, but acting in an opposite direction, the particle would still remain at rest. This force, which is equivalent in its effect to the combined effect of the original forces, is called their *resultant*, and the original forces are called the *components* of the resultant.

13. It will be necessary, then, to begin by deducing rules for the *composition of forces*; that is, for finding their resultant force. After we have determined these, it will be an easy matter to deduce the analytical relations which forces in equilibrium must satisfy, by equating the expression which gives the magnitude of their resultant to zero.

Prop. *To find the resultant of a given number of forces acting upon a particle in the same straight line: and to find the condition that they must satisfy, that they may be in equilibrium.*

14. When two or more forces act on a particle in the same direction, it is evident that the resulting force is equal to their sum, and acts in the same direction.

When two forces act in opposite directions on a particle, it is equally clear that their resultant force is equal to their difference, and acts in the direction of the greater component.

When several forces act in different directions, but in the same straight line, on a particle, the resultant of the forces acting in one direction equals the sum of these forces, and acts in the same direction: and so of the forces acting in the opposite direction. The resultant, therefore, of all the forces equals the difference of these sums, and acts in the direction of the greater.

If the forces acting in one direction are reckoned positive, and those in the opposite direction negative, then their resultant equals their algebraical sum; its sign determining the direction in which it acts.

15. In order that the forces may be in equilibrium, their resultant, and therefore their algebraical sum must equal zero.

Prop. *To find the resultant of two forces acting upon a particle not in the same straight line*[*].

16. Let P and Q represent the magnitudes of the two forces: A the particle (fig. 1.), AP, AQ the directions in which

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[*] The following proof of the Parallelogram of Forces given by M. Duchayla, is well worthy of attention for the simplicity of its demonstration.

1. To find the *direction* of the resultant of two forces acting upon a point.

When the forces are equal, it is clear that the direction of the resultant will *bisect* the angle between the directions of the forces: or, if we represent the forces in magnitude and direction by two lines drawn from the point where they act, the diagonal of the parallelogram described on these lines will be the direction of the resultant.

Let us assume that this is true for two unequal forces, p and m: and also for p and n. We can prove that it must then necessarily be true for two forces, p and $m+n$.

Let A (fig. 2.) be the point on which the forces p and m act, AB, AC their directions and proportional to them in magnitude: complete the parallelogram BC, and draw the diagonal AD: then by hypothesis, the resultant of p and m acts along AD.

Again, take CE in the same ratio to AC that n bears to m. Since it is an experimental fact that the point of application of a force may be transferred to any point of its direction, without disturbing the equilibrium, so long as the two points of application are invariably connected, we may suppose the force n to act at A or C: and therefore the forces p, m and n, in the lines AB, AC, and CE are the same as p and $m+n$ in the lines AB and AE.

Now, replace p and m by their resultant, and transfer its point of application from A to D: then resolve this force at D into two, parallel to AB and AC; these resolved parts must evidently be p and m, p acting in the direction DF, and m in the direction DG. Transfer these two forces, p to C and m to G.

But by the hypothesis, p and n acting at C have a resultant in the direction CG; let then p and n be replaced by their resultant, and transfer its point of application to G. But m acts at G.

Hence by this process we have, without disturbing the equilibrium, removed the forces p and $m+n$ which acted at A to the point G.

Therefore the resultant of p and $m+n$ acts in the direction of the diagonal AG, provided our hypothesis is correct.

But the hypothesis is correct for equal forces, as p, p, and therefore it is true for forces p, $2p$; consequently for p, $3p$, and so it is true for p, $r.p$.

Hence it is true for p, $r.p$ and p, $r.p$, and consequently for $2p$, $r.p$, and so forth; and it is finally true for $s.p$ and $r.p$, r and s being positive integers.

We have still to shew that the Proposition is true for *incommensurable* forces.

Let AB, AC (fig. 3.) represent two such forces. Complete the parallelogram BC. Then if their resultant do not act along AD, suppose it to act along AE; draw EF parallel to BD. Divide AB into a number of equal portions, each less than DE; mark the forces act: α the angle between these directions. Let R represent the magnitude of the resultant, and suppose AR is the direction in which it acts, this line being in the same plane as AP, AQ, and lying between them: let θ be the angle between AP and AR. Draw a line P_2AQ_2 in the plane of the forces through the point A, and perpendicular to AR.

Now let us imagine that P is the resultant of two forces P_1 and P_2 acting in the directions AR, AP_2; and that Q is the resultant of two forces Q_1 and Q_2, acting in the directions AR and AQ_2. Then (Art. 14.)

$$\left.\begin{aligned} R &= P_1 + Q_1 \\ \text{and } 0 &= P_2 - Q_2 \end{aligned}\right\} \cdots\cdots\cdots\cdots\cdots\cdots \text{(1)},$$

P_1 and P_2 are functions of P and θ; and Q_1 and Q_2 are similar functions of Q and $\alpha - \theta$. Since P, P_1, P_2 are merely the numerical ratios which the corresponding forces bear to the unit of force, and since the relation they bear to one another must manifestly be independent of the unit we choose to adopt, the relation between P and P_1 must be of the form

$$\frac{P_1}{P} = \text{function of } \theta = f(\theta) \text{ suppose;}$$

$$\text{and } \therefore \frac{P_2}{P} = f\left(\frac{\pi}{2} - \theta\right).$$

We have, then, to determine the form of $f(\theta)$.

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mark off from CD portions equal to these, and let G be the last division, this evidently falls between D and F; draw GK parallel to AC. Then two forces represented by AC, AG have a resultant in the direction AK, because they are commensurable: and this is nearer to AG than the resultant of the forces represented by AC, AB, which is absurd, since AB is greater than AG.

In the same manner we may shew that every direction besides AD leads to an absurdity, and therefore the resultant must act along AC, whether the forces be commensurable or incommensurable.

2. To find the *magnitude* of the resultant.

Let AB, AC be the directions of the given forces, AD that of their resultant: (fig. 4.) take AE opposite to AD, and of such a length as to represent the *magnitude* of the resultant. Then the forces represented by AB, AC, AE balance each other. Complete the parallelogram BE.

Hence AC is in the same straight line with AF: hence FD is a parallelogram: and therefore $AE = FB = AD$.

Or the resultant is represented in *magnitude* as well as in direction by the diagonal of the parallelogram.

We assume that a force can produce no effect in a direction perpendicular to its own direction.

This principle points out to us two general conditions which P_1 and P_2 must fulfil; for since P can produce no effect in a direction at right angles to its own, it follows that the sum of the resolved parts of P_1 and P_2 in a direction at right angles to that of P must equal zero; and the sum of their resolved parts in the direction of P must equal P.

These conditions furnish the equations

$$P_1 f\left(\frac{\pi}{2} - \theta\right) - P_2 f(\theta) = 0,$$

$$P_1 f(\theta) + P_2 f\left(\frac{\pi}{2} - \theta\right) = P.$$

Then, by putting for P_1 and P_2 their values $Pf(\theta)$ and $Pf\left(\frac{\pi}{2} - \theta\right)$, and dividing by P, we have the first equation identical, and the second gives

$$\{f(\theta)\}^2 + \left\{f\left(\frac{\pi}{2} - \theta\right)\right\}^2 = 1 \text{ ....} \quad (2).$$

This is the equation which $f(\theta)$ is to satisfy; but it admits of an infinite variety of solutions, and we assume (as the result of experiments allows us) that P_1 bears a determinate ratio to P, or $f(\theta)$ has a determinate value, for every value of θ. There must consequently be some other conditions, arising from the nature of the question, which $f(\theta)$ must satisfy; and which are to be our guides in selecting the proper solution of the equation just deduced.

The direct process would be, first to obtain the general solution of our equation, and then to determine the values of the arbitrary quantities involved in the general solution by the particular values of $f(\theta)$ for particular values of θ given by the nature of our problem. We may, however, reverse the process, and first search for the particular values of $f(\theta)$, and use these as our guides in detecting the proper solution.

Now the principle which has hitherto guided us—viz. that a force produces no effect in a direction at right angles to its own—furnishes us with new conditions which point out which of the solutions of equation (2) is to be chosen.

For whenever the direction of P_1 is at right angles to the direction of P, then $P_1 = 0$ and $P_2 = P$ or $-P$; and whenever the direction of P_2 is at right angles to that of P, then $P_2 = 0$ and $P_1 = P$ or $-P$ as exhibited below;

$$\begin{array}{lllll}
\text{when } \theta = 0\,, & P_2 = 0, & P_1 = P; & \therefore & f(0) = 1, \\
\theta = \dfrac{\pi}{2}\,, & P_1 = 0, & P_2 = P; & \therefore & f\left(\dfrac{\pi}{2}\right) = 0, \\
\theta = \pi\,, & P_2 = 0, & P_1 = -P; & \therefore & f(\pi) = -1, \\
\theta = \dfrac{3\pi}{2}, & P_1 = 0, & P_2 = -P; & \therefore & f\left(\dfrac{3\pi}{2}\right) = 0, \\
\theta = 2\pi\,, & P_2 = 0, & P_1 = P; & \therefore & f(2\pi) = 1.
\end{array}$$

. . . .

and all these cases are comprised in the formula

$$f\left(n\,.\,\frac{\pi}{2}\right) = \cos\left(n\,.\,\frac{\pi}{2}\right) \ldots\ldots\ldots\ldots\ldots(3),$$

n being an integer.

These equations (2) and (3) are the only conditions which $f(\theta)$ is to satisfy: and since, as we have observed, $f(\theta)$ must, from the nature of the question, have a determinate form, it follows that there is only one form of $f(\theta)$ which satisfies both equations (2) and (3); consequently if we can find one, this is the solution we are seeking.

Now equation (3) suggests $f(\theta) = \cos\theta$; this fully satisfies both (2) and (3), and is consequently the required solution.

Hence equations (1) become

$$R = P\cos\theta + Q\cos(\alpha - \theta)$$

$$0 = P\sin\theta - Q\sin(\alpha - \theta) \ldots\ldots(4);$$

adding the squares of these,

$$R^2 = P^2 + Q^2 + 2PQ \cos \alpha \ldots\ldots\ldots (5).$$

Equation (4) determines the *direction* of the resultant, and (5) its *magnitude*.

17. These equations point out the following geometrical construction, (fig. 1.)

Take AB, AC in the ratio of P to Q, through B draw BD parallel to AC and cutting AR in D: join CD. Then by Trigonometry,

$$BD = AB \frac{\sin \theta}{\sin (\alpha - \theta)} = AB \frac{Q}{P} \text{ by (4)} = AC$$

by the construction.

Hence BC is a parallelogram: and its diagonal is the *direction* in which the resultant of P and Q acts.

Again, by Trigonometry,

$$AD^2 = AB^2 + AC^2 + 2\,AB\,AC \cos \alpha.$$

Compare this with equation (5), and we see that the diagonal represents the *magnitude* of the resultant on the same scale that the sides of the parallelogram represent the forces P and Q.

This Proposition is, in consequence of the property just proved, called The Proposition of the *Parallelogram of Forces*.

18. Cor. 1. Any force acting on a particle may be replaced by two others, if the sides of a triangle, drawn parallel to the directions of the forces, have the same relative proportion that the forces have.

This is called the *resolution* of a force.

19. Cor. 2. When three forces acting on a particle are in equilibrium, they are respectively in the same proportion as the sines of the angles included by the directions of the other two.

For if we refer to fig. 4, we have

$$P : Q : R :: AB : AC \text{ (or } BD\text{)} \quad AD$$
$$:: \sin ADB : \sin BAD : \sin ABD$$
$$:: \sin CAE : \sin BAE : \sin BAC.$$

R

PROP. *Three forces act upon a particle in directions making right angles with each other: required to find the magnitude and direction of their resultant.*

20. Let AB, AC, AD represent the three forces X, Y, Z in magnitude and direction: fig. 5.

Complete the parallelogram BC, and draw AE: then AE represents the resultant of X and Y in magnitude and direction, by Art. 17. Now the resultant of this force and Z, which are represented by AE, AD, is represented in magnitude and direction by AF, the diagonal of the parallelogram DE.

Hence the resultant of XYZ is represented in magnitude and direction by AF.

Let R be the magnitude of the resultant, and abc the angles the direction of R makes with those of XYZ.

$$\text{Then, since } AF^2 = AE^2 + AD^2$$
$$= AB^2 + AC^2 + AD^2;$$
$$\therefore\ R^2 = X^2 + Y^2 + Z^2.$$

$$\text{Also, } \cos a = \frac{AB}{AF} = \frac{X}{R},$$
$$\cos b = \frac{AC}{AF} = \frac{Y}{R},$$
$$\cos c = \frac{AD}{AF} = \frac{Z}{R}.$$

Whence the magnitude and direction of the resultant are determined.

21. COR. Any force R, the direction of which makes the angles abc with three rectangular axes fixed in space,

may be replaced by the three forces $R \cos a$, $R \cos b$, $R \cos c$, acting simultaneously on the particle on which R acts, and having their directions parallel to the axes of co-ordinates respectively.

PROP. *Any number of forces act upon a particle in any directions: required to find the magnitude and direction of their resultant.* R

22. Let $P P_{,}$ ...... be the forces, and $\alpha\beta\gamma$, $\alpha_{,}\beta_{,}\gamma_{,}$, ... the angles their directions make with three rectangular axes drawn through the proposed point.

Then the component parts of P in the directions of the axes are, by Art. 21,

$$P \cos \alpha, \quad P \cos \beta, \quad P \cos \gamma,$$

$$\text{or } X, \ Y, \ Z, \text{ suppose.}$$

Resolving each of the other forces in the same way, we reduce the system to three forces, by adding those which act in the same lines (Art. 14.), we thus have ?

$$P \cos \alpha + P_{,} \cos \alpha_{,} + \ldots\ldots\ldots \text{ or } \Sigma . P \cos \alpha \text{ or } \Sigma . X,$$

$$P \cos \beta + P_{,} \cos \beta_{,} + \ldots\ldots\ldots \text{ or } \Sigma . P \cos \beta \text{ or } \Sigma . Y,$$

$$\text{and } P \cos \gamma + P_{,} \cos \gamma_{,} + \ldots\ldots\ldots \text{ or } \Sigma . P \cos \gamma \text{ or } \Sigma . Z,$$

acting in the directions of the axes of x, y and z.

The symbol Σ indicates that we are to take the sum of all the quantities in the system which are symmetrical with that before which it is placed.

If we call the resultant R, and the angles which the direction of R makes with the axes a, b, c, we have, by Art. 20.

$$R^2 = (\Sigma . X)^2 + (\Sigma . Y)^2 + (\Sigma . Z)^2,$$

$$\text{and } \cos a = \frac{\Sigma . X}{R}, \quad \cos b = \frac{\Sigma . Y}{R}, \quad \cos c = \frac{\Sigma . Z}{R},$$

to determine the resultant.

R

Prop. *To find the conditions of equilibrium when any number of forces act upon a material particle.*

23. When the forces are in equilibrium, we must have $R = 0$;

$$\therefore\ (\Sigma . X)^2 + (\Sigma . Y)^2 + (\Sigma . Z)^2 = 0;$$

$$\therefore\ \Sigma . X = 0, \quad \Sigma . Y = 0, \quad \Sigma . Z = 0,$$

and these are the conditions among the forces, that they may be in equilibrium. It follows, then, that if a material particle at rest be acted on by forces whose intensities and directions satisfy these three equations it will remain at rest.

These conditions may be expressed under another form: and lead to a principle denominated the Principle of Virtual Velocities.

Prop. *To prove the Principle of Virtual Velocities when forces acting on a particle are in equilibrium.*

24. Let xyz be the co-ordinates to the point of application of the forces, and $x + \delta x$, $y + \delta y$, $z + \delta z$ the co-ordinates to a point near the former. Draw perpendiculars from this latter point upon the directions of the forces P, $P_{,}$...and let δr, $\delta r_{,}$...be the distances of these perpendiculars from the point xyz: hence

$$\delta r = \delta x . \cos \alpha + \delta y . \cos \beta + \delta z . \cos \gamma$$

$$\delta r_{,} = \delta x . \cos \alpha_{,} + \delta y . \cos \beta_{,} + \delta z . \cos \gamma_{,}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

If then we multiply the equations

$$\Sigma . P \cos \alpha = 0, \quad \Sigma . P \cos \beta = 0, \quad \Sigma . P \cos \gamma = 0$$

by δx, δy, δz, and add the equations, bearing in mind that δx, δy, δz are independent of the forces, and may therefore be written inside the symbol Σ, we have

$$\Sigma . P (\delta x \cos \alpha + \delta y \cos \beta + \delta z \cos \gamma) = 0,$$

$$\text{or } \Sigma . P \delta r = 0,$$

which proves the following principle:—That if any number of forces acting upon a particle be in equilibrium, and the point of application be moved geometrically through any small space, then the sum of the products of the forces and the spaces described by the point of application relatively to the directions of the forces will vanish; these spaces being reckoned positive when drawn in the direction in which the force acts, and vice versâ.

This is termed the *Principle of Virtual Velocities*, since the spaces above mentioned measure the relative velocity of the geometric motion in the direction of the forces. We shall see in the next Chapter that this Principle is true for any system of forces.

CHAPTER II.

THE COMPOSITION AND EQUILIBRIUM OF FORCES ACTING ON A RIGID BODY.

25. A SOLID or fluid body is conceived to be an aggregation of indefinitely small material particles or molecules, which are held together by their mutual affinities. This appears to be a safe hypothesis, since experiments shew that any body is divisible into successively smaller and smaller portions without limit, if sufficient force be exerted to overcome the mutual action of the parts of the body.

26. By the term *rigid* we mean to express that the molecules of the body are held together in an invariable form; so that the intensity of the molecular forces is infinitely greater than that of the other forces which act upon the body. Were this not the case, the figure of the body would depend upon the forces which act upon it.

Now, in matter of fact, no body is perfectly rigid; every body yields more or less to the forces by which it is acted on. If, then, in any case this compressibility is of a sensible magnitude, we shall suppose that the body has assumed its figure of equilibrium, and then consider the points of application of the forces as a system of invariable form.

27. We are quite unacquainted at present with the laws according to which the molecules of a mass of matter act upon each other. In consequence of this, we must look for some principle which will enable us to calculate the effect of forces acting upon a rigid body, without bringing the molecular forces into the calculation.

And now we fall upon a case of the action of force totally different from anything we have yet met with. In considering its action on a single particle, the force was supposed to act on the whole of the particle: but now we have to consider the effect of forces acting on individual particles of an assemblage held rigidly together by their mutual affinities. The force which acts upon any particle of the body must in some way have its effect propagated through the whole system of particles, in consequence of their invariable connexion. Sundry experiments have led philosophers to the following principle; which, as will be seen, exactly answers our purpose.

28. *When a force, acting in combination with others, holds a solid body in equilibrium, the equilibrium of the body will not be disturbed if we transfer the point of application of the force to any other point whatever in the line in which the force is acting.*

We shall commence with the simplest case of a rigid body acted on by forces, and so ascend to the most general.

PROP. *Two forces act upon a rigid body in the same plane but not at the same point: required to find their resultant.*

29. Let A, B (fig. 6.) be the points upon which the forces P and Q act: AP, BQ their directions: join AB, and produce PA, QB to cut each other in C, and let $\alpha\beta$ be the angles which AP and BQ make with AB produced: then, by Art. 28, we may suppose P and Q to act at C, the point C being rigidly connected with AB.

Take Ca, Cb along CA, CB, in the ratio of P to Q: and on these describe the parallelogram ab, and draw the diagonal Cd: and let it be produced to cut AB in D.

Then, by Art. 17, Cd represents the resultant of P and Q in magnitude and direction: let R be the resultant: then since

$$Cd^2 = Ca^2 + Cb^2 - 2Ca \cdot Cb \cos(\alpha + \beta);$$

$$\therefore R^2 = P^2 + Q^2 - 2PQ\cos(\alpha + \beta),$$

this gives the *magnitude* of the resultant.

Let θ be the angle which the direction of R makes with AB: then

$$\frac{P}{Q}=\frac{Ca}{Cb}=\frac{Ca}{ad}=\frac{\sin Cda}{\sin aCd}=\frac{\sin DCB}{\sin DCA}$$

$$=\frac{\sin(\beta+\theta)}{\sin(\theta-\alpha)}$$

$$=\frac{\sin\beta+\cos\beta\tan\theta}{\tan\theta\cos\alpha-\sin\alpha};$$

$$\therefore\ \tan\theta\left\{\cos\alpha\frac{P}{Q}-\cos\beta\right\}=\sin\beta+\sin\alpha\frac{P}{Q},$$

$$\tan\theta=\frac{P\sin\alpha+Q\sin\beta}{P\cos\alpha-Q\cos\beta};$$

this determines the *direction* of the resultant.

Let $AD=x$: $AB=a$: then

$$\frac{a-x}{x}=\frac{DB}{DA}=\frac{DB}{DC}\cdot\frac{DC}{DA}$$

$$=\frac{\sin(\beta+\theta)}{\sin\beta}\cdot\frac{\sin\alpha}{\sin(\theta-\alpha)}$$

$$=\frac{\sin\alpha}{\sin\beta}\cdot\frac{P}{Q};$$

$$\therefore\ \frac{x}{a}=\frac{Q\sin\beta}{P\sin\alpha+Q\sin\beta},$$

this determines the point of application of the resultant.

30. Cor. 1. From the value of $\frac{a-x}{x}$ we obtain

$$\frac{P}{Q}=\frac{(a-x)\sin\beta}{x\sin\alpha}$$

$$=\frac{\text{perpendicular from } D \text{ on } Q\text{'s direction}}{\text{perpendicular from } D \text{ on } P\text{'s direction}}.$$

This shews us that if the point D be a fixed *fulcrum*, about which the body can turn, then, in order that P and Q may be in equilibrium about this fulcrum, in which case their resultant must pass through D, they must be inversely proportional in magnitude to the perpendiculars drawn from the fulcrum on their directions.

31. Cor. 2. The product of a force and the distance of its direction from a given point is called *the moment of the force with respect to the point.*

If through the point an *axis* be drawn at right angles to the plane, passing through the point and the direction of the force, this product is called *the moment of the force with respect to the axis.*

R

Hence when P and Q, acting in the plane through D, are in equilibrium about D, we learn by Cor. 1. that their moments with respect to D, or the axis through D, must be equal and opposite.

If the two forces P and Q are parallel to each other, their directions will not meet when produced: and therefore the demonstration of the last article must not be received; but, by a simple artifice, we can easily remedy this difficulty.

Prop. *To find the resultant of two parallel forces acting in the same plane on a rigid body.*

R

32. Let P and Q be the forces; A, B (fig. 7.) their points of application: let P and Q act in the same direction, making angles α with AB. The state of equilibrium of the body will not be altered if we apply two equal and opposite forces, each equal to S, at the points A, B, acting in the line AB.

Then P and S acting at A, are equivalent to some force P' acting in some direction AP'; and Q and S acting at B, are equivalent to some force Q' acting in some direction BQ' inclined to AP.

Produce $P'A$, $Q'B$ to cut each other in C, and draw CD parallel to AP and BQ, and cutting AB in D.

Transfer P' and Q' to C, C being rigidly connected with AB, and resolve them along CD and parallel to AB; the latter parts will be S and S acting in opposite directions, and the sum of the former is $Q + P$.

Hence R, the resultant of P and Q, $= Q + P$. Also since the sides of the triangle ACD are parallel to the directions of the forces P, S, P'; (see Art. 18.)

?

$$\therefore \frac{P}{S} = \frac{CD}{DA},$$

and similarly, $\frac{S}{Q} = \frac{DB}{CD}$;

$$\therefore \frac{P}{Q} = \frac{DB}{DA} = \frac{a - x}{x}$$

if $AB = a$ and $AD = x$;

$$\therefore \frac{x}{a} = \frac{Q}{Q + P},$$

this determines the point of application of the resultant Q.

R

33. If the force P act in a direction opposite to that of Q, (fig. 8.) a similar process will lead us to

$$R = Q - P,$$

$$\frac{x}{a} = \frac{Q}{Q - P},$$

but these are included in the formulæ of last article by putting $-P$ for P.

34. We also observe that the formulæ of Art. 29. comprehend those for parallel forces, although we thought it best not to assume this. We must put $\beta = \pi - a$ or $2\pi - a$ according as P and Q act in the same or opposite directions.

R

35. If $P = Q$ in Art. 33., then $R = 0$ and $x = \infty$, a result perfectly nugatory. It shews us that two equal and opposite parallel forces do not admit of a resultant. In fact the addition of the forces S, S still gives, in this case, two equal forces parallel and opposite in their directions.

Such a system of forces is called a *Couple:* the tendency of a couple is to *twist* the body upon which it acts.

We shall return to this subject, and investigate the laws of the composition and resolution of couples; since to these we shall hereafter reduce the composition and resolution of forces of every description acting upon a rigid body. Previous to this, however, we proceed to determine the resultant of any number of parallel forces.

R

PROP. *To find the resultant of any number of parallel forces acting upon a rigid body.*

36. Let the points of application of the forces be referred to a system of rectangular co-ordinate axes (fig. 9.) $m_1 m_2$ .... the points of application: $x_1 y_1 z_1$, $x_2 y_2 z_2$, ... their co-ordinates. $P_1 P_2$ ...... the forces acting at these points, those being reckoned positive which act in the direction of P_1 and those negative which act in the opposite direction. Join $m_1 m_2$: and take the point n_1 on $m_1 m_2$ such that

$$m_1 n_1 = \frac{P_2}{P_1 + P_2} \cdot m_1 m_2,$$

then n_1 is the point of application of the resultant of P_1 and P_2, that is, of $P_1 + P_2$: see Arts. 32, 33.

Draw $m_1 a$, $n_1 b$, $m_2 c$ perpendicular to the axis of x, and $m_1 de$ parallel to the axis of x, cutting $n_1 b$, $m_2 c$ in d and e.

Then, by similar triangles,

$$\frac{m_1 n_1}{m_1 m_2} = \frac{m_1 d}{m_1 e} = \frac{ab}{ac} = \frac{Ab - x_1}{x_2 - x_1};$$

$$\therefore\ Ab - x_1 = \frac{P_2}{P_1 + P_2} \cdot (x_2 - x_1);$$

$\therefore$ Ab, the abscissa to n_1,

$$= \frac{P_1 x_1 + P_2 x_2}{P_1 + P_2}.$$

Then, supposing P_1 and P_2 to be replaced by $(P_1 + P_2)$ acting at n_1, the abscissa to the point of application of the resultant of $(P_1 + P_2)$, P_3

$$= \frac{(P_1 + P_2) \cdot Ab + P_3 \cdot x_3}{(P_1 + P_2) + P_3}$$

$$= \frac{P_1 x_1 + P_2 \cdot x_2 + P_3 \cdot x_3}{P_1 + P_2 + P_3}.$$

Let R be the resultant of all the forces, and $\bar{x}$, $\bar{y}$, $\bar{z}$ the co-ordinates to its point of application;

$$\therefore\ R = \Sigma . P_1,$$

$$\bar{x} = \frac{\Sigma . P_1 x_1}{\Sigma . P_1}.$$

$$\text{Similarly } \bar{y} = \frac{\Sigma . P_1 y_1}{\Sigma . P_1}, \quad \bar{z} = \frac{\Sigma . P_1 z_1}{\Sigma . P_1}.$$

These determine the magnitude and point of application of the resultant.

R 9.

37. These co-ordinates are independent of the angle which the directions of the forces make with the co-ordinate axes. Hence if these directions be turned about the points of application of the forces, at the same time preserving their parallelism, the point of application of the resultant will not move.

For this reason that point is called the *centre of the parallel forces*.

R

38. A heavy body consists of an aggregation of material particles, each of which, in consequence of the Earth's attraction, tends towards the Earth's centre.

The weight, then, of a body may be considered as the resultant of the weights of the different elementary portions of the body acting in parallel and vertical lines.

In this case the centre of parallel forces is termed *the centre of gravity* of the body. The obvious property of this point is, that if it be fixed, the body will rest in any position; no forces but the body's weight being supposed to act.

R 9

39. The expression $P . x$ is denominated *the moment of the force P with respect to the plane yz.* This must be carefully distinguished from the moment of a force with respect to a point mentioned in Art. 31.

In consequence of the above definition, the equations for determining the position of the centre of parallel forces shew that the sum of the moments of any number of parallel forces with respect to any plane equals the moment of their resultant.

40. We shall now investigate the laws of composition of couples, since we shall hereafter reduce to these the composition of forces of every description acting on a rigid body.

We have already (Art. 35.) mentioned that by a *couple* we mean a system consisting of two equal parallel and opposite forces acting on a body not on the same point. This system does not admit of a single resultant force, as we have shewn: but two or more couples acting upon a body may be replaced by a single couple: this we proceed to demonstrate, after proving some of the properties of couples.

41. Definitions. The *arm* of a couple is the distance between the directions of its forces. R

The *moment* of a couple is the product of the force at either extremity and the arm: (see Art. 31).

The *axis* of a couple is a straight line perpendicular to the plane of the couple and proportional in length to the moment.

Prop. *The effect of a couple upon the equilibrium of a body is not altered, if its arm be turned through any angle about one extremity in the plane of the couple.* R

42. Let the plane of the paper be the plane of the couple (fig. 10.) and AB the arm: AB' its new position: the forces P_1 P_2 are equal, and act on the arm AB.

At A and B' let the two pair of equal and opposite forces $P_3 P_5$, $P_4 P_6$, each $= P_1$ or P_2 be applied, acting perpendicular to AB': this will not affect the equilibrium.

Let BP_2, $B'P_3$ cut in C: join AC: AC manifestly bisects the angle BAB'.

Now P_2 and P_3 are equivalent to some force in direction CA,

P_1 and P_4 .... same force .... AC;

$\therefore P_1\ P_2\ P_3\ P_4$ are in equilibrium with each other;

therefore the remaining forces P_5, P_6 acting at $B'\,A$ produce the same effect as P_1 and P_2 acting on AB. Hence the proposition is true.

Prop. *The effect of a couple on the equilibrium of a body is not altered if we transfer the couple to any plane parallel to its own, the arm remaining parallel to itself.* R

43. See fig. 11. AB the arm: $A'B'$ the new position parallel to AB. Join AB', $A'B$ bisecting each other in G.

At A' B' apply two equal and opposite forces each $= P_1$ or P_2: and let these forces be P_3 P_4 P_5 P_6: this will not alter the effect of the couple.

But P_1 and P_4 are equivalent to $2P_1$ acting at G in direction Ga,

and P_2 and P_3 .... Gb.

Hence P_1 P_2 P_3 P_4 are in equilibrium with each other, and may be removed; therefore the remaining forces P_5 P_6, acting at A' and B' produce the same effect as P_1 and P_2 acting on AB.

Hence the proposition is true.

44. Cor. Combining these two propositions, we see that a couple may be any how transferred so long as its plane remains parallel to itself.

R

Prop. *The effect of a couple on a body at rest will not be altered if we replace it by another whose moment is the same: the plane remaining the same, and the arms being in the same line, and having a common extremity.*

45. Let AB be the arm, (fig. 12.): P, P the forces: and suppose $P = Q + R$: let $AB = a$: and make AC, a new arm, $= b$: at C apply two equal and opposite forces Q_1 Q_2 each $= Q$: this will not alter the effect of the couple.

Now R at A and Q_1 at C will balance $Q + R$ or P at B,

$$\text{if } AB : BC :: Q_1 : R \text{ (Art. 32.)},$$

$$\text{or if } AB : AC :: Q_1 : Q_1 + R = P,$$

$$\text{or if } Q \cdot b = P \cdot a,$$

?

we then have remaining the couple Q_1 Q_2 acting on the arm AC.

Hence the couple P, P acting on AB, may be replaced by the couple Q_1 Q acting on AC, if $Q \cdot b = P \cdot a$; that is, if their moments are the same.

R

Prop. *To find the resultant of any number of couples acting upon a body, the planes of the couples being parallel to each other.*

46. First suppose the couples all transferred to the same plane (Art. 43.): next let them all be transferred so as

to have their arms in the same straight line, and one extremity common (Art. 42.): and lastly let them all be replaced by others having the same arm (Art. 45).

Thus if P, Q, R, S, ... be the forces, and

a, b, c, d, ... be their arms,

we shall have replaced them by the following forces, (supposing a the length of the common arm)

$$P\cdot\frac{a}{a},\quad Q\cdot\frac{b}{a},\quad R\cdot\frac{c}{a},\ \ldots\ldots\ldots\ldots \text{ acting on the arm } a.$$

?

Hence their resultant will be a couple whose force

$$= P\cdot\frac{a}{a} + Q\cdot\frac{b}{a} + R\cdot\frac{c}{a} + \ldots\ldots\ldots\ldots \text{ and arm} = a,$$

or whose moment $= P\cdot a + Q\cdot b + R\cdot c + \ldots\ldots\ldots$

Hence the moment of the resultant couple is equal to the sum of the moments of the original couples.

If one of the couples, as (S, S), act in a direction opposite to the couple (P, P), then the force at each extremity of the arm of the resultant couple will be

$$P\cdot\frac{a}{a} + Q\cdot\frac{b}{a} + R\cdot\frac{c}{a} - S\cdot\frac{d}{a} + \ldots\ldots\ldots$$

and the moment of the resultant couple will be

$$P\cdot a + Q\cdot b + R\cdot c - S\cdot d + \ldots\ldots\ldots\ldots$$

or the algebraical sum of the moments of the original couples; the moments of those couples which tend in the direction opposite to the couple (P, P) being reckoned negative.

PROP. *To find the resultant of two couples not acting in the same plane.*

R

47. Let the planes of the couples intersect in the line AB, which is perpendicular to the plane of the paper (fig. 13.), and let the couples be referred to the common arm AB, and let their forces, thus altered, be P and Q.

D

This Theorem is called the □^{m} of couples

In the plane of the paper draw Aa, Ab perpendicular to the planes of the couples (P, P) and (Q, Q): and equal in length to their *axes*, (Art. 41).

Let R be the resultant of the forces P, Q at A; and P, Q at B.

? Since AP, AQ are parallel to BP, BQ respectively, therefore AR is parallel to BR.

Hence the two couples are equivalent to the single couple (R, R) acting on the arm AB.

Draw Ac perpendicular to the plane of (R, R), and in the same proportion to Aa, Ab that the moment of the couple (R, R) has to those of (P, P), (Q, Q).

Then Ac is the *axis* of (R, R).

? Now the three lines Aa, Ac, Ab make the same angles with each other that AP, AR, AQ make with each other; also they are in the same proportion in which

$$AB \cdot P, \quad AB \cdot R, \quad AB \cdot Q \text{ are,}$$

or in which P, R, Q are.

But R is the resultant of P and Q;

therefore Ac is the diagonal of the parallelogram on Aa, Ab (see Art. 17).

Hence if two straight lines, having a common extremity, represent the axes of two couples, that diagonal of the parallelogram described on these lines, which passes through their common extremity is equal in magnitude and direction to the axis of the resultant couple.

R

PROP. *To find the magnitude and position of the couple which is the resultant of three couples which act in planes at right angles to each other.*

48. Let AB, AC, AD be the axes of the given couples, (fig. 5). Complete the parallelogram CB: and draw AE the diagonal. Then AE is the axis of the couple which is the resultant of the two couples whose axes are AB, AC.

Complete the parallelogram DE, and draw AF the diagonal. Then AF is the axis of the couple which is the resultant of the couples whose axes are AE, AD, or of those whose axes are AB, AC, AD.

$$\text{Now } AF^2 = AE^2 + AD^2$$
$$= AB^2 + AC^2 + AD^2.$$

Let G be the moment of the resultant couple, L, M, N those of the given couples;

$$\therefore\ G^2 = L^2 + M^2 + N^2;$$

and if λ, μ, ν be the angles the axis of the resultant makes with those of the components

$$\cos\lambda = \frac{AB}{AF} = \frac{L}{G}; \quad \cos\mu = \frac{M}{G}; \quad \cos\nu = \frac{N}{G}.$$

49. Cor. Hence conversely any couple may be replaced by three couples acting in planes at right angles to each other, their moments being

$$G\cos\lambda, \quad G\cos\mu, \quad G\cos\nu,$$

where G is the moment of the given couple, and λ, μ, ν the angles its axis makes with the axes of the three couples.

Prop. *To find the resultant of any number of forces acting on a rigid body in the same plane.*

50. Let the system be referred to any pair of rectangular co-ordinate axes Ax, Ay in the given plane: (fig. 14).

Let P, P_1, P_2, ... be the forces,

a, a_1, a_2, ... the angles which their directions make with the axis of x.

xy, x_1y_1, x_2y_2, ... the co-ordinates to their points of application.

Let B be the point of application of P: join BA: the points B and A are rigidly connected. At A apply two equal and opposite forces, each equal and parallel to P. This will not affect the equilibrium. Draw Ap perpendicular to PB produced if necessary.

R

? Hence P acting at B is replaced by P acting at A, together with a couple (P, P) acting on the arm Ap, or a couple whose moment $= P \, . \, Ap$, and tending to turn the body from the axis of x to the axis of y.

?
$$\text{Now } Ap = x \sin \alpha - y \cos \alpha.$$

Hence the moment of the couple (P, P)

$$= P \, . \, (x \sin \alpha - y \cos \alpha).$$

The moments of those couples are reckoned positive that tend to turn the body from the axis of x to the axis of y: and those negative that tend the other way.

The other forces may be similarly replaced.

Hence our system is reduced to the forces

$$P, \; P_1, \; P_2 \, \ldots\ldots\ldots \text{ acting at } A$$

in directions parallel to those of the original forces; and the couples whose moments are

$$P \, \{x \sin \alpha - y \cos \alpha\},$$
$$P_1 \, \{x_1 \sin \alpha_1 - y_1 \cos \alpha_1\},$$
$$P_2 \, \{x_2 \sin \alpha_2 - y_2 \cos \alpha_2\},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

acting in the plane of the paper.

Let R be the resultant of the forces acting at A, a the angle which R makes with the axis of x; G the moment of the resultant couple: then, by Art. 22,

?
$$R \cos a = \Sigma \, . \, P \cos \alpha,$$
$$R \sin a = \Sigma \, . \, P \sin \alpha,$$

$$\text{and, by Art. 46, } G = \Sigma \, . \, P \, (x \sin \alpha - y \cos \alpha),$$

and if $P \cos \alpha = X$, and $P \sin \alpha = Y$, these may be written

$$R^2 = (\Sigma \, . \, X)^2 + (\Sigma \, . \, Y)^2, \quad \tan a = \frac{\Sigma \, . \, Y}{\Sigma \, . \, X},$$

$$\text{and } G = \Sigma \, . \, \{Y \, . \, x - X \, . \, y\}.$$

R

51. Let the arm of the resultant couple be turned in the plane of the forces and about its extremity A, till it is perpendicular to the direction of R. Art. 42: (fig. 15).

Let AR be the direction of R: $AB = a$, the arm of the resultant couple, and, consequently, $\frac{G}{a}$ the force at each extremity: let this $= R'$.

?

Hence the forces are all reduced to a force $R + R'$ acting at A in the direction AR, and R' acting at B in the direction BR', parallel to AR. The resultant of these is R, acting at a point C in the direction CR parallel to AR, the distance AC being $= \frac{R'}{R} AB$ (by Art. 32.) $= \frac{G}{R}$.

Wherefore the resultant of all the forces $P, P_1, \ldots\ldots$ is a force R acting in the straight line whose equation is

$$y + AC \cos a = \tan a\,(x - AC \sin a),$$

which simplified becomes

$$x \tan a - y = \frac{AC}{\cos a},$$

$$\text{or } x \sin a - y \cos a = AC,$$

$$\text{or } x \,.\, \Sigma \,.\, Y - y \,.\, \Sigma \,.\, X = G,$$

the *direction* in which R acts will be determined by the sign of $\tan a$.

52. Cor. If it should happen that the forces are such that $R = 0$, then we are left with the couple whose moment is G, and there is not a single resultant force.

Prop. *To find the conditions of equilibrium of any number of forces acting on a rigid body in the same plane.*

R

53. We have shewn in the last Prop. that the resultant of any number of forces acting in the same plane on a rigid body equals a force R acting about the origin of co-ordinates, at a distance

$$= \frac{G}{R} \text{ where } R^2 = (\Sigma \,.\, X)^2 + (\Sigma \,.\, Y)^2,$$

$$\text{and } G = \Sigma \,.\, \{Yx - Xy\}.$$

Now when the forces are in equilibrium their resultant must vanish, therefore $R = 0$; also $G = 0$, since the distance $\frac{G}{R}$ must be indeterminate and not infinite; for if it were infinite, then the resultant of the forces would be a couple: see Art. 35.

Hence the conditions of equilibrium are

$$\Sigma . X = 0, \quad \Sigma . Y = 0,$$

$$\text{and } \Sigma . \{Y . x - X . y\} = 0.$$

These may be written

$$\Sigma . P \cos \alpha = 0, \quad \Sigma . P \sin \alpha = 0,$$

$$\text{and } \Sigma . P (x \sin \alpha - y \cos \alpha) = 0.$$

Prop. *To find the two resultants of any number of forces acting upon a rigid body in any directions.*

54. Let the forces be referred to three rectangular axes Ax, Ay, Az; and suppose $PP_1P_2\ldots$ are the forces, xyz, $x_1y_1z_1$, $x_2y_2z_2\ldots$ the co-ordinates to their points of application, and $\alpha\beta\gamma$, $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2\ldots$ the angles their directions make with the axes: fig. 16.

Let m be the point of application of P, mP its direction $Ar = x$, $rn = y$, $nm = z$: An in the plane xy, also ns parallel to Ax.

Now P may be replaced by its three components

$$P \cos \alpha, \quad P \cos \beta, \quad P \cos \gamma, \quad (\text{or } X, Y, Z \text{ suppose})$$

parallel to the axes, (Art. 21.)

Z produces the same effect if it be transferred to n. Now the equilibrium of the body will not be disturbed if we apply at A and also at r two opposite forces, each equal and parallel to Z. Then Z at m is equivalent to Z at A, and the two couples of which the moments are $Z . rn$ and $Z . Ar$ and the axes coincide respectively with the co-ordinate axes of x and y.

Hence Z at m is replaced by Z at A, and the two couples $Z . y$ and $-Z . x$ acting in the planes perpendicular to x and y

respectively: the moments of those couples which tend to turn the body from the axis of x to that of y about the axis of z, from y to z about x, and from z to x about y, are reckoned positive, and those in the opposite direction negative.

In the same manner we may substitute for Y and X.

Wherefore the force P acting at m may be replaced by X, Y, Z acting at A along the axes, together with the couples

$Z.y$ and $-Y.z$ in plane perpendicular to axis of x

$X.z$ and $-Z.x$ .... y

$Y.x$ and $-X.y$ .... z

or, by adding the moments of the couples acting in the same or parallel planes (Art. 46.)

P is replaced by X, Y, Z acting at A and the couples whose moments are

$Z.y - Y.z$ in plane perpendicular to axis of x

$X.z - Z.x$ .... y

$Y.x - X.y$ .... z.

By a similar resolution of all the forces, we shall have them replaced by the forces

$$\Sigma.X, \quad \Sigma.Y, \quad \Sigma.Z,$$

acting at A along the axes: and the couples

$\Sigma.\{Z.y - Y.z\} = L$ acting in the plane perpendicular to axis of x

$\Sigma.\{X.z - Z.x\} = M$ .... y

$\Sigma.\{Y.x - X.y\} = N$ .... z.

Let R be the resultant of the forces acting at A; a, b, c the angles its direction makes with the axes of co-ordinates: then (Art. 22.)

$$R^2 = \{\Sigma.X\}^2 + \{\Sigma.Y\}^2 + \{\Sigma.Z\}^2$$

$$\cos a = \frac{\Sigma.X}{R}, \quad \cos b = \frac{\Sigma.Y}{R}, \quad \cos c = \frac{\Sigma.Z}{R}.$$

Let G be the moment of the couple which is the resultant of the three couples above mentioned; λ, μ, ν the angles its axis makes with the axes of co-ordinates; then (Art. 48.)

$$G^2 = L^2 + M^2 + N^2,$$

$$\cos\lambda = \frac{L}{G}, \quad \cos\mu = \frac{M}{G}, \quad \cos\nu = \frac{N}{G}.$$

55. We may still further reduce the forces in the following manner.

Let the plane of the couple be turned round its axis till the projection of the direction of R on this plane is perpendicular to the arm. Let a be the length of the arm chosen arbitrarily. Then $\frac{G}{a}$ is the force at each extremity. Also let θ be the angle between the directions of R and the axis of G. Hence the whole force at A is =

$$\sqrt{R^2 + \frac{G^2}{a^2} + \frac{2RG}{a}\sin\theta}, \text{ (Art. 17.)}$$

$$\text{where } \cos\theta = \cos a \cos\lambda + \cos b \cos\mu + \cos c \cos\nu:$$

and the second force is $\frac{G}{a}$, acting at the other extremity of the arm.

These two forces cannot in general be reduced to a single force, since their directions do not meet.

In one case, viz., when the directions of these two forces meet, they can be reduced to a single force.

PROP. *To prove that* G *is the principal Moment of the Forces.*

56. The Principal Moment means the Moment of greatest Magnitude.

The quantities L, M, N are the sums of the moments of the forces with respect to the axes of x, y, z respectively (Art. 31), and they are equivalent to $\sqrt{L^2 + M^2 + N^2}$ $(= G)$, the moment of the forces with respect to an axis which makes angles with the axes whose cosines are $\frac{L}{G}$, $\frac{M}{G}$, $\frac{N}{G}$.

Now G, the resultant moment of the forces, must be independent of the *directions* of the axes of co-ordinates*: but L, M, N depend upon these directions. But since $L^2 + M^2 + N^2 = G^2$, it shews that the greatest value of L is G, in which case $M = 0$ and $N = 0$. Consequently G is the principal moment about the given centre.

57. The values of L, M, N in the general case (Art. 54.) shew that the moment about an axis through the given centre, and making an angle ϕ with the axis of principal moments, equals $G \cos \phi$.

PROP. *To find the locus of the centres which give the least principal moments; the magnitude of these moments and the position of their axes.*

58. Let $x_1 y_1 z_1$ be the co-ordinates to a centre which gives a minimum principal moment: let $L_1 M_1 N_1 G_1$ be the values of $LMNG$ at that point: then these are found by putting $x - x_1$, $y - y_1$, $z - z_1$ for xyz in $LMNG$;

$$\therefore L_1 = L - y_1 \Sigma . Z + z_1 \Sigma . Y,$$

$$M_1 = M - z_1 \Sigma . X + x_1 \Sigma . Z,$$

$$N_1 = N - x_1 \Sigma . Y + y_1 \Sigma . X,$$

$$G_1^2 = L_1^2 + M_1^2 + N_1^2$$

---

* We might prove this in the following manner:

Let r, r', ... be the distances of the points of application of the forces P, P', ... from the origin of co-ordinates; that is, from the centre of moments: $l m n$, $l' m' n'$,...... the angles that r, r',...... make with the axes: also let (PP'), (Pr), ... represent the angles between straight lines drawn through the origin parallel to the directions of P and P', of P and r, and so on. Then

$$L = \Sigma . Pr (\cos m \cos \gamma - \cos n \cos \beta), \quad M = \Sigma . Pr (\cos n \cos \alpha - \cos l \cos \gamma),$$

$$N = \Sigma . Pr (\cos l \cos \beta - \cos m \cos \alpha);$$

$$\therefore G^2 = L^2 + M^2 + N^2 \text{ (after reduction)}$$

$$= \Sigma . P^2 r^2 \sin^2 (Pr) + 2\Sigma . PP'rr' \{ \cos (PP') \cos (rr') - \cos (Pr') \cos (P'r) \},$$

and this is independent of the *directions* of the axes of co-ordinates, though it does depend on the situation of the centre of moments.

$$= (L - y_1\Sigma . Z + z_1\Sigma . Y)^2 + (M - z_1\Sigma . X + x_1\Sigma . Z)^2$$
$$+ (N - x_1\Sigma . Y + y_1\Sigma . X)^2.$$

When this is a minimum, its three partial differential coefficients with respect to $x_1 y_1 z_1$ must vanish: hence three equations which may easily be written in the form

$$R^2 . x_1 = (x_1\Sigma . X + y_1\Sigma . Y + z_1\Sigma . Z)\,\Sigma . X + N\Sigma . Y - M\Sigma . Z,$$
$$R^2 . y_1 = (x_1\Sigma . X + y_1\Sigma . Y + z_1\Sigma . Z)\,\Sigma . Y + L\Sigma . Z - N\Sigma . X,$$
$$R^2 . z_1 = (x_1\Sigma . X + y_1\Sigma . Y + z_1\Sigma . Z)\,\Sigma . Z + M\Sigma . X - L\Sigma . Y.$$

If we multiply these respectively by $\Sigma . X$, $\Sigma . Y$, and $\Sigma . Z$, we find an identical equation: which shews that these three are equivalent to only two equations: and since they are simple equations in $x_1 y_1 z_1$, we learn that the centres of minimum principal moments lie in a straight line, any two of the above equations being the equations to this line.

It is evident that G_1 increases indefinitely with $x_1 y_1 z_1$, and therefore does not admit of a maximum value.

59. If we eliminate the second terms of the above equations, they become of the ordinary form of equations to a line: we have

$$L - y_1\Sigma . Z + z_1\Sigma . Y = \frac{(L\Sigma . X + M\Sigma . Y + N\Sigma . Z)\,\Sigma . X}{R^2},$$

$$M - z_1\Sigma . X + x_1\Sigma . Z = \frac{(L\Sigma . X + MZ . Y + N\Sigma . Z)\,\Sigma . Y}{R^2},$$

$$N - x_1\Sigma . Y + y_1\Sigma . X = \frac{(L\Sigma . X + M\Sigma . Y + N\Sigma . Z)\,\Sigma . X}{R^2}.$$

60. The minimum principal moment is

$$G_1 = \frac{L\Sigma . X + M\Sigma . Y + N\Sigma . Z}{R}.$$

61. Let $\alpha_1\, \beta_1\, \gamma_1$ be the angles which the axis of G_1 makes with lines parallel to the co-ordinate axes: then

$$\cos \alpha_1 = \frac{L_1}{G_1}, \quad \cos \beta_1 = \frac{M_1}{G_1}, \quad \cos \gamma_1 = \frac{N_1}{G_1};$$

and these become, by the above equations,

$$\cos \alpha_1 = \frac{\Sigma . X}{R}, \quad \cos \beta_1 = \frac{\Sigma . Y}{R}, \quad \cos \gamma_1 = \frac{\Sigma . Z}{R},$$

which shew that the axes of all the minima principal moments are parallel to each other, and to the direction of the resultant.

PROP. *Required to find the condition among the forces that they may have a single resultant.*

62. In order that this may be the case, it is clear that the force R must be in the plane of the couple G. For then the force resulting from the composition of R with one of the forces of the couple will, when produced, meet the other force of the couple, and, being compounded, will thus produce a single resultant. Now this condition is satisfied when the angle between R and the axis of G equals 90^0: or when the cosine of this angle equals zero: that is, when

$$\cos a \cos \lambda + \cos b \cos \mu + \cos c \cos \nu = 0;$$

therefore the condition is that

$$\frac{(\Sigma . X) L + (\Sigma . Y) M + (\Sigma . Z) N}{R . G} = 0,$$

$$\text{or } (\Sigma . X) L + (\Sigma . Y) M + (\Sigma . Z) N = 0,$$

unless R or G vanishes.

This is no condition when $R = 0$: that is, when $\Sigma . X = 0$, $\Sigma . Y = 0$, $\Sigma . Z = 0$, for the above equation is then identical.

In fact we then have only the couple G: which does not admit of a single resultant.

Also this is no condition when $G = 0$, for then $L = 0$, $M = 0$, $N = 0$, and the equation is again identical.

But in this case it is evident we have a single resultant R.

PROP. *When the forces are reducible to a single resultant, required the magnitude of this force and the equations to the line in which it acts.*

63. In this case the force R is in the plane of the couple of which the moment is G.

Let the arm of the couple be turned about its extremity A (see fig. 15), and in the plane of the couple, till it is perpendicular to the force R: and let $AB = a$ be the arm of the couple: then the force of the couple $(R') = \frac{G}{a}$; and the single resultant equals R acting at C in the direction CR parallel to AR, C being in BA, produced and determined by the equation

$$AC = \frac{R'}{R} AB = \frac{G}{R}.$$

64. We must now find the equations to the line in which this resultant acts.

Let $x_1 y_1 z_1$ be the co-ordinates to some point in this line; then, transferring the origin to this point, it is clear that the body must have no tendency to revolve about the origin.

Therefore the new values of $L\,M\,N$ when we put $x_1 + x$, $y_1 + y$, $z_1 + z$ for $x\,y\,z$ must $= 0$;

$$\therefore\ 0 = \Sigma . P . \{(y_1 + y) . \cos\gamma - (z_1 + z) \cos\beta\},$$

$$\text{or } 0 = L + y_1 \Sigma . Z - z_1 \Sigma . Y \text{ .... } (1).$$

Similarly,

$$0 = M + z_1 \Sigma . X - x_1 \Sigma . Z \text{ .... } (2),$$

$$0 = N + x_1 \Sigma . Y - y_1 \Sigma . X \text{ .... } (3).$$

These three equations are equivalent to only two: for if we eliminate z_1 from (1) and (2), we have

$$0 = L\Sigma . X + M\Sigma . Y - x_1 \Sigma . Z . \Sigma . Y + y_1 \Sigma . Z . \Sigma . X.$$

But $L\Sigma . X + M\Sigma . Y + N\Sigma . Z = 0$, by Art. 62;

$$\therefore\ 0 = N + x_1 \Sigma . Y - y_1 \Sigma . X;$$

and therefore equation (3) is a necessary consequence of (1) and (2): wherefore any two of equations (1), (2), (3) are the equations to the line in which the single resultant acts.

Prop. *To find the conditions of equilibrium of any number of forces acting upon a rigid body in any directions.*

65. We have shewn that the forces are in the general case reducible to two acting in different planes. These forces, then, must each vanish when there is equilibrium.

Hence (Art. 55.)

$$R^2 + \left(\frac{G}{a}\right)^2 + \frac{2RG}{a} \sin\theta = 0,$$

$$\text{and } \frac{G}{a} = 0, \ a \text{ being arbitrary};$$

$$\therefore \ R^2 = 0, \text{ and } G^2 = 0, \text{ or}$$

$$(\Sigma . X)^2 + (\Sigma . Y)^2 + (\Sigma . Z)^2 = 0 \text{ and } L^2 + M^2 + N^2 = 0,$$

and these lead to the six conditions

$$\Sigma . X = 0, \quad \Sigma . Y = 0, \quad \Sigma . Z = 0,$$

$$\Sigma . (Zy - Yz) = 0, \quad \Sigma . (Xz - Zx) = 0, \quad \Sigma . (Yx - Xy) = 0.$$

These may be thus written:

$$\Sigma . P \cos\alpha = 0, \quad \Sigma . P \cos\beta = 0, \quad \Sigma . P \cos\gamma = 0,$$

$$\Sigma . P(y \cos\gamma - z\cos\beta) = 0, \quad \Sigma . P(z\cos\alpha - x\cos\gamma) = 0,$$

$$\Sigma . P(x\cos\beta - y\cos\alpha) = 0.$$

66. If we derive the conditions of equilibrium from the case where the forces admit of a single resultant, we shall arrive at the same conclusion. For we must have the force $R = 0$, and also the distance $\frac{G}{R}$ at which it acts must be arbitrary and not necessarily infinite: hence also $G = 0$, and the conclusions are the same as before*.

---

* We have remarked in Art. 25, that the property of the divisibility of matter, leads us to the supposition that every body consists of an assemblage of material particles, or molecules, which are held together by their mutual attraction. Now we are totally unacquainted with the nature of these molecular forces: if, however, we assume the two hypotheses, that the action of any two molecules on each other is the same, and also that it acts in the line joining their centres, two suppositions which appear to be perfectly legitimate, then we shall be able to deduce the conditions of equilibrium of a rigid body from those of a single particle.

PROP. *To find the conditions of equilibrium of a rigid body from those of a single molecule.*

Let

PROP. *To find the conditions of equilibrium of forces acting upon a rigid body when one point is fixed.*

67. Let the fixed point be taken as the origin of co-ordinates.

Let the body be referred to three rectangular co-ordinate axes: and let xyz be the co-ordinates to one of its constituent particles: XYZ the sums of the resolved parts parallel to the axes of the forces which act upon this particle, neglecting the molecular forces: P, P' ... the molecular forces acting on this particle; $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$, ... the angles their respective directions make with the three axes of co-ordinates.

Then we may suppose the rest of the body to be removed, and this particle held in equilibrium by the above forces. Hence by Art. 23.

$$\left.\begin{aligned} X + P\cos\alpha + P'\cos\alpha' + \dots\dots &= 0 \\ Y + P\cos\beta + P'\cos\beta' + \dots\dots &= 0 \\ Z + P\cos\gamma + P'\cos\gamma' + \dots\dots &= 0 \end{aligned}\right\} \dots\dots (\alpha).$$

We shall have a similar system of equations for each particle in the body: if there be n particles, we shall have $3n$ equations. These $3n$ equations will be connected one with another, since any molecular force which enters into one system of equations must enter into a second system; this is in consequence of the *mutual* action of the molecules.

There are two considerations which will enable us to deduce from these $3n$ equations, *six* equations of condition, independent of the molecular forces. These will be the equations which the other forces must satisfy, in order that the equilibrium may be established.

The first consideration is this, that the molecular actions are *mutual;* and that, consequently, if $P\cos\alpha$ represent the resolved part parallel to the axis of x of any one of the molecular forces involved in the $3n$ equations, we shall likewise meet with the term $-P\cos\alpha$ in another of those equations which have reference to the axis of x. Consequently, if we add all those equations together which have reference to the same axis, we have the three following equations of condition independent of the molecular forces

$$\Sigma . X = 0, \qquad \Sigma . Y = 0, \qquad \Sigma . Z = 0.$$

The second consideration is this:—that the straight lines joining the different particles are the directions in which the molecular forces act.

Thus let P be the molecular action between the particles whose co-ordinates are (xyz) and $(x_1 y_1 z_1)$:

$$\begin{array}{ccc} P\cos\alpha, & P\cos\beta, & P\cos\gamma, \\ -P\cos\alpha, & -P\cos\beta, & -P\cos\gamma, \end{array}$$

the corresponding resolved parts of P for the two particles.

Then
$$\cos\alpha = \frac{x_1 - x}{\sqrt{(x_1-x)^2 + (y_1-y)^2 + (z_1-z)^2}},$$

$$\cos\beta = \frac{y_1 - y}{\sqrt{(x_1-x)^2 + (y_1-y)^2 + (z_1-z)^2}},$$

$$\cos\gamma = \frac{z_1 - z}{\sqrt{(x_1-x)^2 + (y_1-y)^2 + (z_1-z)^2}}.$$

These

Now the action of the forces on the body will produce a pressure on the fixed point, and this will act in some definite direction.

Let $X'Y'Z'$ be the resolved parts of this pressure parallel to the axes.

If then we consider the forces $-X'$, $-Y'$, $-Z'$ in connexion with the given forces, we may suppose the body to be free, and the equations of equilibrium give

$$\Sigma . X - X' = 0, \quad \Sigma . Y - Y' = 0, \quad \Sigma . Z - Z' = 0,$$

$$L = 0, \quad M = 0, \quad N = 0.$$

These enable us to obtain three more equations free from molecular forces: for if we multiply the first and second of equations (α) by y and x respectively, and then subtract them, we have

$$Yx - Xy + \dots\dots\dots + P\{x \cos\beta - y\cos\alpha\} + \dots\dots\dots = 0,$$

and by the same process, we obtain from the system of equations which refer to the particle $(x_1 y_1 z_1)$,

$$Y_1 x_1 - X_1 y_1 + \dots\dots\dots - P\{x_1 \cos\beta - y_1 \cos\alpha\} + \dots\dots\dots = 0.$$

But the values of $\cos\alpha$ and $\cos\beta$, given above, lead to the condition

$$(x_1 - x)\cos\beta - (y_1 - y)\cos\alpha = 0.$$

Wherefore the equation

$$\left.\begin{array}{l} Y . x \;\; - X . y + \dots\dots\dots \\ + Y_1 . x_1 - X_1 y_1 + \dots\dots\dots \end{array}\right\} = 0,$$

will not involve P, the molecular action between the particles whose co-ordinates are $x y z$ and $x_1 y_1 z_1$, respectively.

It follows readily from what we have shewn, that if we form all the equations

$$Y_2 . x_2 - X_3 . y_3 + \dots\dots\dots\dots\dots\dots\dots\dots\dots = 0,$$

$$Y_3 . x_3 - X_3 . y_3 + \dots\dots\dots\dots\dots\dots\dots\dots\dots = 0,$$

……………………………………………………

and add them to those above, we shall have a final equation

$$\Sigma . (Y . x - X . y) = 0,$$

independent of the molecular forces.

In like manner we should obtain

$$\Sigma . (X . z - Z . x) = 0,$$

$$\Sigma . (Z . y - Y . z) = 0.$$

These six equations are the only conditions which can be obtained independent of the molecular forces: they must be satisfied by the forces which hold in equilibrium the assemblage of molecules, whatever be the laws of their molecular action.

Now in the case of a *rigid* body, the molecular forces are supposed to be themselves in equilibrium independently of the extraneous forces; hence the above six equations express the conditions of equilibrium of a rigid body.

The first three equations give the resolved parts of the pressure on the fixed point: and the last three are the only conditions to be satisfied by the given forces.

PROP. *To find the conditions of equilibrium of a body which has two points in it fixed.*

68. Let the axis of z pass through the two fixed points: and let the distances of the points from the origin be z' and z''. Also let $X'Y'Z'$, $X''Y''Z''$ be the resolved parts of the pressures on these points.

Then, as in the last Prop. the equations of equilibrium will be

$$\Sigma . X - X' - X'' = 0, \quad \Sigma . Y - Y' - Y'' = 0, \quad \Sigma . Z - Z' - Z'' = 0,$$

$$L - Y' . z' - Y'' . z'' = 0, \quad M + X' . z' + X'' z'' = 0,$$

$$N = 0.$$

The first, second, fourth and fifth of these equations will determine $X'X''Y'Y''$: and the third equation gives $Z' + Z''$, shewing that the pressures on the fixed points in the direction of the line joining them are indeterminate, being connected by one equation only.

The last is the only condition of equilibrium, viz. $N = 0$.

PROP. *To find the conditions of equilibrium of a rigid body resting on a plane.*

69. Let this plane be the plane of xy: and let $x'y'$ be the co-ordinates to one of the points of contact, R' the pressure which the body exerts against the plane at that point. Then the force $-R'$, and similar forces for the other points of contact, taken in connection with the given forces ought to satisfy the equations of equilibrium.

$$\text{Hence } \Sigma . X = 0, \quad \Sigma . Y = 0, \quad \Sigma . Z - \Sigma . R' = 0,$$

$$L + \Sigma . R'y' = 0, \quad M - \Sigma . R'x' = 0, \quad N = 0.$$

If only one point be in contact with the plane, then the third equation gives the pressure, and we have five equations of condition,

$$\Sigma . X = 0, \quad \Sigma . Y = 0, \quad L + y' \Sigma . Z = 0,$$
$$M - x' \Sigma . Z = 0, \quad N = 0.$$

If two points be in contact, then

$$R'y' + R''y'' = -L, \quad R'x' + R''x'' = M,$$

give $R' = -\dfrac{Lx'' + My''}{y'x'' - x'y''}$, $\quad R'' = \dfrac{Lx' - My'}{y'x'' - x'y''}$,

and the equations of condition are

$$\Sigma . X = 0, \quad \Sigma . Y = 0, \quad \Sigma . Z + \frac{L(x'' - x') + M(y'' + y')}{y'x'' - x'y''} = 0,$$

and $N = 0$.

If three points be in contact, then the pressures are determined from the equations

$$\left. \begin{aligned} R' + R'' + R''' &= \Sigma . Z \\ R'y' + R''y'' + R'''y''' &= -L \\ R'x' + R''x'' + R'''x''' &= M \end{aligned} \right\}$$

and the conditions of equilibrium are

$$\Sigma . X = 0, \quad \Sigma . Y = 0, \quad N = 0.$$

If more than three points are in contact, then the pressures are indeterminate, since they are connected by only three equations: but the conditions of equilibrium are still

$$\Sigma . X = 0, \quad \Sigma . Y = 0, \quad N = 0.$$

CHAPTER III.

THE EQUILIBRIUM OF A SYSTEM OF RIGID BODIES.

70. In order to obtain the conditions of equilibrium of two or more rigid bodies connected together in any way whatever, we must substitute unknown forces in the place of the mutual actions at the points of connexion, and then write down the equations of equilibrium of each body. These systems of equations will be connected together, since a force depending on the mutual actions of any two of the bodies must enter *both* the systems of equations, which correspond to those bodies.

Prop. *To find the conditions of equilibrium of a system of bodies acted on by given forces.*

71. Let any one of the bodies A be acted on by the given forces $X_1\, Y_1\, Z_1$, ... at the points $(x_1\, y_1\, z_1)$, ... also suppose that in consequence of the connexion of the system that a mutual force P acts between the bodies A and B, making angles $\alpha\,\beta\,\gamma$ with the axes: and let $x_1'\,y_1'\,z_1'$ be its point of application in A, and $x_2'\,y_2'\,z_2'$ the point of application in B.

Now if we suppose the force P to act on A and analogous forces for all the other mutual actions arising from the connexion of the system, the body A may be supposed to be in equilibrium under the action of these forces and the forces $X_1 Y_1 Z_1$ ...... Hence, by Art. 65,

$$\Sigma . X_1 + P\cos\alpha + \ldots\ldots = 0, \quad \Sigma . Y_1 + P\cos\beta + \ldots\ldots = 0,$$

$$\Sigma . Z_1 + P\cos\gamma + \ldots\ldots = 0,$$

$$\Sigma . (Z_1 y_1 - Y_1 z_1) + P(y_1'\cos\gamma - z_1'\cos\beta) + \ldots\ldots = 0,$$

$$\Sigma \,.\, (X_1 z_1 - Z_1 x_1) + P\,(z_1' \cos \alpha - x_1' \cos \gamma) + \ldots\ldots = 0,$$
$$\Sigma \,.\, (Y_1 x_1 - X_1 y_1) + P\,(x_1' \cos \beta - y_1' \cos \alpha) + \ldots\ldots = 0.$$

In the same manner for the body B,

$$\Sigma \,.\, X_2 - P \cos \alpha + \ldots\ldots = 0, \quad \Sigma \,.\, Y_2 - P \cos \beta + \ldots\ldots = 0,$$
$$\Sigma \,.\, Z_2 - P \cos \gamma + \ldots\ldots = 0,$$
$$\Sigma \,.\, (Z_2 y_2 - Y_2 z_2) - P\,(y_2' \cos \gamma - z_2' \cos \beta) + \ldots\ldots = 0,$$
$$\Sigma \,.\, (X_2 z_2 - Z_2 x_2) - P\,(z_2' \cos \alpha - x_2' \cos \gamma) + \ldots\ldots = 0,$$
$$\Sigma \,.\, (Y_2 x_2 - X_2 y_2) - P\,(x_2' \cos \beta - y_2' \cos \alpha) + \ldots\ldots = 0.$$

If we add the second set of equations to the first set, each to each, we shall have six equations free from P: for P evidently vanishes from the first three; and it enters the fourth in the form $P\,\{(y_1' - y_2') \cos \gamma - (z_1' - z_2') \cos \beta\}$: and this vanishes when the bodies are in contact, because then $y_1' = y_2'$ and $z_1' = z_2'$: also it vanishes when the bodies are not in contact, because then P must act in the line passing through the points $(x_1' y_1' z_1')$, $(x_2' y_2' z_2')$; and then, r being the distance between these points,

$$r \cos \gamma = z_1' - z_2', \quad r \cos \beta = y_1' - y_2';$$
$$\therefore\ (y_1' - y_2') \cos \gamma - (z_1' - z_2') \cos \beta = 0.$$

and in the same way P disappears from the fifth and sixth equations.

Hence the six final equations are free from P: and, by adding together the equations referring to all the bodies, each to each, we shall have finally

$$\Sigma \,.\, X = 0, \quad \Sigma \,.\, Y = 0, \quad \Sigma \,.\, Z = 0,$$
$$\Sigma \,.\, (Zy - Yz) = 0, \quad \Sigma \,.\, (Xz - Zx) = 0, \quad \Sigma \,.\, (Yx - Xy) = 0,$$

free from all the mutual actions of the bodies of the system.

We might have been led to this conclusion by remembering that the equilibrium of the system would not be disturbed by supposing the bodies, when at rest, to become united rigidly at the points of mutual action, and so considering the system as one rigid body.

PROP. *To prove that the Principle of Virtual Velocities is true of any system of forces which keep any material system in equilibrium.*

R

72. We shall first enunciate this Principle.

Suppose a material system is held in equilibrium by the action of a system of forces: suppose the points of application of the forces are geometrically moved through very small spaces in a manner consistent with the connexion of the parts of the system one with another. Suppose perpendiculars drawn from the new positions of the points upon the directions of the forces acting at the points in their positions of equilibrium. The distances of any perpendicular from the original point of application of the corresponding force is called the *virtual velocity of the point with respect to that force*, and is estimated positive or negative, according as the perpendicular falls on the side of the point towards which the force acts or the opposite side: then the Principle is this,

The algebraical sum of the products of each force of the system and the corresponding virtual velocity vanishes.

73. I. Suppose the system consists of only one rigid body.

We must cause the different particles to describe small spaces consistent with their connection; this will, in the case of a rigid body, be as well accomplished by supposing the co-ordinate axes to receive a slight alteration of position.

Suppose the axes to revolve round z through a small angle θ: then

$$x, y, z \text{ become } x + y\theta, \quad y - x\theta, \quad z,$$

neglecting small quantities of the second and higher orders.

Next, suppose these new axes to revolve through a small angle ϕ about the new axis of y: by these means the original values x, y, z become

$$(x + y\theta) - z\phi, \quad y - x\theta, \quad z + (x + y\theta)\,\phi,$$

$$\text{or } x + y\theta - z\phi, \quad y - x\theta, \quad z + x\phi.$$

Next, suppose the axes to revolve about the new axis of x, through a small angle ψ, and the co-ordinates become

$$x + y\theta - z\phi, \quad y - x\theta + z\psi, \quad z + x\phi - y\psi,$$

omitting small quantities of the second and higher orders.

Lastly, let the origin be shifted to a point whose co-ordinates are a, b, c: hence, if δx, δy, δz be the total changes in x, y, z produced by these changes of axes,

$$\delta x = a + y\theta - z\phi \ldots\ldots\ldots\ldots (1),$$

$$\delta y = b + z\psi - x\theta \ldots\ldots\ldots\ldots (2),$$

$$\delta z = c + x\phi - y\psi \ldots\ldots\ldots\ldots (3).$$

Now multiply the equations of equilibrium

$$\Sigma . X = 0, \quad \Sigma . Y = 0, \quad \Sigma . Z = 0,$$

$$\Sigma . (Xy - Yx) = 0, \quad \Sigma . (Zx - Xz) = 0, \quad \Sigma . (Yz - Zy) = 0,$$

by a, b, c, θ, ϕ, ψ respectively, and add;

$$\therefore \Sigma . \{X(a + y\theta - z\phi) + Y(b + z\psi - x\theta) + Z(c + x\phi - y\psi)\} = 0;$$

and, consequently,

$$\Sigma . \{X\delta x + Y\delta y + Z\delta z\} = 0.$$

74. Let R be the force, of which XYZ are the components: a, b, c the angles which the direction of R makes with the axes;

$$\therefore \; X = R\cos a, \quad Y = R\cos b, \quad Z = R\cos c.$$

Also let δs be the small geometric displacement of the point of application of R, of which δx, δy, δz are the resolved parts: a', b', c' the angles δs makes with the axes: then

$$\delta x = \delta s . \cos a', \quad \delta y = \delta s . \cos b', \quad \delta z = \delta s . \cos c';$$

$$\therefore X\delta x + Y\delta y + Z\delta z = R\,\delta s\,(\cos a \cos a' + \cos b \cos b' + \cos c \cos c') = R\,\delta r$$

where δr is the resolved part of δs in the direction of R's action; that is, the virtual velocity of the point $(x\,y\,z)$ with respect to R.

$$\text{Hence } \Sigma . R\,\delta r = 0,$$

and the Principle of Virtual Velocities is true of a system of forces holding a rigid body in equilibrium.

75. II. Suppose the system consists of any number of rigid bodies.

Let P be the mutual action of any two of the rigid bodies, whether by contact or by any means of connexion whatever: let $\alpha\,\beta\,\gamma$ be the angles which its direction makes with the axes, and let $x\,y\,z$, $x'\,y'\,z'$ be the co-ordinates to the points where the force P acts.

Now each of these bodies is in equilibrium under the action of its own forces, together with the force P, and the mutual actions it has with the other bodies of the system. Hence, by the first case,

$$\Sigma\,.\,R\,\delta r + P\,(\delta x \cos\alpha + \delta y \cos\beta + \delta z \cos\gamma) + \ldots = 0 \ldots (1);$$

also for the other body on which P acts,

$$\Sigma\,.\,R'\,\delta r' - P\,(\delta x' \cos\alpha + \delta y' \cos\beta + \delta z' \cos\gamma) + \ldots = 0 \ldots (2);$$

and P will not occur in any of the equations that have reference to the other bodies.

Adding equations (1) and (2), will give

$$\Sigma\,.\,R\,\delta r + \Sigma\,.\,R'\,\delta r' + P\,\{(\delta x - \delta x')\cos\alpha$$
$$+ (\delta y - \delta y')\cos\beta + (\delta z - \delta z')\cos\gamma\} + \ldots\ldots = 0.$$

Now in consequence of the geometric displacement of the system, suppose the points (xyz) and $(x'y'z')$ describe the small spaces δt and $\delta t'$, making respectively with the axes the angles m, m', m'' and n, n', n'': hence

$$\delta x = \delta t \cos m, \quad \delta y = \delta t \cos m', \quad \delta z = \delta t \cos m'',$$
$$\delta x' = \delta t' \cos n, \quad \delta y' = \delta t' \cos n', \quad \delta z' = \delta t' \cos n''.$$

Hence $(\delta x - \delta x')\cos\alpha + (\delta y - \delta y')\cos\beta + (\delta z - \delta z')\cos\gamma$

$= \delta t\,(\cos m \cos\alpha + \cos m' \cos\beta + \cos m'' \cos\gamma)$

$\qquad - \delta t'\,(\cos n \cos\alpha + \cos n' \cos\beta + \cos n'' \cos\gamma)$

= resolved part of δt − resolved part of $\delta t'$ in direction of P

= sum of virtual veloc. of the pnts. (xyz), $(x'y'z')$ with resp. to P

$= \delta p + \delta p'$ suppose.

Wherefore if we form the equations analogous to equations (1) and (2) for all the bodies, and add them together, we shall have, supposing Σ now to extend through the whole system,

$$\Sigma \,.\, R\delta r + \Sigma \,.\, P(\delta p + \delta p') = 0.$$

and the Principle of Virtual Velocities is still true.

76. Cor. 1. If the force P be the mutual normal pressure of two surfaces in contact, then by giving the system such a geometric motion that these surfaces shall remain in contact, we shall cause P to disappear from this equation, because then

$$\delta p + \delta p' = 0.$$

77. Cor. 2. If the points (xyz) and $(x'y'z')$ are connected invariably, as for instance by an inextensible string, then P disappears from the equation of Virtual Velocities, since

$$\delta p + \delta p' = 0.$$

Prop. *When a system of rigid bodies is in equilibrium under the action of no forces but their weights, mutual forces, and pressures upon smooth immoveable surfaces, then the centre of gravity is in the lowest or highest position it can possibly attain by moving the system consistently with the connexion of its parts one with another.*

R

78. For let the axis of z be taken vertical: and let P_1, P_2,...be the vertical forces with which the different particles tend downwards by reason of the attraction of the Earth: $z_1 z_2$...the vertical ordinates to their points of application, $\bar{z}$ the vertical ordinate to the centre of gravity (see Art. 36.)

$$\therefore\ \bar{z} = \frac{P_1 z_1 + P_2 z_2 + P_3 z_3 + \ldots\ldots}{P_1 + P_2 + P_3 + \ldots\ldots}.$$

Now suppose the system to receive a slight displacement of its parts consistent with their connexion, and let $\delta z_1 \delta z_2 \delta z_3 \ldots$ be the vertical displacement of the points of application of $P_1 P_2 P_3 \ldots$ (these are the virtual velocities of the points); and let $\bar{z}$ become $\bar{z} + \delta \bar{z}$;

$$\therefore\ \bar{z} + \delta\bar{z} = \frac{P_1 z_1 + P_2 z_2 + P_3 z_3 + \ldots\ldots}{P_1 + P_2 + P_3 + \ldots\ldots}$$

$$+\frac{P_1\delta z_1 + P_2\delta z_2 + P_3\delta z_3 + \ldots\ldots}{P_1 + P_2 + P_3 + \ldots\ldots};$$

$$\therefore\ \delta\bar{z} = \frac{P_1\delta z_1 + P_2\delta z_2 + P_3\delta z_3 + \ldots\ldots}{P_1 + P_2 + P_3 + \ldots\ldots}.$$

But by the principle of Virtual Velocities, the numerator of this fraction vanishes when there is equilibrium;

$$\therefore\ \delta\bar{z} = 0,$$

and $\bar{z}$ is a maximum or minimum: and the centre of gravity is in its highest or lowest position.

R

PROP. *To prove that when the centre of gravity has its lowest position the equilibrium is* stable, *and when it has its highest position the equilibrium is* unstable.

79. When a system of bodies is in equilibrium and an indefinitely small motion is given to the parts of the system so as to disturb the state of rest, the equilibrium is said to be *stable* or *unstable* according as the parts of the system do or do not return to their original positions of rest.

Now suppose the pressures (mentioned in the last Prop.) and the weight of the parts of the system are not in equilibrium. We shall prove that the centre of gravity cannot ascend, but must descend.

The resultant of the weights of the different parts of the system passes through the centre of gravity of the system. Let W be the weight of the whole system: and suppose the centre of gravity would move in the direction GG_1 (fig. 17.) making an angle θ with the vertical drawn downwards from G, if not prevented by a force P acting in the opposite direction and combining with the pressures to preserve equilibrium: $GG_1 = a$: then by Virtual Velocities,

?

$$W \cdot a\cos\theta - P \cdot a = 0;$$

$$\therefore\ \cos\theta = \frac{P}{W};$$

therefore $\cos\theta$ cannot be negative, or θ cannot lie between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$; that is, G cannot move upwards but must move downwards when the system is not in equilibrium.

Now if the system be in equilibrium with its centre of gravity as high as possible, any slight disturbance must bring it lower; and since, by what we have just proved, it can never rise again, it follows that the equilibrium will be *unstable*. But if the equilibrium be such that the centre of gravity is in its lowest position, any disturbance must raise it higher; and since when left to itself it must fall, it follows that the centre of gravity will return to its former position, or the equilibrium is *stable*.

80. We have in the foregoing part of this work deduced the conditions of equilibrium of a material system from the simplest principles, commencing with the equilibrium of a single material particle: and we have from these conditions proved the Principle of Virtual Velocities. But we might have pursued an inverse course and commenced with proving the Principle of Virtual Velocities, and thence deducing the conditions of equilibrium of a material system.

Prop. *To prove the Principle of Virtual Velocities independently of the Parallelogram of Forces.*

81. The following is Lagrange's Proof of this Principle. Let us suppose that the forces are P_1, P_2, P_3 ... and that they are commensurable and in the proportion of the numbers $n_1, n_2, n_3, \ldots$ let $A_1, A_2, A_3, \ldots$ (fig. 18.) be their points of application: A_1a_1, A_2a_2, A_3a_3,... their directions.

Now imagine a_1 and b_1 to be two blocks consisting each of n_1 wheels of equal size, the wheels in the same block turning freely about the same axis: and let the centres of these blocks be in the straight line A_1a_1 produced. Let a_1 be connected with A_1 by an inextensible string: and suppose b_1 is firmly fixed to an immoveable beam B_1; and a_1, b_1 connected by an inextensible string passing round their wheels alternately, one end of the string being attached to a fixed point M any where in the plane of the first wheel of b_1 over which it passes; and the other end being carried (as represented in the figure) to another system of blocks corresponding to the force P_2, each block having n_2 wheels; and so on: and lastly, let the string be passed over a simple wheel at C and be stretched by a weight W hanging by it.

The string is imagined to be perfectly flexible, and the wheels perfectly smooth: consequently the string will be stretched uniformly throughout, with a tension equal to the weight W. It is very evident, then, that since the wheels of a_1 and b_1 are all equal, the portions of string connecting them are parallel, and (they being $2n_1$ in number) the tension of A_1a_1 equals the weight $2n_1W$; in the same manner the tension of A_2a_2 is $2n_2W$; and so on.

Consequently by this imaginary contrivance the weight W produces forces at the points A_1A_2 .... in the directions A_1a_1, A_2a_2 ...... and in the proportion of n_1n_2 ...... ; that is, in the proportion of P_1P_2 ......

But P_1P_2 ...... are in equilibrium: and since the unit of force may be any force, a system of forces *in the same proportion as* P_1P_2 ...... acting at the same points and in the same directions as P_1P_2 ...... will be in equilibrium.

Hence if we remove the forces P_1P_2 ...... and replace them in the manner described above, W will be at rest: and this will be the case of whatever magnitude W be, since by increasing or diminishing W, the forces P_1P_2 ...... are altered so as to retain their proportion unchanged.

Wherefore, however much we alter W, we cannot thereby cause the moveable block (a_1) of any of the systems (as a_1b_1) to move.

This shews that the relation of the magnitudes of the forces P_1P_2..., their directions, and points of application is such, that if we forcibly make the block a_1, or any other block, to approach or recede from the other block b_1 of the system by an indefinitely small space, then the other moveable blocks will so shift, that on the whole the length of string given off from the blocks which approach will exactly equal the length of string taken in by the blocks which separate. If this were not the case, this indefinitely small displacement of the system would give W an indefinitely small motion, and this would shew conversely, that it is possible to move W, which (as we have proved) cannot be done, however much we alter W in magnitude.

Hence, if $\delta p_1\,\delta p_2$ ...... be the spaces through which $a_1\,a_2$ ...... move in consequence of the indefinitely small displacement, those being reckoned positive when the blocks approach, or string is given off, and the others negative. Then $n_1\delta p_1$, $n_2\delta p_2$, will be the lengths of string given off or taken on the wheels, according as they are positive or negative;

$$\therefore\ n_1\delta p_1 + n_2\delta p_2 + \ldots\ldots\ldots\ldots = 0,$$

$$\text{or}\ P_1\delta p_1 + P_2\delta p_2 + \ldots\ldots\ldots\ldots = 0,$$

which is the Principle of Virtual Velocities.

82. The displacements δp_1, δp_2 ...... must be taken indefinitely small, otherwise the equilibrium will be sensibly disturbed, and W will not remain at rest. In fact the best way of representing the principle is this; that when any part of the system is moved through a space less than any assignable quantity, then W will move through a small space which varies as the square or some higher power of the disturbance, so that it vanishes in the limit.

Prop. *To obtain the equations of equilibrium of a rigid body from the Principle of Virtual Velocities.*

83. By this principle we have $\Sigma\,.\,P\delta p = 0$. Let XYZ be the resolved parts of P: and δx, δy, δz the virtual velocities of the point (xyz) with respect to P;

$$\therefore\ \Sigma\,.\,(X\delta x + Y\delta y + Z\delta z) = 0.$$

Now, by Art. 73, we must put

$$\delta x = a + y\theta - z\phi,\quad \delta y = b + z\psi - x\theta,\quad \delta z = c + x\phi - y\psi,$$

in which a, b, c, θ, ϕ, ψ are arbitrary small quantities: hence

$$a\Sigma\,.\,X + b\Sigma\,.\,Y + c\Sigma\,.\,Z$$
$$+\,\psi\Sigma\,.\,(Yz - Zy) + \phi\Sigma\,.\,(Zx - Xz) + \theta\Sigma\,.\,(Xy - Yx) = 0,$$

and because a, b, c, θ, ϕ, ψ are arbitrary,

$$\Sigma\,.\,X = 0,\quad \Sigma\,.\,Y = 0,\quad \Sigma\,.\,Z = 0,$$
$$\Sigma\,.\,(Yz - Zy) = 0,\quad \Sigma\,.\,(Zx - Xz) = 0,\quad \Sigma\,.\,(Xy - Yx) = 0,$$

which are the six equations of equilibrium deduced in Art. 65.

CHAPTER IV.

CENTRE OF GRAVITY.

84. It was shewn in Art. 37. that there is a point in every body such that, if the particles of the body be acted on by parallel forces and this point be fixed, the body will rest in whatever position it be placed.

85. Now the weight of a body may be considered as the resultant of the weights of the different elementary portions of the body acting in parallel and vertical lines. In this case the point above described, the centre of parallel forces, is called *the centre of gravity of the body*. We intend to devote the present Chapter to the determination of this point in bodies of various forms.

86. We shall first give a few geometrical calculations of the position of the centre of gravity.

Ex. 1. *To find the centre of gravity of a triangular figure of uniform thickness and density.*

Let ABC be one surface of the triangular figure: fig. 19. Bisect AC in D; join BD: draw adc parallel to ADC cutting BD in d. Then by similar triangles

$$ad : AD :: Bd : BD$$

$$\text{and } dc : DC :: Bd : BD$$

$$\therefore ad : AD :: dc : DC$$

$$\text{but } AD = DC; \quad \therefore ad = dc.$$

Hence BD bisects every line parallel to the side AC: and therefore each of these lines will balance on BD, and consequently the whole triangle will balance on BD: and therefore the centre of gravity must be in the line BD.

Bisect AB in E and join CE; let this cut BD in F. Then, as before, the centre of gravity must be in CE: but it must be in BD: and therefore F is the centre of gravity.

Join DE.

$$\text{Then } \because AD = DC \text{ and } AE = EB;$$

$$\therefore DE \text{ is parallel to } BC \text{ and } BC = 2 \,.\, DE,$$

and by similar triangles

$$\frac{DF}{DE} = \frac{BF}{BC}; \quad \therefore DF = \tfrac{1}{2} BF;$$

$$\therefore DF = \tfrac{1}{3} DB.$$

Hence to find the centre of gravity of a triangle, bisect any side, join the point of bisection with the opposite angle, and the centre of gravity lies a third of the way up this line.

Ex. 2. *To find the centre of gravity of a pyramid on a triangular base.*

Let ABC be the base; V the vertex: fig 20, bisect AC in D; join BD, DV: take $DF = \frac{1}{3} \,.\, DB$, then F is the centre of gravity of ABC. Join FV: and draw abc parallel to ABC cutting VF in f: join bf; and produce it to meet DV in d.

Then by similar triangles, we easily see that $ad = dc$: also

$$\frac{bf}{BF} = \frac{Vf}{VF} = \frac{df}{DF}: \text{ but } DF = \tfrac{1}{3} BF;$$

$$\therefore df = \tfrac{1}{3} bf;$$

therefore f is the centre of gravity of the triangle abc: and if we suppose the pyramid to be made up of an infinitely great number of infinitely thin triangular figures parallel to the base, each of these has its centre of gravity in VF. Hence the centre of gravity of the pyramid is in VF.

Again, take $DH = \frac{1}{3} DV$: join HB cutting VF in G. Then as before, the centre of gravity of the pyramid must be in BH: but it is in VF: hence G, the point of intersection of these lines, is the centre of gravity.

Join FH: then FH is parallel to VB:

$$\text{also } \because\ DF = \tfrac{1}{3} DB; \quad \therefore\ FH = \tfrac{1}{3} VB:$$

$$\text{and } \frac{FG}{FH} = \frac{VG}{VB} \text{ but } FH = \tfrac{1}{3} VB;$$

$$\therefore\ FG = \tfrac{1}{3} GV = \tfrac{1}{4} FV.$$

Hence the centre of gravity is found to be one-fourth of the way up the line joining the centre of gravity of the base with the vertex.

Ex. 3. *To find the centre of gravity of any pyramid having a plane base.*

Divide the base into triangles: if any part of the base is curvilinear, then suppose the curve to be divided into an indefinitely great number of indefinitely short straight lines. Join the vertex of the pyramid with the centres of gravity of all the triangles, and also with all their angles. Draw a plane parallel to the base at a distance from the base equal to one-fourth of the distance of the vertex from the base: then this plane cuts every line drawn from the vertex to the base in parts, having the same ratio 3 : 1; and therefore the triangular pyramids have their centres of gravity in this plane, and therefore the whole pyramid has its centre of gravity in this plane.

Again, join the vertex with the centre of gravity of the base: then every section of the pyramid parallel to the base will be similar to the base, and will have its centre of gravity in this line. Hence the whole pyramid has its centre of gravity in this line.

Wherefore the centre of gravity is one-fourth of the way up the line joining the centre of gravity of the base with the vertex.

Ex. 4. *To find the centre of gravity of the frustrum of a pyramid, formed by parallel planes.*

Let $ABC\,abc$ be the frustrum, fig. 20: G, g the centres of gravity of the pyramids $VABC$, $Vabc$: it is clear that the centre of gravity of the frustrum must be in gG produced; at G' suppose.

Let $G'F = x$; $Ff = c$; $AB = a$, $ab = b$.

Now the smaller pyramid and the frustrum supposed to act at their centres of gravity are in equilibrium about G: hence by Art. 32.

$$\frac{GG'}{Gg} = \frac{\text{weight of smaller pyd.}}{\text{weight of frustrum}}$$

$$= \frac{\text{vol. of small pyd.}}{\text{vol. of large}-\text{vol. of small pyd.}} = \frac{b^3}{a^3 - b^3},$$

$$Gg = VG - Vg = \tfrac{3}{4}(VF - Vf) = \frac{3c}{4};$$

$$\therefore\ GG' = \frac{3c}{4}\,\frac{b^3}{a^3 - b^3}.$$

Also $GF = \frac{1}{4}VF = \frac{1}{4}(VF - Vf)\dfrac{a}{a-b}$ by similar figures,

$$= \frac{c}{4}\,\frac{a}{a-b};$$

$$\therefore\ FG' = FG - G'G = \frac{c}{4}\left\{\frac{a}{a-b} - \frac{3b^3}{a^3 - b^3}\right\}$$

$$= \frac{c}{4}\,\frac{a^2 + 2ab + 3b^2}{a^2 + ab + b^2}.$$

This is true of a frustrum of a pyramid on any base, a and b being homologous sides in the two ends.

We proceed now to the analytical calculations.

Prop. *To obtain formulæ for the calculation of the co-ordinates of the centre of gravity of a body.*

R

87. Let xyz be the rectangular co-ordinates to an elementary parallelopiped of the body, the mass of the element being dm: then if g be the constant ratio of the mass of a body to its weight, gdm is the weight of this element: or the force with

which it presses downwards in a vertical line: $g\,dm\,.\,x$ is the moment of this force with respect to the plane of yz (see Art. 39.), and $\int.\,gx\,dm$ is the sum of the moments of the forces which the parts of the body exert downwards in vertical lines: also $\int g\,dm$ is the sum of the forces. Hence if $\bar{x}$ be that coordinate of the centre of gravity of the body which is parallel to the axis of x, (Art. 36.)

$$\bar{x} = \frac{\int.\,gx\,dm}{\int.\,g\,dm} = \frac{\int.\,x\,dm}{\int.\,dm}.$$

Similarly, $\bar{y} = \dfrac{\int.\,y\,dm}{\int.\,dm}$, $\bar{z} = \dfrac{\int.\,z\,dm}{\int.\,dm}$,

the limits of integration being determined by the form of the body.

When the body is not bounded by continuous surfaces, these formulæ cannot be used, except in some particular cases, as we shall see when we come to apply them to examples. When these formulæ will not apply we must divide the body into distinct portions, of which the respective centres of gravity can be calculated by the above formulæ; and must finally find the centre of gravity of the whole body by considering these constituent portions as condensed, each into its centre of gravity, and so forming an assemblage of particles to which the formulæ of Art. 36. can be applied.

R

Ex. 1. *A straight rod of uniform thickness and density:* (fig. 21.)

AB the rod: P, Q two transverse sections, $AP = x$, $PQ = dx$, M the mass of the whole rod and l its length: then the mass of $PQ = M\dfrac{dx}{l}$;

$$\therefore\ \bar{x} = \frac{\int_0^l .\,x\,dx}{\int_0^l .\,dx}, \text{ since } \frac{M}{l} \text{ divides out}$$

$$= \frac{\frac{1}{2}l^2}{l} = \tfrac{1}{2}l = AG.$$

Ex. 2. *A curved line of uniform density and thickness, the curvature lying in one plane:* (fig. 22.)

R

$$\text{Let } AP = s, \; PQ = ds;$$

the mass of $PQ = M\dfrac{ds}{l}$ and $ds = \sqrt{1 + \dfrac{dy^2}{dx^2}}\,dx$,

x and y being co-ordinates to P;

$$\therefore\; \bar{x} = \frac{\int x\,ds}{\int ds}, \quad \bar{y} = \frac{\int y\,ds}{\int ds}$$

between the proper limits. The two following examples are applications of these formulæ.

Ex. 3. *A quadrant of a circle:* (fig. 23.)

R

Here $y^2 = a^2 - x^2$, the centre B being origin: BA axis of x.

$$\frac{dy}{dx} = \frac{-x}{\sqrt{a^2 - x^2}}; \qquad \frac{ds}{dx} = \frac{a}{\sqrt{a^2 - x^2}}$$

the limiting values of x are 0 and $AB = a$;

$$\therefore\; \bar{x} = \frac{\int_0^a \dfrac{ax\,dx}{\sqrt{a^2 - x^2}}}{\int_0^a \dfrac{a\,dx}{\sqrt{a^2 - x^2}}} = \frac{a^2}{\frac{1}{2}\pi a} = \frac{2a}{\pi} = BH,$$

?

$$\bar{y} = \frac{\int_0^a \dfrac{ay\,dx}{\sqrt{a^2 - x^2}}}{\int_0^a \dfrac{a\,dx}{\sqrt{a^2 - x^2}}} = \frac{\int_0^a dx}{\int_0^a \dfrac{dx}{\sqrt{a^2 - x^2}}} = \frac{2a}{\pi} = HG.$$

?

Ex. 4. *The arc of a semi-cycloid:* (fig. 23.)

R

The axis AB being the axis of x and the vertex A the origin,

$$y = a \text{ vers.}^{-1}\frac{x}{a} + \sqrt{2ax - x^2}.$$

$$\frac{dy}{dx} = \frac{2a - x}{\sqrt{2ax - x^2}} = \sqrt{\frac{2a - x}{x}}, \quad \frac{ds}{dx} = \sqrt{\frac{2a}{x}},$$

the limits of x are 0 and AB or $2a$,

$$x = \frac{\int_0^{2a} x \sqrt{\frac{2a}{x}}\, dx}{\int_0^{2a} \sqrt{\frac{2a}{x}}\, dx} = \frac{2a}{3} = AH,$$

$$\bar{y} = \frac{\int_0^{2a} y \sqrt{\frac{2a}{x}}\, dx}{\int_0^{2a} \sqrt{\frac{2a}{x}}\, dx} = \frac{2x^{\frac{1}{2}} y - 2\int x^{\frac{1}{2}} \frac{dy}{dx}\, dx}{\int \frac{dx}{\sqrt{x}}}$$

between the proper limits

$$= \frac{(2a)^{\frac{3}{2}}\pi - 2\int_0^{2a} \sqrt{2a - x}\, dx}{2\sqrt{2a}} = \pi a - \frac{4a}{3} = HG.$$

Ex. 5. *A curve line of double curvature.*

In this case $ds = \sqrt{1 + \frac{dy^2}{dx^2} + \frac{dz^2}{dx^2}}\, dx$, $x\, y\, z$ being co-ordinates to the variable extremity of s: this value of ds put in $\bar{x}\, \bar{y}\, \bar{z}$ will give the required co-ordinates.

Ex. 6. *Any portion of a helix, or the curve of the thread of a screw:* (fig. 24.)

The equations are $y = \sqrt{a^2 - x^2}$, $z = na \cos^{-1} \frac{x}{a}$;

$$\therefore \frac{dy}{dx} = \frac{-x}{\sqrt{a^2 - x^2}}, \quad \frac{dz}{dx} = \frac{-na}{\sqrt{a^2 - x^2}};$$

$$\therefore 1 + \frac{dy^2}{dx^2} + \frac{dz^2}{dx^2} = \frac{a^2(4 + n^2)}{a^2 - x^2},$$

and the limits of x are a and x;

$$\bar{x} = \frac{\int_a^x \frac{x\,dx}{\sqrt{a^2 - x^2}}}{\int_a^x \frac{dx}{\sqrt{a^2 - x^2}}} = \frac{\sqrt{a^2 - x^2}}{\cos^{-1}\frac{x}{a}},$$

$$\bar{y} = \frac{\int_a^x dx}{\int_a^x \frac{dx}{\sqrt{a^2 - x^2}}} = \frac{a - x}{\cos^{-1}\frac{x}{a}},$$

$$\bar{z} = \frac{\int_a^x na\cos^{-1}\frac{x}{a}\frac{dx}{\sqrt{a^2 - x^2}}}{\int_a^x \frac{dx}{\sqrt{a^2 - x^2}}} = \frac{na}{2}\cos^{-1}\frac{x}{a}.$$

Ex. 7. *A body of uniform thickness and density bounded by a plane curve and its ordinate.* R

Let the plane parallel to the plane faces of the body and bisecting it be the plane of xy: the centre of gravity is evidently in this plane: M the mass of the body, and A its area: then the mass of an elementary portion of the area at the point P, of which the co-ordinates are xy, is $M\frac{dxdy}{A}$; and since $\frac{M}{A}$ divides both numerator and denominator, the co-ordinates of the centre of gravity become

$$\bar{x} = \frac{\iint x\,dx\,dy}{\iint dx\,dy}, \quad \bar{y} = \frac{\iint y\,dx\,dy}{\iint dx\,dy}$$

between proper limits.

We shall sometimes find it convenient to use polar coordinates: (fig. 25.) R

Let $AP = r$, $xAP = \theta$: dr, $rd\theta$ the sides of the elementary portion of the area at P; then $M\frac{r\,dr\,d\theta}{A}$ is the mass of the element at P: and $x = r\cos\theta$, $y = r\sin\theta$; ?

$$\therefore \bar{x} = \frac{\iint r^2\cos\theta\,dr\,d\theta}{\iint r\,dr\,d\theta}, \quad \bar{y} = \frac{\iint r^2\sin\theta\,dr\,d\theta}{\iint r\,dr\,d\theta},$$

between the proper limits: the following examples are applications of these; we shall sometimes use rectangular and sometimes polar co-ordinates.

R

Ex. 8. *Let the curve be the semi-parabola* AC, (fig. 23.)

$AM = x$, $MP = y$: $QM^2 = 4mx$, the equation to AC. Now x and y are independent variables, we may consequently integrate our expressions, first considering y variable and x constant, and then with regard to x. This admits of an easy explanation. Integrating our expressions on the supposition that x is constant and y variable is the same as calculating the expressions only for the elementary masses which lie in a *strip* of the area, like QM in the figure, in which PM or y is different for each element, but AM or x is the same: the limiting values of y in this integration will be $y = 0$ and $y = MQ = 2\sqrt{mx}$, and the result will therefore be a function of x only: then integrating this result with respect to x is the same as adding together the expressions for all the strips like QM, of which the area consists: the limits of x are 0 and AB or a;

$$\therefore \iint x\,dx\,dy \text{ between proper limits} = \int_0^a x\,(y + X)\,dx$$

$$(X \text{ a function of } x) = \int_0^a 2\sqrt{m}\,x^{\frac{3}{2}}\,dx = \frac{4}{5}\sqrt{m}\,a^{\frac{5}{2}},$$

$$\iint dx\,dy \text{ between limits} = \int_0^a (y + X')\,dx = \frac{4}{3}\sqrt{m}\,a^{\frac{3}{2}};$$

$$\therefore \bar{x} = \frac{3a}{5} = AH,$$

$$\text{in like manner } \bar{y} = \frac{3\sqrt{ma}}{4} \text{ or } \frac{3\,BC}{8} = GH.$$

If we had taken the double area CAC', the limits of y would have been $y = -MQ' = -2\sqrt{mx}$, and $y = MQ = 2\sqrt{mx}$, and we should have found $\bar{x} = \frac{3a}{5} = AH$, $\bar{y} = 0$, and therefore H is the centre of gravity of the whole.

M. S.

Ex. 9. *Let* CAC′ *be a circular area:* (fig. 23.)

$y^2 = 2ax - x^2$: and if we take a portion ACB, the limits of y are 0, and $\sqrt{2ax - x^2}$, and those of x are 0 and AB or a

$$\bar{x} = \frac{\int_0^a\int_0^y x\,dx\,dy}{\int_0^x\int_0^y dx\,dy} = \frac{\int_0^a x\sqrt{2ax - x^2}\,dx}{\int_0^x \sqrt{2ax - x^2}\,dx}$$

$$= \frac{\int_0^a \dfrac{2ax^2 - x^3}{\sqrt{2ax^2 - x^3}}\,dx}{\int_0^a \dfrac{2ax - x^2}{\sqrt{2ax - x^2}}\,dx} = \frac{4a}{3\pi} = AH^*.$$

$$\text{Also } \bar{y} = \frac{\int_0^a\int_0^y y\,dx\,dy}{\int_0^a\int_0^y dx\,dy} = \frac{\frac{1}{2}\int_0^a y^2\,dx}{\int_0^a y\,dx} = \frac{\frac{1}{2}\int_0^a (2ax - x^2)\,dx}{\int_0^a \sqrt{2ax - x^2}\,dx}$$

$$= \frac{\frac{1}{2}(a^3 - \frac{1}{3}a^3)}{\frac{\pi}{4}a^2} = \frac{4a}{3\pi} = GH.$$

Ex. 10. *Let* CAC′ *be an ellipse.*

R

Then if we take the quadrant ACB, $AB = a$, $BC = b$,

$$\bar{x} = \frac{4a}{3\pi}, \quad \bar{y} = \frac{4b}{3\pi}.$$

Ex. 11. *Let* CAC′ *be a cycloid:* $AB = 2a$,

a

$$\bar{x} = AH = \frac{7a}{6}, \quad \bar{y} = HG = a\left(\frac{\pi}{2} - \frac{8}{9\pi}\right)$$

Ex. 12. *A triangle:* (fig. 26.)

Draw AD perpendicular to BC; A the origin, AD the axis of x: $DAB = \alpha$, $DAC = \beta\,(= A - \alpha)$, $AD = e$; $x \tan\alpha = y'$, $x \tan\beta = y''$, the limits of y;

---

[*] The general form is

$$\int_0^x \frac{x^n\,dx}{\sqrt{2ax - x^2}} = \frac{2n-1}{n}\,a\int_0^x \frac{x^{n-1}\,dx}{\sqrt{2ax - x^2}} - \frac{x^{n-1}\sqrt{2ax - x^2}}{n}.$$

$$\therefore\ \bar{x} = \frac{\int_0^e x^2 (\tan\alpha + \tan\beta)\, dx}{\int_0^e x (\tan\alpha + \tan\beta)\, dx} = \frac{2e}{3} = AH.$$

$$\bar{y} = \frac{\frac{1}{2}\int_0^e x^2 (\tan^2\alpha - \tan^2\beta)\, dx}{\int_0^e x (\tan\alpha + \tan\beta)\, dx} = \frac{e}{3} (\tan\alpha - \tan\beta) = HG.$$

$$\therefore\ AH = \tfrac{2}{3} AD, \qquad GH = \tfrac{1}{3}(BD - CD),$$

$$\text{and } \therefore\ \tfrac{2}{3} DE = GH;$$

$$\therefore\ BD - CD = 2DE;$$

$$\therefore\ BD - DE = CD + DE;$$

$$\text{and } BE = CE,$$

and $AG = \frac{2}{3} AE$, as in Ex. 1. Art. 86.

As an instance of the application of polar co-ordinates, we will take the following.

R

Ex. 13. *A semi-ellipse* CBC′: (fig. 25.)

Let H be its centre of gravity: $AP = r$, $BAP = \theta$.

In this case we must integrate first with respect to r and then with respect to θ; but not first with respect to θ and then with respect to r. For if we first integrate with respect to r, we take the sum of the elements in AQ, and the whole area can be divided into strips like AQ: but if we begin by integrating with respect to θ, we take the elements in an annular strip through P: and the area cannot be divided into strips described after the same law, hence we should be unable to integrate again with respect to r.

The limits of r are 0 and AQ or $\dfrac{b}{\sqrt{1 - e^2 \cos^2\theta}}$, the limits of θ are $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$;

$$\therefore\ \bar{x} = \frac{\iint r^2 \cos\theta\, d\theta\, dr}{\iint r\, d\theta\, dr} \text{ between these limits}$$

$$= \frac{2b}{3} \, \frac{\displaystyle\int_{-\alpha}^{\alpha} \frac{\cos\theta\, d\theta}{(1 - e^2\cos^2\theta)^{\frac{3}{2}}}}{\displaystyle\int_{-\alpha}^{\alpha} \frac{d\theta}{1 - e^2\cos^2\theta}}, \quad \alpha = \frac{\pi}{2}$$

$$= \frac{2b}{3}\,\frac{\int_{-\alpha}^{\alpha}\frac{d.\tan\theta}{(1-e^2+\tan^2\theta)^{\frac{3}{2}}}}{\int_{-\alpha}^{\alpha}\frac{d.\tan\theta}{1-e^2+\tan^2\theta}},\quad \alpha=\frac{\pi}{2}$$

$$= \frac{2b}{3}\,\frac{\frac{1}{1-e^2}\,\frac{\tan\alpha}{\sqrt{1-e^2+\tan^2\alpha}}}{\frac{1}{\sqrt{1-e^2}}\tan^{-1}\frac{\tan\alpha}{\sqrt{1-e^2}}} = \frac{2b}{3}\,\frac{1}{\frac{\pi}{2}\sqrt{1-e^2}} = \frac{4a}{3\pi}.$$

Ex. 14. *The sector of a circle:* (fig. 25.) R

Let $BP'A$ be the sector: $\angle\, BAP' = \alpha$.

It matters not in this example whether we integrate with respect to r first or θ first; since the area may be made up of either strips like AQ, or of annular strips like that passing through P.

$$\bar{x} = \frac{\int_0^\alpha\int_0^a r^2\cos\theta\, d\theta\, dr}{\int_0^\alpha\int_0^a r\, d\theta\, dr} = \frac{2a\int_0^\alpha \cos\theta\, d\theta}{3\int_0^\alpha d\theta} = \frac{2a\,(\sin\alpha)}{3\alpha},$$?

$$\bar{y} = \frac{\int_0^\alpha\int_0^a r^2\sin\theta\, d\theta\, dr}{\int_0^\alpha\int_0^a r\, d\theta\, dr} = \frac{2a\int_0^\alpha \sin\theta\, d\theta}{3\int_0^\alpha d\theta} = \frac{2a\,(1-\cos\alpha)}{3\alpha};$$

$$\therefore\ \angle\, GAB = \tan^{-1}\left(\frac{\bar{y}}{\bar{x}}\right) = \tan^{-1}\frac{1-\cos\alpha}{\sin\alpha} = \frac{\alpha}{2},$$

$$AG = \sqrt{\bar{x}^2+\bar{y}^2} = \frac{2a}{3\alpha}\sqrt{2-2\cos\alpha} = \frac{4a\sin\frac{\alpha}{2}}{3\alpha}.$$

Ex. 15. *A surface of revolution:* (fig. 27.) R

Let $AM = x$, $MP = y$; $MM' = dx$: through M and M' draw two planes at right angles to the axis of the figure; that is, the axis of x. Now every portion of the surface between these planes is equally distant from the axis, and therefore the centre of gravity of the surface $PQQ'P'$ is at M ultimately: let M be the mass of the whole surface (the thickness and density being uniform) and S the whole surface;

$$\therefore \text{ mass of the surface } PQQ'P' = M\,\frac{2\pi y\,ds}{S};$$

$$\therefore\ \bar{x} = \frac{\int_0^a xy\,\frac{ds}{dx}\,dx}{\int_0^a y\,\frac{ds}{dx}\,dx} = AG,\quad AB = a.$$

It is evident that $\bar{y} = 0$.

R

Ex. 16. *The surface of a portion of a sphere.*

$$y^2 = 2ax - x^2,\quad \frac{ds}{dx} = \frac{a}{\sqrt{2ax - x^2}}$$

$$\bar{x} = \frac{\int_0^x ax\,dx}{\int_0^x a\,dx} = \frac{ax^2}{2ax} = \frac{x}{2} = AG.$$

R ?

Ex. 17. *The surface of a cone.*

$$y = ax,\quad \frac{ds}{dx} = \sqrt{1 + a^2};$$

$$\therefore\ \bar{x} = \frac{\int_0^x x^2\,dx}{\int_0^x x\,dx} = \frac{2x}{3} = AG.$$

Ex. 18. *Let the body be any surface of uniform thickness and density:* (fig. 28.)

Let xyz be co-ordinates to any point of the surface: the area of a small portion of the surface at that point is

$$\sqrt{1 + \frac{dz^2}{dx^2} + \frac{dz^2}{dy^2}}\,dx\,dy;$$

therefore mass of the corresponding element

$$= \frac{M}{S}\sqrt{1 + \frac{dz^2}{dx^2} + \frac{dz^2}{dy^2}}\,dx\,dy;$$

$$\therefore\ \bar{x} = \frac{\iint x\sqrt{1 + \dfrac{dz^2}{dx^2} + \dfrac{dz^2}{dy^2}}\,dx\,dy}{\iint \sqrt{1 + \dfrac{dz^2}{dx^2} + \dfrac{dz^2}{dy^2}}\,dx\,dy}$$

between proper limits, and similar expressions for $\bar{y}$, $\bar{z}$.

Ex. 19. *The surface of an eighth part of a sphere.*

The origin being at the centre, $x^2 + y^2 + z^2 = a^2$ (fig. 28.)

$$\sqrt{1 + \frac{dz^2}{dx^2} + \frac{dz^2}{dy^2}} = \frac{a}{z} = \frac{a}{\sqrt{a^2 - x^2 - y^2}}.$$

We shall consider y to vary, x remaining constant: that is, we shall take all the elements in the strip QQ': hence the limits of y are 0 and $Q'M$ or $\sqrt{a^2 - x^2}$, which is obtained from the equation to the surface by putting $z = 0$: then the limits of x are 0 and AB or a;

$$\therefore\ \bar{x} = \frac{\int_0^a \int_0^{y'} \cdot \dfrac{x\,dx\,dy}{\sqrt{a^2 - x^2 - y^2}}}{\int_0^a \int_0^{y'} \dfrac{dx\,dy}{\sqrt{a^2 - x^2 - y^2}}}, \quad y' = \sqrt{a^2 - x^2}.$$

$$= \frac{\int_0^a \cdot \dfrac{\pi}{2}\,x\,dx}{\int_0^a \cdot \dfrac{\pi}{2}\,dx} = \frac{a}{2},$$

in the same manner $\bar{y} = \dfrac{a}{2}$, $\bar{z} = \dfrac{a}{2}$.

Ex. 20. *Let the body be a solid formed by the revolution of a plane curve about the axis of* x: (fig. 27.)

R

The centre of gravity of the slice PQ' is evidently at M when the thickness of the slice is diminished indefinitely: and the mass of this slice $= \dfrac{M}{V}\pi y^2 dx$, $V =$ whole volume;

$$\therefore\ \bar{x} = \frac{\int_0^x xy^2\,dx}{\int_0^x y^2\,dx}.$$

R

Ex. 21. *Let the body be a hemisphere.*

$$y^2 = 2ax - x^2$$

$$\bar{x} = \frac{\int_0^a (2ax^2 - x^3)\,dx}{\int_0^a (2ax - x^2)\,dx} = \frac{5a}{8}.$$

R

Ex. 22. *Let the body be a paraboloid.*

$$y^2 = 4mx$$

$$\bar{x} = \frac{\int_0^x x^2\,dx}{\int_0^x x\,dx} = \frac{2x}{3}.$$

R

Ex. 23. *The frustrum of a paraboloid.*

Let a and b be the radii of the larger and smaller ends: α and β the values of x measured from the vertex to the ends:

$$\text{then } \alpha = \frac{a^2}{4m},\quad \beta = \frac{b^2}{4m},$$

$$\bar{x} = \frac{\int_\beta^\alpha x^2\,dx}{\int_\beta^\alpha x\,dx} = \frac{2}{3}\,\frac{\alpha^3 - \beta^3}{\alpha^2 - \beta^2} = \frac{2}{3}\,\frac{\alpha^2 + \alpha\beta + \beta^2}{\alpha + \beta};$$

therefore the distance from smaller end

$$= \bar{x} - \beta = \frac{2\alpha^2 - \alpha\beta - \beta^2}{3(\alpha + \beta)}$$

$$= (\alpha - \beta)\,\frac{2\alpha + \beta}{3(\alpha + \beta)} = \frac{c}{3}\,\frac{2a^2 + b^2}{a^2 + b^2}$$

c = length of the frustrum.

R

Ex. 24. *Frustrum of a cone.*

$$\text{Distance from smaller end} = \frac{c}{4}\,\frac{b^2 + 2ab + 3a^2}{b^2 + ab + a^2},$$

as in Ex. 4. Art. 86.

Ex. 25. *Let the body be any solid.*

We shall first suppose the body referred to rectangular co-ordinates, as in fig. 29.

Let the body be divided into *slices*, like $Q'N''M$, by planes parallel to the plane yz: let these slices be divided into *prisms*, like QN, by planes parallel to the plane zx: and let these prisms be divided into *parallelopipeds*, like PP', by planes parallel to the plane xy. In this manner the body is divided into a number of elementary parallelopipeds: those at the extremities of the prisms will not be perfect; but when the distance of the cutting planes is diminished indefinitely, the sum of these imperfect portions vanishes.

Let $x\,y\,z$ be co-ordinates to P; dx, dy, dz the sides of the parallelopiped at P: then $dx\,dy\,dz$ is the volume of this figure, and V being the volume and M the mass of the whole body, supposed homogeneous, the mass of the element at P $= \dfrac{M}{V}\,dx\,dy\,dz$;

$$\therefore\ \bar{x} = \frac{\iiint x\,dx\,dy\,dz}{\iiint dx\,dy\,dz}, \quad \bar{y} = \frac{\iiint y\,dx\,dy\,dz}{\iiint dx\,dy\,dz},$$

$$\text{and } \bar{z} = \frac{\iiint z\,dx\,dy\,dz}{\iiint dx\,dy\,dz},$$

between the proper limits.

We shall now suppose the body is referred to polar co-ordinates: as in fig. 30.

Let the body be divided into *slices*, such as $CN'NA$, by planes passing through AC: let these slices be divided into *pyramids* having their vertices in A, like AQ, by the rotation of rays like AQ about AC, preserving a constant inclination to AC during the rotation: lastly, let each of these pyramids be divided into *six-sided figures*, like PP', by planes perpendicular to its length. In this manner the body is divided into a number of six-sided figures which become parallelopipeds ultimately when the distance of the cutting planes is diminished indefinitely.

Let $CAP = \theta$, $AP = r$, $BAN = \phi$;

therefore the sides of the figure at P are dr, $rd\theta$, $r\sin\theta d\phi$, and its volume ultimately equals the product of these

$$= r^2 \sin\theta\, dr d\theta d\phi.$$

Also $x = r\sin\theta\cos\phi$, $\quad y = r\sin\theta\sin\phi$, $\quad z = r\cos\theta$;

therefore, supposing the body homogeneous,

$$\bar{x} = \frac{\iiint r^3 \sin^2\theta \cos\phi\, dr d\theta d\phi}{\iiint r^2 \sin\theta\, dr d\theta d\phi},$$

$$\bar{y} = \frac{\iiint r^3 \sin^2\theta \sin\phi\, dr d\theta d\phi}{\iiint r^2 \sin\theta\, dr d\theta d\phi}.$$

$$\bar{z} = \frac{\iiint r^3 \sin\theta \cos\theta\, dr d\theta d\phi}{\iiint r^2 \sin\theta\, dr d\theta d\phi},$$

between the proper limits.

Ex. 26. *The eighth part of a sphere:* (fig. 29).

Now xyz, being co-ordinates to any point P in the body, are independent variables: we may therefore integrate with respect to z, considering x and y not to vary: that is the same as taking all the elements of the mass in a given prism QN, since although z or PN is different for each element, yet x and y remain the same: the limiting values of z are 0 and $QN = \sqrt{a^2 - x^2 - y^2}$ ($= z'$ suppose) obtained from the equation to the surface. This integration with respect to z between limits will leave a result a function of x and y without z. We shall then integrate with respect to y, considering x constant; this is the same as taking all the prisms in the same slice as $Q'N''$; since, although MN or y is different for each prism, yet AM or x is the same. The limits of y are 0 and MN' or $\sqrt{a^2 - x^2}$ ($= y'$ suppose) obtained from the equation to the line BN'. We shall finally integrate with respect to x from $x = 0$ to $x = AB$ or a, which is the same as taking all the slices, and therefore the whole body;

$$\therefore\ \bar{x} = \frac{\int_0^a \int_0^{y'} \int_0^{z'} x\, dx dy dz}{\int_0^a \int_0^{y'} \int_0^{z'} dx dy dz} = \frac{\int_0^a \int_0^{y'} x\sqrt{a^2 - x^2 - y^2}\, dx dy}{\int_0^a \int_0^{y'} \sqrt{a^2 - x^2 - y^2}\, dx dy}$$

$$= \frac{\int_0^a \cdot \frac{\pi}{4} x (a^2 - x^2)\, dx}{\int_0^a \frac{\pi}{4} (a^2 - x^2)\, dx} = \frac{\frac{1}{2} a^4 - \frac{1}{4} a^4}{a^3 - \frac{1}{3} a^3} = \frac{3a}{8};$$

in like manner we shall find $\bar{y} = \frac{3a}{8}$, $\bar{z} = \frac{3a}{8}$.

Ex. 27. *The same as last example, but referred to polar co-ordinates:* (fig. 30.)

We shall integrate first with respect to r, then θ, and lastly ϕ. The limits of r are 0 and AQ or a; those of θ are 0 and $\frac{\pi}{2}$; those of ϕ are 0 and $\frac{\pi}{2}$;

$$\therefore\ \bar{x} = \frac{\int_0^\alpha \int_0^\alpha \int_0^a r^3 \sin^2\theta \cos\phi\, d\phi\, d\theta\, dr}{\int_0^\alpha \int_0^\alpha \int_0^a r^2 \sin\theta\, d\phi\, d\theta\, dr}, \quad \alpha = \frac{\pi}{2}$$

$$= \frac{3a}{4} \frac{\int_0^\alpha \int_0^\alpha \sin^2\theta \cos\phi\, d\phi\, d\theta}{\int_0^\alpha \int_0^\alpha \sin\theta\, d\phi\, d\theta}$$

$$= \frac{3a}{8} \frac{\int_0^\alpha \cdot \frac{\pi}{2} \cos\phi\, d\phi}{\int_0^\alpha d\phi} = \frac{3a}{8}:$$

So also $\bar{y} = \frac{3a}{8}$, $\bar{z} = \frac{3a}{8}$.

Ex. 28. *A hemisphere in which the density varies as the* nth *power of the distance from the centre.*

We shall use polar co-ordinates.

The volume of an element at $P = r^2 \sin\theta\, d\phi\, d\theta\, dr$; and if ρ be the density at a distance a the density at a distance $r = \rho \left(\frac{r}{a}\right)^n$;

$$\therefore\ \text{mass of element at } P = \frac{\rho}{a^n} r^{n+2} \sin\theta\, d\phi\, d\theta\, dr.$$

The limits of r are 0 and a; of θ, 0 and π; of ϕ, $-\frac{\pi}{2}$ and $\frac{\pi}{2}$;

$$\therefore\ \bar{x} = \frac{\int_{-\alpha}^{\alpha}\int_0^{\pi}\int_0^{a} r^{n+3}\sin^2\theta\cos\phi\,d\phi\,d\theta\,dr}{\int_{-\alpha}^{\alpha}\int_0^{\pi}\int_0^{a} r^{n+2}\sin\theta\,d\phi\,d\theta\,dr},\quad \alpha = \frac{\pi}{2}$$

$$= \frac{n+3}{n+4}\cdot\frac{a}{2}.$$

GULDINUS's PROPERTIES.

Prop. *To prove that if any plane figure revolve about an axis lying in its own plane, the content of the solid generated by this figure in revolving through any angle is equal to a prism, of which the base is the revolving figure and height the length of the path described by the centre of gravity of the area of the plane figure.*

?

88. Let the axis of revolution be the axis of x, and the plane of the revolving figure in its initial position to be the plane of xy; we shall suppose the figure to be wholly on one side of the axis of x: θ the angle through which the figure revolves.

Then the elementary area $dx\,dy$ of the plane figure in revolving through an angle $d\theta$, generates the elementary solid whose volume is $y\,d\theta\,dx\,dy$; therefore whole solid

$$= \int_0^{\theta}\iint y\,d\theta\,dx\,dy,$$

the limits of x and y depend upon the equation to the curve

$$= \theta\iint y\,dx\,dy$$

between the proper limits.

But if $\bar{y}$ be the ordinate to the centre of gravity of the plane figure, then by Art. 87. Ex. 7.

$$\bar{y} = \frac{\iint y\,dx\,dy}{\iint dx\,dy}$$

the limits the same as before;

therefore whole solid $= \theta \iint y\,dx\,dy = \bar{y}\theta \cdot \iint dx\,dy$

$= $ arc descd. by centre of gravity × area of figure.

Hence the Prop. is true.

Prop. *To prove that the surface of the solid generated is equal in area to the rectangle of which the sides are the length of the perimeter of the generating figure and the length of the path of the centre of gravity of the perimeter.* R

89. The surface generated by the arc ds of the figure revolving through an angle $d\theta$ equals $y\,d\theta\,ds$;

$\therefore$ whole surface $= \int_0^\theta \int y\,d\theta\,ds = \theta \int y\,ds$ between proper limits.

But $\bar{y} =$ ordinate to centre of gravity of perimeter

$$= \frac{\int y\,ds}{\int ds} \text{ between same limits as before;}$$

therefore whole surface $= \bar{y}\theta \cdot \int ds$

$=$ arc descd. by centre of gravity × length of perimeter.

Hence the Prop. is true.

90. It is evident that these theorems are true also when the generating figure is bounded by a line not of continuous curvature.

Ex. 1. *To find the solid content and the surface of the ring of an anchor.* R

Let the radius of the axis be a, and the radius of a transverse section be b: then the length of the path of the centre of gravity of the area of the generating figure $= 2\pi a$, and the area of the figure $= \pi b^2$; ?

$$\therefore \text{ content of solid} = 2\pi^2 a b^2.$$

Also path of centre of gravity of the perimeter $= 2\pi a$, and the length of the perimeter $= 2\pi b$;

$$\therefore \text{ surface} = 4\pi^2 a b.$$

R

Ex. 2. *To find the centre of gravity of the area and also of the arc of a semi-circle.*

A semi-circle by revolving about its diameter generates a sphere: the content of the sphere $= \frac{4\pi}{3} a^3$, a the radius: the surface $= 4\pi a^2$:

the area of the semi-circle $= \frac{\pi}{2} a^2$; the perimeter $= \pi a$;

therefore distance of centre of gravity of area from diameter

$$= \frac{\text{content of sphere}}{2\pi \,.\, \text{area of } \frac{1}{2} \text{ circle}} = \frac{4a}{3\pi};$$

and distance of centre of gravity of arc from diameter

$$= \frac{\text{surface of sphere}}{2\pi \,.\, \text{arc of } \frac{1}{2} \text{ circle}} = \frac{2a}{\pi}.$$

CHAPTER V.

MACHINES. FRICTION.

91. A Machine is an instrument, or a system of solid bodies, for the purpose of transmitting force from one part to another of the system.

It would be endless to describe all the machines that have been invented; we shall consequently confine ourselves to those of simple construction. The most simple species of machines are denominated the Mechanical Powers. These we shall explain, and also a few combinations of them.

92. A *Lever* is an inflexible rod moveable only about a fixed axis; which is called the *fulcrum*. The portions of the lever into which the fulcrum divides it are called the *arms* of the lever: when the arms are in the same straight line, it is called a *straight lever;* in other cases a *bent lever*.

Two forces act upon the lever about the fulcrum, called the *power* and the *weight:* the power is the force applied by the hand (or other means) to sustain or overcome the other force, or the weight. There are three species of levers: the first has the fulcrum between the power and weight; in the second the weight acts between the fulcrum and the power; and in the third the power acts between the fulcrum and the weight.

Prop. *To find the conditions of equilibrium of two forces acting in the same plane on a lever.*

93. Let the plane of the paper be the plane in which the forces act, and also be perpendicular to the axis, of which C is the projection, and about which the lever can move (fig. 31.), A, B the points of application of the forces P, W; α, β the angles which the directions of the forces make with any line aCb drawn through C on the paper. Let R be the pressure

upon the fulcrum, and θ the angle which it makes with the line aCb; then if we apply a force R in the direction CR, we may suppose the fulcrum removed, and the body to be held in equilibrium by the forces P, W, R.

We shall resolve these forces in directions parallel and perpendicular to aCb; and also take their moments about C: then, by Art. 53, we have the following equations of condition:

$$P\cos\alpha - W\cos\beta - R\cos\theta = 0 \ \ldots\ldots\ldots\ (1),$$

$$P\sin\alpha + W\sin\beta - R\sin\theta = 0 \ \ldots\ldots\ldots\ (2),$$

$$\text{and } P\,.\,CD - W\,.\,CE = 0 \ \ldots\ldots\ldots\ (3).$$

CD and CE being drawn perpendicular to the directions of P and W.

These three equations determine the ratio of P to W when there is equilibrium; and the magnitude and direction of the pressure on the fulcrum.

For equation (3) gives

$$\frac{P}{W} = \frac{CE}{CD} = \frac{\text{perpendicular on direction of } W}{\text{perpendicular on direction of } P}.$$

Also by transposing the last terms of (1) and (2), we have

$$R\cos\theta = P\cos\alpha - W\cos\beta,$$

$$R\sin\theta = P\sin\alpha + W\sin\beta.$$

Add their squares;

$$\therefore\ R^2 = P^2 + W^2 - 2PW\cos(\alpha + \beta),$$

which gives the magnitude of R.

Take the ratio of the above equations;

$$\therefore\ \tan\theta = \frac{P\sin\alpha + W\sin\beta}{P\cos\alpha - W\cos\beta},$$

which gives the direction of the pressure.

If we suppose B to be the fulcrum and take the moments about B instead of C, we have instead of equation (3) the following*:

---

* This is not a new equation of condition; but is a consequence of the three already given, (1), (2), (3). To shew this imagine AD and BE produced to meet CR:

$$\frac{P}{R} = \frac{\text{perpendicular on direction of } R}{\text{perpendicular on direction of } P}.$$

It follows, then, that *the condition of equilibrium in a lever of any species is that the two forces must be inversely as the perpendiculars drawn upon their directions from the fulcrum.*

94. This property of the lever renders it a useful instrument in multiplying the power of a force. For any two forces, however unequal in magnitude, may be made to balance each other simply by fixing the fulcrum so that the ratio of its distances from the directions of the forces shall be equal to the ratio of the forces; an adjustment which can always be made. If the fulcrum be moved from this position, then that force will preponderate from which the fulcrum is moved and the equilibrium will be destroyed. We are thus led to understand how mechanical advantage is gained by using a crow-bar to move heavy bodies, as large blocks of stone: a poker to raise the coals in a grate: scissors, shears, nippers, and pincers; these last consisting of two levers of the first kind. The brake of a pump is a lever of the first kind. In the Stanhope printing-press we have a remarkable illustration of the mechanical advantage that can be gained by levers. The frame-work in which the paper to be printed is fixed, is acted upon by the shorter arm of a lever, the other arm being connected to a second lever, the longer arm of which is worked by the pressman. These levers are so adjusted that at the instant the paper comes in contact with the types, the perpendiculars from the fulcra upon the directions of the forces acting at the shorter arms are exceedingly short, and consequently the levers multiply the force exerted by the pressman to an enormous extent.

CR: they will meet this line in the same point, since the distances by these two constructions are CD cosec $(\theta - \alpha)$ and CE cosec $(\theta + \beta)$; and these are made equal, by equations (1), (2), (3), if we eliminate P, Q, W. Suppose, then, F to be the point in which these lines meet. By multiplying (1), (2), respectively by sin β and cos β, and adding, we have

$$\frac{P}{R} = \frac{\sin(\theta+\beta)}{\sin(\alpha+\beta)} = \frac{FB\sin(\theta+\beta)}{FB\sin(\alpha+\beta)} = \frac{\text{perpendicular on direction of } R}{\text{perpendicular on direction of } P};$$

therefore this equation is a consequence of the equations (1), (2), (3), as might have been anticipated.

As examples of levers of the second kind, we may mention a wheelbarrow, an oar, a chipping-knife, a pair of nutcrackers.

It must be observed, however, that as the lever moves about the fulcrum the space through which the weight is moved is, in the first and second species of lever, smaller than the space passed through by the power: and therefore what is gained in power is lost in despatch. For example in the case of the crow-bar: to raise a block of stone through a given space by applying the hand at the further extremity of the lever, we must move the hand through a greater space than that which the weight describes.

But in the third species of lever the reverse is the case. The power is nearer the fulcrum than the weight, and is consequently greater; but the motion of the weight is greater than that of the power. In this kind of lever despatch is gained at the expense of power. An excellent example is the treddle of a turning lathe. But the most striking example of levers of the third kind is found in the animal frame, in the construction of which it seems to be a prevailing principle to sacrifice power to readiness and quickness of action. The limbs of animals are generally levers of this description. The condile of the bone rests in its socket as the fulcrum; a strong muscle attached to the bone near the condile is the power, and the weight of the limb together with any resistance opposed to its motion is the weight. A slight contraction of the muscle gives a considerable motion to the limb. A drawing of the human arm is given as an illustration of these remarks: (fig. 32.)

95. The lever is applied to determine the weight of substances. Under this character it is called a Balance. The Common Balance has its two arms equal, with a scale suspended from each extremity; the fulcrum being *above* the line joining the extremities of the arms. The substance to be weighed is placed in one scale, and weights placed in the other till the beam remains in equilibrium in a perfectly horizontal position; in which case the weight of the substance is indicated by the weights by which it is balanced. If the weights differ ever so slightly, the horizontality of the beam will be disturbed, and after oscillating for some time (in consequence of the fulcrum being placed *above* the line joining the points of support of the

scales) it will, on attaining a state of rest, form an angle with the horizon, the extent of which is a measure of the sensibility of the balance.

In the construction of a balance the following requisites should be attended to. 1. When loaded with equal weights the beam should be perfectly horizontal. 2. When the weights differ, even by a slight quantity, the *sensibility* should be such as to detect this difference. 3. When the balance is disturbed it should readily return to its state of rest, or it should have *stability*. We shall now consider how these may be fulfilled.

PROP. *To find how the requisites of a good balance may be satisfied.*

96. Let P and Q be the weights in the scales (fig. 33.): $AB = 2a$: C the fulcrum: h its distance from the line joining A, B: W the weight of the beam and scales: k the distance of the centre of gravity of these from C measured downwards: θ the angle the beam makes with the horizon when there is equilibrium.

Let us take the moments of P, Q, W about C: their sum equals zero since there is equilibrium (Art. 53.) Then

since the distance of P's direc. from $C = a\cos\theta - h\sin\theta$

.... Q's .... $= a\cos\theta + h\sin\theta$

.... W's .... $= k\sin\theta$,

we have

$$P(a\cos\theta - h\sin\theta) - Q(a\cos\theta + h\sin\theta) - Wk\sin\theta = 0;$$

$$\therefore \tan\theta = \frac{(P-Q)a}{(P+Q)h + Wk}.$$

This determines the position of equilibrium. The first requisite—the horizontality when P and Q are equal—is satisfied by making the arms equal.

For the second we observe that for a given difference of P and Q the sensibility is greater the greater $\tan\theta$ is; and for a given value of $\tan\theta$, the sensibility is greater the smaller the difference of P and Q is: hence $\frac{\tan\theta}{P-Q}$ is a correct measure of the sensibility: and therefore the second requisite is fulfilled by making $(P+Q)\frac{h}{a}+W\frac{k}{a}$ as small as possible.

The stability is greater the greater the moment of the forces which tend to restore the equilibrium when it is destroyed. Suppose $P=Q$, then P and Q may be placed at the mid-point between A and B: and the moment of the forces tending to restore equilibrium equals $\{(P+Q)h+Wk\}\sin\theta$. Hence to satisfy the third requisite, this must be made as large as possible. This is, in part, at variance with the second requisite. They may, however, both be satisfied by making $(P+Q)h+Wk$ large, and a large also: that is, by increasing the distances of the fulcrum from the beam and from the centre of gravity of the beam and scales, and by lengthening the arms.

It must be remarked that the sensibility of a balance is of more importance than the stability, since the eye can judge pretty accurately whether the index of the beam makes equal oscillations on each side of the vertical line; that is, whether the position of rest would be horizontal: if this be not the case, then the weights must be altered till the oscillations are nearly equal.

97. Another kind of balance is that in which the arms are unequal, and the same weight is used to weigh different substances by varying its point of support, and observing its distance from the fulcrum by means of a graduated scale. The common steelyard is of this description.

Prop. *To shew how to graduate the common steelyard.*

98. Let AB be the beam of the steelyard (fig. 34.) A the fixed point from which the substance to be weighed is suspended, Q being its weight: C the fulcrum: W the weight of the beam together with the hook or scale-pan suspended from A; G the centre of gravity of these.

Suppose that P suspended at N balances Q suspended from A, then taking the moments of P, Q, W about C, we have

$$Q \cdot AC - W \cdot CG - P \cdot CN = 0;$$

$$\therefore\ Q = \frac{CN + \frac{W}{P} \cdot CG}{AC} \cdot P.$$

Take the point D, so that $CD = \frac{W}{P} CG$;

$$\therefore\ Q = \frac{CN + CD}{AC} \cdot P = \frac{DN}{AC} P.$$

Now let the arm DB be graduated by taking Da_1, Da_2, Da_3,......equal respectively to AC, $2AC$, $3AC$,......let the figures 1, 2, 3, 4,......be placed over the points of graduation, and let subdivisions be made between these. Then by observing the graduation at N we know the ratio of Q to P; and this latter being a given weight we know the weight of Q. In this way any substance may be weighed.

99. There is a remarkable balance called after its inventor Roberval's Balance: a representation of it is given in fig. 35. DC' is a frame of which the opposite sides are equal, and the extremities are connected by pins at D, C, D', C' so as to allow of free motion: this frame is supported by a stand $EE'A$, being connected to it by pins at E and E' so as to allow of free motion about those points: EE' must be parallel to DC and $D'C'$, but not necessarily equi-distant from them: arms are fixed at right angles to the sides DD', CC' to support weights Q and P. The peculiarity of this machine is, that if P and Q balance in any given position on the horizontal arms, the equilibrium will remain undisturbed if we shift P or Q or both of them along their arms in either direction: also if we push one arm down and consequently raise the other the whole will remain at rest in the position in which it is left. We shall prove these facts, and explained the paradoxical character of the machine in the Chapter of Problems. We may however easily prove by the Principle of Virtual Velocities the facts mentioned above, though the paradox will not be removed.

If we lower the arm on which P acts through a space x, then D' sinks through a space x, and D, and therefore the arm on which Q acts, rises through a space $\frac{a'x}{a}$, which is independent of the distances of P and Q along their arms: a and a' are the lengths DE and ED'.

$$\text{Then } P.x - Q.\frac{a'x}{a} = 0 \text{ by Virtual Velocities,}$$

$$\text{or } \frac{P}{Q} = \frac{a'}{a},$$

for all positions of the frame and of the weights.

It will be seen upon referring to the Chapter of Problems that although the equilibrium remains undisturbed when P and Q have different positions, yet the strains at the joints D, D', C, C', E, E' and the point of application (B in figure) of the downward-pressure undergo changes.

It is on the principle of this machine that the balances used of late years in shops are constructed: the scales rest each by one point upon the extremities of a lever below them, and the only motion they are capable of is in a vertical direction.

100. The second of the Mechanical Powers is the Wheel and Axle. This machine consists of two cylinders fixed together with their axes in the same line: the larger is called the wheel and the smaller the axle: the axis of the axle is generally much larger than that of the wheel. The cord by which the weight is suspended is fastened to the axle, and then coiled round it, while the power which supports the weight acts by a cord coiled round the circumference of the wheel; by spokes acted on by the hand, as in the *capstan*; or by the hand acting on a handle as in the *windlass*.

Prop. *To find the ratio of the power and weight in the Wheel and Axle when in equilibrium.*

101. Let AD be the wheel and $CC'B$ the axle (fig. 36.) P the power represented by a weight suspended from the cir-

cumference of the wheel at A: W the weight hanging from the axle at B.

Then since the axis of the machine is fixed, the condition of equilibrium is that the sum of the moments of the forces about this axis vanishes (Art. 68.);

$$\therefore\ P \cdot AC - W \cdot C'B = 0;$$

$$\therefore\ \frac{W}{P} = \frac{AC}{BC'} = \frac{\text{rad. of wheel}}{\text{rad. of axle}}.$$

It will be seen that this machine is only a modification of the lever. In short it is an assemblage of levers all having the same axis: and as soon as one has been in action the next comes into play; and in this way an endless leverage is obtained. In this respect, then, the wheel and axle surpasses the common lever in mechanical advantage. It is much used in docks, and in shipping.

102. The third Mechanical Power is the Toothed Wheel. It is extensively applied in all machinery; in cranes, steam-engines, and particularly in clock and watch work. If two circular hoops of metal or wood having their outer circumferences indented, or cut into equal teeth all the way round, be so placed that their edges touch, one tooth of one circumference lying between two of the other (as represented in the figure 87.); then if one of them be turned round by any means, the other will be turned round also. This is the simple construction of a pair of toothed wheels.

Prop. *To find the relation of the power and weight in Toothed Wheels.*

103. Let A and B be the fixed centres of the toothed wheels on the circumferences of which the teeth are arranged. C the point of contact of two teeth: QCQ a normal to the surfaces in contact at C. Suppose an axle is fixed on the wheel B, and the weight W suspended from it at E by a cord: also suppose the power P acts by an arm AD: draw Aa, Bb perpendicular to QCQ. Let the mutual pressure at C be Q. Then since the wheel A is in equilibrium about the fixed axis A, the sum of the moments about A equals zero:

$$\therefore\ P \,.\, AD - Q \,.\, Aa = 0.$$

Also since the wheel B is in equilibrium about B, the sum of the moments about B equals zero:

$$\therefore\ Q \,.\, Bb - W \,.\, BE = 0.$$

Then by eliminating Q from these two equations,

$$\frac{P}{W} = \frac{P}{Q} \cdot \frac{Q}{W}$$

$$= \frac{Aa}{AD} \cdot \frac{BE}{Bb};$$

$$\text{or}\quad \frac{\text{moment of } P}{\text{moment of } W} = \frac{Aa}{Bb};$$

when the teeth are small this ratio

$$= \frac{\text{rad. of wheel } A}{\text{rad. of wheel } B} \text{ very nearly.}$$

104. Wheels are in some cases turned by means of straps passing over their circumferences. In such cases the minute protuberances of the surfaces prevent the sliding of the straps, and a mutual action takes place such as to render the calculation exactly analogous to that in the Proposition.

For the calculation of the best forms for the teeth, the reader is referred to a Paper of Mr Airy's, in the Camb. Phil. Trans. Vol. II. p. 277.

105. The fourth Mechanical Power is the Pully. There are several species of pullies: we shall mention them in order. The simple pully is a small wheel moveable about its axis: a cord passes over part of its circumference. If the axis is fixed the effect of the pully is only to change the direction of the cord passing over it: if, however, the axis be moveable then, as will be presently seen, a mechanical advantage may be gained.

Prop. *To find the ratio of the power and weight in the single moveable Pully.*

106. I. Suppose the parts of the cord divided by the pully are parallel (fig. 38.)

Let the cord ABP have one extremity fixed at A, and after passing under the pully at B suppose it held by the hand exerting a force P. The weight W is suspended by a cord from the centre C of the pully.

Now the tension of the cord ABP is the same throughout. Hence the pully is acted on by three parallel forces, P, P, and W: hence

$$2P - W = 0;$$

$$\therefore \frac{W}{P} = 2.$$

II. Suppose the portions of cord are not parallel (fig. 39.)

Let a and a' be the angles which Aa and Pb make with the vertical.

Now the pully is held in equilibrium by W in CW, P in aA, P in bP. Hence by Art. 53,

$$\text{horizontal forces, } P\sin a - P\sin a' = 0\ldots(1)$$

$$\text{and vertical forces, } P\cos a + P\cos a' - W = 0\ldots(2)$$

the equation of moments is identical.

$$\text{By (1), } \sin a = \sin a' \text{ and } a = a';$$

$$\therefore \text{ by (2), } \frac{W}{P} = 2\cos a \text{ which is the relation required.}$$

Prop. *To find the ratio of the power and weight in a system of pullies, in which each pully hangs by a separate string, one end being fastened in the pully above it and the other end on a fixed beam: all the strings being parallel.*

107. Let n be the number of pullies (fig. 40.)

I. Let us neglect the weights of the pullies themselves.

Then the tension of $b_1W = W$; $\therefore$ the tension of $a_1b_1b_2 = \frac{1}{2}W$;

$$\therefore \text{ tension of } a_2 b_2 b_3 = \frac{1}{2^2} W, \text{ tension of } a_3 b_3 c = \frac{1}{2^3} W,$$

and so on, and the tension of the string passing under the n^{th} pully $= \frac{1}{2^n} W$, and this $= P$;

$$\therefore \frac{W}{P} = 2^n.$$

II. Let us suppose the weights of the pullies to be considered: and let $\omega_1 \omega_2 \omega_3 \dots \omega_n$ be these weights.

Then if $p_1 p_2 p_3 \dots p_n$ be the weights which they would sustain at P and P_1 the weight W would sustain at P, we have

$$p_1 = \frac{\omega_1}{2^n},\ p_2 = \frac{\omega_2}{2^{n-1}}, \dots\dots p_n = \frac{\omega_n}{2},\ P_1 = \frac{W}{2^n};$$

$$\therefore P = p_1 + p_2 + \dots\dots + p_n + P_1,$$

$$\text{or } P = \frac{1}{2^n} \left\{ W + \omega_1 + 2\omega_2 + 2^2\omega_3 + \dots\dots + 2^{n-1}\omega_n \right\}.$$

If $\omega_1 = \omega_2 = \omega_3 = \dots\dots = \omega_n$,

$$P = \frac{1}{2^n} \left\{ W + (2^n - 1)\,\omega_1 \right\}.$$

Prop. *To find the ratio of the power and weight when the system is the same as in the last Proposition; but the strings are not parallel.*

108. We shall neglect the weights of the blocks. The pullies will evidently so adjust themselves that the string at their centre will bisect the angle between the strings touching their circumference.

Let $\alpha_1 \alpha_2 \alpha_3 \dots\dots \alpha_n$ be the angles included between the strings touching the first, second, third, ... n^{th} pullies respectively: (fig. 41.)

Then, by Art. 106, tension of $a_1 b_1 b_2 = \frac{W}{2 \cos \alpha_1}$;

$$\text{therefore tension of } a_2 b_2 b_3 = \frac{W}{2^2 \cos \alpha_1 \cos \alpha_2},$$

$$\text{tension of } a_3 b_3 c = \frac{W}{2^3 \cos \alpha_1 \cos \alpha_2 \cos \alpha_3},$$

$$\text{tension of the last string} = \frac{W}{2^n \cos \alpha_1 \cos \alpha_2 \cos \alpha_3 \ldots\ldots \cos \alpha_n},$$

and this $= P$;

$$\therefore \frac{W}{P} = 2^n \cos \alpha_1 \cos \alpha_2 \cos \alpha_3 \ldots\ldots \cos \alpha_n.$$

Prop. *To find the relation of the power and weight in a system of pullies where the same string passes round all the pullies.*

109. This system consists of two blocks; each containing a number of the pullies with their axes coincident. The weight is suspended from the lower block, which is moveable, and the power acts at the loose extremity of the string, which passes round the respective pullies of the upper and lower block alternately.

Since the same string passes round all the pullies, its tension will be everywhere the same, and equal to the power P. Let n be the number of portions of string at the lower block; then $n \cdot P$ will be the sum of their tensions;

$$\therefore W = n \cdot P.$$

If we take into account the weight of the lower block, and call it B, then

$$W + B = n \cdot P.$$

If the strings at the lower block are not vertical, we must take the sum of the parts resolved vertically, and equate it to W. But, in general, this deviation from the vertical is so slight, that it is neglected.

110. As the weight is rising or falling, it will be observed that in general the pullies move with different angular motions. The degree of angular motion of each pully depends upon the

magnitude of its radius. Mr James White took advantage of this to choose the radii of the pullies in such a manner as to give them the same angular motion, and so prevented the wear and resistance caused by the friction of the pullies against each other. This being the case, the pullies might be fastened together. Instead of this, however, the pullies were cut in the same block.

It will be seen without much difficulty, that the angular motions of the pullies of the upper block will be as the series of odd integers 1, 3, 5, ... and those of the pullies of the lower block as the series of even integers 2, 4, 6, ...

These series, then, determine the relative sizes of the pullies in the two blocks.

Prop. *To find the ratio of the power to the weight when all the strings are attached to the weight.*

111. If we neglect the weight of the pullies (fig. 42.) the tension of the strings $b_1 a_1 = P$; the tension of $a_2 b_2 = 2P$; and so on: if there be n pullies, then the sum of the tensions of the strings attached to the weight

$$= P + 2P + 2^2 P + \dots\dots + 2^{n-1} P = (2^n - 1) P;$$

$$\therefore \frac{W}{P} = 2^n - 1.$$

If we suppose the weights of the pullies are $\omega_1 \omega_2 \omega_3 \dots\dots\dots$ reckoning from the lowest, and $\omega' \omega'' \omega''' \dots\dots$ the portions of W which they respectively support (since they evidently assist P), and W' the portion of W supported by P; then

$$W' = (2^n - 1) P,$$

$$\omega' = (2^{n-1} - 1) \omega_1,$$

$$\omega''' = (2^{n-2} - 1) \omega_2,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\omega^{(n-1)} = (2 - 1) \omega_{n-1};$$

$$\therefore W = W' + \omega' + \dots\dots = (2^n - 1) P + (2^{n-1} - 1) \omega_1$$
$$+ (2^{n-2} - 1) \omega_2 + \dots\dots + (2 - 1) \omega_{n-1}.$$

If $\omega_1 = \omega_2 = \omega_3 \dots\dots\dots$

$$W = (2^n - 1)\,P + \left\{2^{n-1} + 2^{n-2} + \dots\dots + 2 - (n-1)\right\}\omega_1$$

$$= (2^n - 1)\,P + (2^n - n - 1)\,\omega_1.$$

112. The fourth Mechanical Power is the Inclined Plane. By an inclined plane we mean a plane inclined to the horizon. A weight W may be supported on an inclined plane by a power P less than W.

PROP. *To find the ratio of the power and weight in the inclined plane.*

113. Let AB be the inclined plane (fig. 43.): α the angle it makes with the horizon. Let the power P act on the weight in the direction CP, making an angle ϵ with the plane. Now the weight at C is held at rest by P in CP, W in the vertical CW, and a pressure R in CR, at right angles to the plane.

Hence, by Art. 23, if we resolve these forces perpendicular and parallel to the plane, we have

$$R + P\sin\epsilon - W\cos\alpha = 0 \dots\dots\dots (1),$$

$$P\cos\epsilon - W\sin\alpha = 0 \dots\dots\dots (2).$$

The second gives the required relation $\frac{P}{W} = \frac{\sin\alpha}{\cos\epsilon}$; and the first equation gives the magnitude of the pressure R.

COR. 1. If P act horizontally, $\epsilon = -\alpha$, and $P = W\tan\alpha$.

COR. 2. If P act parallel to the plane, $\epsilon = 0$, $P = W\sin\alpha$.

COR. 3. If P act vertically, $\epsilon = \frac{\pi}{2} - \alpha$, $P = W$.

114. The fifth mechanical power is the Wedge. This is a triangular prism, and is used to separate obstacles by introducing its edge between them and then thrusting the wedge forward. This is effected by the blow of a hammer or other such means, which produces a violent pressure for a short time, sufficient to overcome the greatest forces.

PROP. *An isosceles wedge being introduced between two obstacles, required to find its tendency to separate the obstacles when the wedge is prevented from being thrust back by a given force.*

115. Let $2P$ be the force acting at the back of the wedge (fig. 44.) In the figure we suppose the obstacles to be the two halves of a tree. The portions of the tree we suppose similarly situated on the two sides of the wedge: let A and A' be the points of contact between the wedge and the obstacles: AN, $A'N'$ normals to the wedge at A and A': R, R the mutual resistances of the wedge and obstacles at A and A'.

Now if the wedge were to move backwards or to be thrust forwards the points A and A' would move in some unknown curve line: the nature of this curve would depend upon the elasticity and strength of the material of the obstacles and upon other circumstances. Draw AT, $A'T'$ tangents to these curves at the points A and A'. Then it will be seen that the parts of the pressures at A and A' which measure the tendency of the obstacles to separate will be their resolved parts along these tangent lines; since if they separate it must be by A and A' moving along these lines. The resolved parts perpendicular to these tangents are counteracted by the resistance of the ground at E.

Let W be the resolved part of R along the tangent on either side: and suppose the angle $NAT = i$. Also let the angle of the wedge be $2a$.

Then the wedge being sustained by the forces $2P$, R and R; we have by resolving them vertically

$$2P - 2R \sin a = 0 \text{ .... } (1);$$

the horizontal parts counteract each other of necessity, also the equation of moments is an identical equation.

$$\text{Again } W = R \cos i \text{ .... } (2);$$

$$\therefore \frac{P}{W} = \frac{\sin a}{\cos i}.$$

If, then, we know the angle i we shall know W: but we have no means of ascertaining the value of i, and consequently the preceding calculation is of little importance.

When i is very small, then $W = \dfrac{P}{\sin \alpha}$ nearly.

116. The last Mechanical Power is the Screw.

This machine in its simple construction consists of a cylinder (fig. 45.) AB with a uniform projecting thread $abcd$ ... traced round its surface, and making a constant angle α with lines parallel to the axis of the cylinder. This cylinder fits into a block D pierced with an equal cylindrical aperture, on the inner surface of which is cut a groove the exact counterpart of the projecting thread $abcd$.

It is easily seen from this description, that when the cylinder is introduced into the block, the only manner in which it can move is backwards or forwards by revolving about its axis, the thread sliding in the groove. Suppose W is the weight acting on the cylinder (including the weight of the cylinder itself) and P is the power acting at the end of an arm AC at right angles to the axis of the cylinder: the block D is supposed to be firmly fixed, and the axis of the cylinder to be vertical.

Prop. *To find the ratio of the power and weight in the Screw when they are in equilibrium.*

117. Let $AC = a$: rad. of cylinder $= b$.

Now the forces which hold the cylinder in equilibrium are W, P and the reactions of the pressures of the various portions of the thread on the corresponding portions of the lower surface of the groove in which the thread rests: these reactions are indeterminate in their number; but they all act in directions perpendicularly to the surface of the groove, and therefore their directions make a constant angle α with a horizontal plane. If, then, R be one of these reactions, $R \sin \alpha$, $R \cos \alpha$ are the resolved parts vertically and horizontally: the horizontal portions of the reactions act each at right angles to a radius of the cylinder. Hence resolving the forces vertically, and also taking the moments of the forces in horizontal planes, we have

$$W - \Sigma . R \sin \alpha \quad = 0 \;\ldots\ldots\ldots\; (1),$$

$$Pa - \Sigma . R \cos \alpha \, b = 0 \;\ldots\ldots\ldots\; (2),$$

we might write down the other four equations of equilibrium; but they introduce unknown quantities with which we are unconcerned in our question.

$$\text{Hence } \frac{W}{P}=\frac{a\sin\alpha\,\Sigma\,.\,R}{b\cos\alpha\,\Sigma\,.\,R}, \text{ because } b \text{ and } \alpha \text{ are constant:}$$

$$=\frac{a\sin\alpha}{b\cos\alpha}=\frac{2\pi a}{2\pi b\cot\alpha}$$

$$=\frac{\text{circumference of circle whose rad. is } AC}{\text{vertical dist. between two successive winds of the thread}}.$$

The screw is used to gain mechanical power in many ways. In excavating the Thames Tunnel the heavy iron frame-work which supported the workmen was gradually advanced by means of large screws.

FRICTION.

118. In the investigations of this Chapter we have supposed that the surfaces of the bodies in contact are perfectly smooth. Now in practice this is not the fact; for no surface can be so entirely freed from roughness and asperities as to be perfectly smooth, although their effect may in many cases be greatly diminished. By a *smooth* surface is meant a surface which opposes no resistance whatever to the motion of a body upon it, and therefore the resistance is wholly perpendicular to the surface. A surface which does oppose a resistance to the motion of a body upon it is said to be *rough*.

The friction of a body on a surface is measured by the least force which will put the body in motion along the surface.

Coulomb made a series of experiments upon the friction of bodies against each other and deduced the following laws: *Mémoires des Savans Etrangers*, Tom. x.

(1) *The friction varies as the pressure, when the materials of the surfaces in contact remain the same.* When the pressures are very great indeed it is found that the friction is somewhat less than this law would give.

(2) *The friction is independent of the extent of the surfaces in contact so long as the pressure remains the same.* When the surfaces in contact are extremely small, as for instance a cylinder resting on a surface, this law gives the friction much too great.

These two laws are true when the body is on the point of moving and also when it is actually in motion: but in the case of motion the magnitude of the friction is much less than when the body is in a state bordering upon motion.

(3) *The friction is independent of the velocity when the body is in motion.*

It follows from these laws that if P be the normal pressure of the body upon the surface then the friction $= \mu . P$, where μ is a constant quantity for the same materials, and is called *the coefficient of friction.*

In the state bordering on motion and when the surfaces in contact are of finite extent we have the following results from experiment:

$\mu = \frac{1}{2}$ surfaces wood, the grain being in same direction.

$= \frac{1}{4}$ .... opposite ......

$= \frac{1}{4}$ metallic surfaces.

$= \frac{1}{5}$ one surface wood and the other metal.

Oil and grease considerably diminish friction; fresh tallow reduces it to half its value.

In the state bordering on motion and when the surfaces in contact are single lines, then $\mu = \frac{1}{12}$ for wood. When the surface in contact is a physical point the statical friction is inconsiderable.

But for full particulars on this subject we refer the reader to Coulomb's papers, and also to two Memoirs recently published in the *Mémoires de l'Institut.* by M. Morin.

Prop. *To find the greatest angle which the direction of the mutual pressure of two surfaces in contact may make with their common normal at the point where the pressure acts without sliding; the coefficient of friction being given.*

119. Let P be the mutual pressure, its direction making an angle β with the normal. Then $P\cos\beta$ is the *direct* or normal pressure of the surfaces, and $P\sin\beta$ is the force balanced by the friction acting wholly or in part.

Hence the greatest value of the ratio $\dfrac{P\sin\beta}{P\cos\beta}$ is μ;

$$\therefore\ \beta = \tan^{-1}\mu \text{ is the greatest value of } \beta.$$

CHAPTER VI.

ROOFS, ARCHES AND BRIDGES.

120. In the present Chapter we shall apply the principles of equilibrium to explain in what manner the thrusts, strains, and pressures in general act in roofs and arches. We refer the reader to Robison's *Mechanical Philosophy*, Vol. I. for two Articles on Roofs and Arches, which contain many interesting details that would be entirely out of place in these pages.

Prop. *In a simple isosceles truss-roof required to calculate the tension of the tie-beam.*

121. Let AB, BC be two beams of the roofing connected by the tie-beam AC (fig. 46.): the truss resting on walls, as drawn in the figure. Let B be the weight of each sloping beam and the portion of the tiling or thatching supported by the beam: let G be the point at which this weight acts. Also suppose that the weight on the vertex arising from other appendages equals W: let α be the angle the roof makes with the horizon: $AG = b$, $AB = a$.

Now the forces acting on AB are a pressure at A perpendicular to the wall, $= R$ suppose: the tension of the tie-beam acting at A in the direction AC, $= T$ suppose: the weight B acting vertically at G: and lastly, some force P acting at B in direction BP making an unknown angle θ with the beam in the plane of the paper, and arising from the weight W and the action of the beam BC.

In order to find the connexion between P and W, we remark that the point B is held at rest by the force W downwards, and the two reactions P, P acting along the dotted lines.

For the equilibrium of AB we have

$$\text{horizontal forces} = T - P\cos(\alpha - \theta) = 0 \text{ .... } (1),$$

$$\text{vertical forces} = R - B - P\sin(\alpha - \theta) = 0 \text{ .... } (2),$$

$$\text{moments about } A = Bb\cos\alpha - Pa\sin\theta = 0 \text{ .... } (3).$$

For the equilibrium of the point B,

$$\text{vertical forces} = 2P\sin(\alpha - \theta) - W = 0 \text{ .... } (4).$$

Here, then, we have four equations and four unknown quantities T, P, R, θ: and therefore we can determine the unknown quantities, and therefore T.

By (1) (4) $T = \frac{1}{2} W \cot(\alpha - \theta)$, eliminating P:

and by (3) (4), eliminating P, we have

$$Bb\cos\alpha = \tfrac{1}{2} Wa \frac{\sin\theta}{\sin(\alpha - \theta)} = \tfrac{1}{2} Wa \frac{\sin\{\alpha - (\alpha - \theta)\}}{\sin(\alpha - \theta)}$$

$$= \tfrac{1}{2} Wa \{\sin\alpha \cot(\alpha - \theta) - \cos\alpha\};$$

$$\therefore \cot(\alpha - \theta) = \frac{2Bb + Wa}{Wa} \cot\alpha;$$

$$\therefore T = \frac{2Bb + Wa}{2a} \cot\alpha.$$

This measures the horizontal thrust of the roofing against the supporting walls supposing the tie-beams to give way: and we learn that this will be less the larger α is, or the steeper the roof is, the other quantities remaining the same. Also the smaller b is in proportion to a, or the nearer G is to A, the smaller is this thrust.

PROP. *To explain the manner in which buttresses act in supporting a roof: and to calculate their angle of elevation.*

122. Let (as before) AB, BC (fig. 47.) be two beams of the roofing: AD a piece of timber firmly attached to AB running down the inside of the wall, and resting on a *corbel* E: FH a beam to strengthen the attachment of AD, AB. Let R be the pressure on the top of the wall and corbel; N the point at which the resultant of the reactions of all the hori-

zontal pressures on the wall acts; T this resultant; $AN = x$; G the point through which the weight on the slanting timbers acts; $AG = b$, $AB = a$.

Now the action of the beams on each other at B must be in a horizontal direction, since we suppose there is no extra weight acting at B, as in the last Proposition; let P be this mutual pressure.

For the equilibrium of BAD,

$$\text{horizontal forces} = T - P = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad (1),$$

$$\text{vertical forces} = R - B = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad (2),$$

$$\text{moments about } A = Tx - Bb\cos a + Pa\sin a = 0 \quad\ldots\ldots\quad (3).$$

Here we have three equations and four unknown quantities T, P, R, x: and another relation connecting these quantities cannot be found; hence the problem is indeterminate: we shall see in the solution what quantities are indeterminate.

By (2) $R = B$, and is therefore not indeterminate.

By (1) (3), eliminating P, we have

$$T = \frac{Bb\cos a}{x + a\sin a}, \; = P \text{ by } (1).$$

Hence P, T, and x are indeterminate: but when a value is given to one of them, then, that the equilibrium may subsist, the other two must satisfy the two conditions just deduced.

Let ϕ be the angle which the resultant of R and T makes with the vertical; then

$$\tan\phi = \frac{T}{R} = \frac{b\cos a}{x + a\sin a}.$$

Now our object is to find the least angle at which a buttress need be built to support the roof. If the roof be on the point of sinking it must be so by tending to turn about the extremities D, D of the framework: in which case the force T acts at D, and x will then have its greatest value: also we perceive that both T and also ϕ are smaller the greater x is: hence the least angle at which the buttress need be built is given by the equation

$$\tan\phi = \frac{A'G'}{AD + AA'} = \frac{A'G'}{A'D}$$

BA' being horizontal; AA', GG' vertical.

Hence the dotted line $G'D$ represents the limiting angle which the buttress must make with the vertical in order that the roof may not fall.

This calculation shews the great use of the part of the framework which runs down the wall.

We have drawn in our figure but one connecting beam HF: but there might have been more, and the calculation would have been precisely the same, supposing that ABD is a *rigid* framework: and GG' the vertical line in which the weight of this framework and the superincumbent tiling or other covering acts. The simple rule is to draw a horizontal BG' from the vertex of the roof, cutting GG' in G' and join G' with the lowest point D of the framework: $G'D$ gives the least inclination of the buttress. Also the buttress need not extend higher up the wall than the level of D.

The roofs of Westminster Hall and of Trinity College Hall, Cambridge, are good illustrations of this kind of roof.

Before quitting this subject we will investigate the following Proposition.

PROP. *To calculate the conditions of equilibrium of any number of beams forming a framework in a vertical plane, symmetrical with respect to a vertical line through the highest point.*

123. Let the lengths of the beams be $a_1 a_2 a_3 \ldots$ reckoning from the lowest: $G_1 G_2 G_3 \ldots$ the points at which the weights of the beams and the weights with which they may be loaded act; $b_1 b_2 b_3 \ldots$ the distances of these points from the lower extremities of the beams; $\alpha_1 \alpha_2 \alpha_3 \ldots$ the angles which the beams make with the horizon; R the vertical pressure on the walls of support of each of the two lowest beams; T the horizontal thrust of these beams on the walls; or the tension of the tie-beam connecting them if there be one.

Now the actions of any two beams on each other at the points of junction must be in the same line, since there is no third force to keep them in equilibrium. Let then $P_1 P_2 \ldots$ be these mutual actions between the first and second, the second and third beams, and so on; $\theta_1 \theta_2 \ldots$ the angles which the directions of these forces make with the horizon.

For the equilibrium of the lowest beam

$$T - P_1 \cos\theta_1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$B_1 - R + P_1 \sin\theta_1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$B_1 b_1 \cos\alpha_1 - P_1 a_1 \sin(\alpha_1 - \theta_1) = 0 \ldots\ldots (3).$$

For the equilibrium of the second beam

$$P_1 \cos\theta_1 - P_2 \cos\theta_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$B_2 - P_1 \sin\theta_1 + P_2 \sin\theta_2 = 0 \ldots\ldots\ldots\ldots (5)$$

$$B_2 b_2 \cos\alpha_2 - P_2 a_2 \sin(\alpha_2 - \theta_2) = 0 \ldots\ldots (6),$$

and so on, till we come to the highest beam in which the angle θ_n must $= 0$, since the two highest beams which form the vertex have no third force at their point of junction to keep P_n and P_n in equilibrium. Hence for the last (the n^{th}) beam

$$P_{n-1} \cos\theta_{n-1} - P_n = 0 \ldots\ldots\ldots\ldots (3n-2)$$

$$B_n - P_{n-1} \sin\theta_{n-1} = 0 \ldots\ldots\ldots\ldots (3n-1)$$

$$B_n b_n \cos\alpha_n - P_n a_n \sin\alpha_n = 0 \ldots\ldots (3n).$$

Also we have the following analytical relation connecting $\alpha_1 \alpha_2 \ldots$, $a_1 a_2 \ldots$

$$a_1 \cos\alpha_1 + a_2 \cos\alpha_2 + \ldots + a_n \cos\alpha_n = D \ldots (3n+1)$$

$2D$ being the distance of the opposite walls from each other.

We have then $3n+1$ equations from which to eliminate the quantities $P_1 P_2 \ldots P_n$, $\theta_1 \theta_2 \ldots \theta_{n-1}$, R, T, which are $2n+1$ in number and we have n equations remaining to determine the n angles $\alpha_1 \alpha_2 \alpha_3 \ldots \alpha_n$; and these being known we know the position of equilibrium of the beams.

If we add together all the second equations we have

$$R = B_1 + B_2 + \ldots\ldots + B_n$$

wherefore R is known.

Now by (2) (3) eliminating P_1 we have

$$\frac{\sin(a_1 - \theta_1)}{\sin\theta_1} = \frac{B_1 b_1 \cos a_1}{(R - B_1)\,a_1};$$

$$\therefore\ \sin a_1 \cot\theta_1 = \left\{1 + \frac{B_1 b_1}{(R - B_1)\,a_1}\right\}\cos a_1;$$

$$\therefore\ \cot\theta_1 = \frac{(R - B_1)\,a_1 + B_1 b_1}{(R - B_1)\,a_1}\cot a_1;$$

$$\therefore\ T = (R - B_1)\cot\theta_1 \text{ by (1) (2)}$$

$$= \frac{(R - B_1)\,a_1 + B_1 b_1}{a_1}\cot a_1 = \left\{B_2 + \ldots + B_n + B_1\frac{b_1}{a_1}\right\}\cot a_1$$

whence T is known.

Again, by adding equations (1), (4); (2) (5) respectively we have

$$T - P_2\cos\theta_2 = 0$$

$$B_1 + B_2 - R + P_2\sin\theta_2 = 0$$

also by (6) $B_2 b_2 \cos a_2 - P_2 a_2 \sin(a_2 - \theta_2) = 0$.

Hence, as by solving (1) (2) (3), we have

$$T = \left\{B_3 + \ldots + B_n + B_2\frac{b_2}{a_2}\right\}\cot a_2,$$

and in the same manner we should obtain

$$T = \left\{B_4 + \ldots + B_n + B_3\frac{b_3}{a_3}\right\}\cot a_3$$

……………………………………

$$T = \left\{B_n\frac{b_n}{a_n}\right\}\cot a_n.$$

These n values of T, being equated, give $n-1$ relations connecting the angles $\alpha_1\alpha_2\ldots\alpha_n$; and these combined with equation $(3n+1)$ determine these angles.

The relation of one angle α_m to the preceding α_{m-1} is given by

$$\tan\alpha_m = \frac{B_{m+1}+\ldots+B_n+B_m\dfrac{b_m}{a_m}}{B_m+\ldots+B_n+B_{m-1}\dfrac{b_{m-1}}{a_{m-1}}}\tan\alpha_{m-1}.$$

This shews that every one of the angles $\alpha_1\alpha_2\ldots\alpha_n$ is greater or every one less than 90^0 and equation $(3n+1)$ shews that they must be all less than 90^0, otherwise we should have $D=$ a sum of negative quantities.

Also the beams must be less and less inclined to the horizon as we ascend, since $\tan\alpha_m$ is less than $\tan\alpha_{m-1}$.

124. By an *Arch* is meant an assemblage of bodies supported, as represented in fig. 48, by their mutual pressures and the pressures of the two extreme bodies against fixed obstacles.

We shall suppose the bodies to have the usual form, that of truncated wedges; and to be placed so as to have their sides which are in contact perpendicular to the same vertical plane. These bodies are then called *voussoirs:* the highest voussoir is called the *key-stone* of the arch: the surfaces which separate the voussoirs are called the *joints:* the external curve of the arch is called the *extrados;* the internal curve the *intrados:* the solid mass against which the lowest voussoir on each side rests is called the *pier* or *abutment.*

125. It is found in practice that the friction of the voussoirs against each other is so great that they are incapable of sliding past each other; and in many cases all possibility of sliding is prevented by the voussoirs being *joggled;* that is, being united by a piece of stone or iron which is partly imbedded in one voussoir and partly in the voussoir in contact with it.

In consequence of this the conditions of equilibrium of an arch reduce themselves to the condition that the arch shall not break at any part by the rotation of one voussoir upon another: or, which is the same thing, by the opening of any of the joints.

Prop. *To explain how an arch is supported; and in what manner friction tends to preserve the equilibrium.*

126. When two bodies with *smooth* surfaces in contact are pressing against each other, then, in order that they may not slide upon each other, the mutual pressure must act in a line which is perpendicular to the two surfaces. But if the surfaces in contact be *rough*, the mutual pressure need not act in a direction perpendicular to the surfaces to prevent sliding, but may act in any line making an angle with the perpendicular less than a certain finite angle, the magnitude of which depends on the degree of roughness of the surfaces. See Art. 119.

Suppose fig. 48. represents an arch in equilibrium, the extreme voussoirs G_1 and G_7 resting on the piers A and B: $G_1 G_2 G_3$......are the centres of gravity of the voussoirs: the weights of the voussoirs act in the vertical lines $G_1 b$, $G_1 d$, $f G_3$.

Now any voussoir is held in equilibrium by the action of its weight in a vertical direction downwards, and by the pressures of the contiguous voussoirs or abutment on its two joints. The mutual pressures of any two voussoirs at a joint must evidently be equal to some single force acting at some unknown point.

Suppose the pressure of the pier A acts on the voussoir G_1 at the point a in the direction ab cutting $G_1 b$ in some point b: then the third force which supports G_1 must act through b in some determinate direction dcb; let this cut the vertical through G_2 in d: this is the direction of the mutual pressures of the voussoirs G_1 and G_2, and c is the point at which they act: then the third force which supports G_2 must act through d in some direction fed; and so on: and it follows that the mutual pressures of the voussoirs act in some determinate (though unknown) line $abdfhjlno$ made up of portions of straight line. This is called *the line of pressure.*

Now if the surfaces of the voussoirs were *smooth* it would be necessary for the equilibrium, that this line of pressure should be perpendicular to the joints at the points a, c, e, g, i, k, m, o. This condition establishes certain relations between the weights of the voussoirs and the angles which the joints make with the horizon. These relations may be easily calculated, but since the voussoirs never are smooth in practice the calculation would be useless. But when the surfaces of the voussoirs are *rough*, the only conditions for the equilibrium are that the angles which the line of pressure makes with the perpendiculars to the joints at the points a, c, e, g, i, k, m, o should not exceed a certain finite angle, which can be made as large as we please by having the stones *rough-hewn* or by *joggling* them (Art. 125.)

We see, then, that in consequence of friction the weights of the voussoirs and the angles of inclination of their joints need not fulfil those exact relations, which would be necessary if the surfaces were smooth.

127. But the great advantage of friction in the support of an arch is yet to appear. For in order that an arch in a bridge may be of service, it must be able to sustain weights (not immoderate in their magnitude) placed on different parts without breaking.

Let us now suppose a weight W to be placed on the voussoir G_2. This weight adds to the weight of G_2, and consequently disturbs the line of pressure and shifts it to some new position $a'b'd'f'h'j'l'n'o'$ as represented in fig. 49. But the arch will still stand if the angles at a', c', e', g', i', k', m', o' do not exceed the finite limit. Whereas the equilibrium of the arch would certainly be disturbed by W if the surfaces were smooth.

In this way we see, then, the important aid that friction affords in the support of an arch.

Prop. *To find the conditions of equilibrium of an arch.*

128. Since we suppose the friction of the voussoirs against each other to be so great that they cannot slide upon each other, it follows that the arch can fall only in consequence of its breaking at the upper or lower extremities of some of the joints. And, since we suppose the piers A and B to be immoveable, simple geometrical considerations shew, that if the arch break it must break in at least three pieces, four joints

at least opening, the points where they open being alternately in the extrados and intrados of the arch. So long as the line of pressure cuts all the joints (as is represented in figures 48, 49.) the arch must stand, because in that case the joints are prevented from opening by the pressure acting along that line.

But by continually loading the arch in the same part we may gradually shift the line of pressure till it passes through the extremity of one of the joints, as g' in fig. 50: and now the pressure acting through g' will not prevent the joint $g'H$ opening at H, although other circumstances may.

From what we have already said about the arch not falling till two joints at least in the extrados and two joints at least in the intrados open, it follows, that the arch will certainly stand as we continually load it till the line of pressure passes through the extremities of at least four joints, the extremities being alternately in the intrados and extrados.

If, then, our arch be such, that by loading it we cannot shift the line of pressure into this position, the arch will sustain any load without falling. Nevertheless when the arch is much loaded, and the line of pressure passes through the extremity of any joint, there will be a great strain at that point. This explains the fact observed by Professor Robison, who constructed some chalk models and found that chips fell off from three or four of the extremities of the joints.

If, however, the arch be of such a form that we can place on it a sufficient load to cause the line of pressure to pass through at least four extremities of joints situated alternately in the intrados and extrados, there is a possibility of the arch breaking and falling by the opening of the joints.

Let D, K, L, M (fig. 51.) be the points through which the line of pressure passes in this case: join them by straight lines. Then the arch may be supposed to be a system of heavy beams DK, KL, LM. In order to determine the conditions that the equilibrium of the arch shall be stable, suppose the joints are forcibly opened through very small angles, the parts of the arch being sustained in the position represented in fig. 51. That the equilibrium of the arch may be stable, the joints, when the arch thus sustained in a broken form is left to itself, ought to collapse and not open wider; a condition which is satisfied if the system of beams *DK*, *KL*, *LM* be such that when left to themselves the point *K* shall ascend and the point *L* descend.

Hence to ascertain whether an arch of certain dimensions and figure will sustain any weight placed on it, we must consider all the ways in which the arch can break and find whether in each case the system of beams is of the nature just described. If this be the case we may be assured that the arch will support any weight.

PROP. *To prove that an arch, in which a tangent line drawn at the highest point of the intrados and produced to the abutments lies wholly within the voussoirs, will sustain any weight placed on any part of the extrados without breaking: also if the arch be of greater span than this, it will bear any weight placed on those parts of the extrados from which straight lines can be drawn through the voussoirs to both abutments.*

129. We will take the arch of greatest length under the first conditions mentioned in the enunciation. Let *F* be the highest point of the intrados (fig. 52.); then the tangent at *F* passes through the highest point *C* and *C′* of the extreme joints. Hence from any point *G* in the extrados a straight line can be drawn to each pier lying wholly within the mass of the voussoirs. This would not be the case if the arch were the least portion longer without having the voussoirs proportionably lengthened. For suppose the left hand abutment (in the figure) had the position of the dotted line, then no straight line can be drawn from a point *c* through the voussoirs to the right hand abutment.

It appears, then, that if two straight lines *Ga*, *Gb* can be drawn from *G* through the voussoirs to the piers, the tangent at *F* must be wholly in the voussoirs: and, this being the case, any weight placed on *G* will be sustained, since the portions of the arch on the right and left of *G* will act like beams *Gb*, *Ga*, of which the points *b* and *a* cannot slip, because we suppose the friction of the voussoirs, or at any rate the joggling, sufficient to prevent sliding.

The second part of the Proposition is evidently true after what has been already written.

130. Cor. 1. The principle of this Proposition seems to have been used by Mylne in the construction of Blackfriars Bridge, London, one of the arches and piers of which we have represented in fig. 53. AVY is a circular arc of which C is the centre, and the radius 56 feet: OV height above low-water = 40 feet: VK = 6 feet 7 inches. AB, YE are circular arcs of radius 35 feet: ab = 19 feet: YI = about 8 or 9 feet. All the joints are joggled: and a line from K to the middle point of the joint YI lies wholly within the masonry; and does not even pass near the extremities of the joints, so that chipping of the voussoirs cannot take place. Therefore the portion YVL cannot break however great the load on or near the crown of the bridge, except by the crushing of the materials: and it would require an enormous pressure on the *haunches* (near Y) to raise the crown, since the weight of KY is about 2000 tons.

The tangent at Y falls within the foot of the pier F: and the pier itself is like one solid mass by having the stones and oaken planks below ab (low-water mark) well joggled, and by having each of the voussoirs between Y and a projecting over the one below it, and so giving each a firm hold of the rubble-work in the centre of the pier (as represented in the figure). The rubble-work itself is held down in its place by the small inverted arch IG. Since, then, the tangent at Y falls within the foot F of the pier, and the pier is as one solid mass, the arch $BAVE$ would stand of itself even were the other arches to fall, since if KY were on the point of falling the pressure would act through Y. This gives additional security to the bridge.

131. Cor. 2. From this Proposition we learn how it is that the Gothic Arch will sustain such enormous weights upon its crown, as we see is the case in many of our ecclesiastical buildings. The stone steeple of St Dunstan's in the East, London, is supported by four semi-pointed arches. In fact, it is a principle that a pointed arch must have a great pressure upon its crown to prevent its falling; for we may consider it as consisting of the two extreme portions of a very large circular arch brought together, so that the pressure on the crown must

at least equal the pressure of the portion of the circular arch which is removed. Flying buttresses always have a great pressure upon their highest part.

But besides this the pointed arch, for the reason explained in the Proposition, will sustain almost any weight on its crown provided the lowest stones do not give way: and consequently the Gothic arch is stronger for lofty buildings than the circular: but the circular arch is far better adapted than the Gothic arch for bridges, since the pressure of weights passing over may act upon any part of the arch, not only on the crown.

An arch built in a wall is almost sure to stand of whatever form it be, so long as its foundation is firm: for suppose the haunches were about to fall in, the crown rising; then, in order that the crown may rise, the whole of the masonry or brick work above the black line (in fig. 54.) must be moved upwards, a weight sufficient to prevent the crown from rising. Moreover, suppose the crown would rise; then directly it had risen and thrust the masonry above it through a small space, the pressure which caused the haunches to sink will cease to act in consequence of the *dove-tailing* together (so to speak) of the stones or bricks, which lie above the haunches. In the the same manner we see that the crown could not sink.

PROP. *To explain the manner in which a dome is supported.*

132. A Dome or Cupola is an assemblage of stones, bricks, or other materials in equilibrium, of which the intrados and extrados are surfaces of revolution having a common vertical axis.

If we consider any given horizontal course of stones, it is evident that this course cannot fall inwards, since all the stones tend equally towards the center, and consequently wedge each other in. But the form of the dome might be such that the weight of the superincumbent courses should thrust out the course under consideration and the courses below. In this way, and in this way only, can the dome fall.

It is very easily seen that a conical dome is secure, and will bear any weight on its upper course, provided the lowest course is kept from bursting outwards. A dome with its convexity inwards would be still more secure: for every stone is pressed inwards, since it forms part of an arch with its convexity inwards, and extremities in the highest and lowest horizontal courses: consequently the stones of each horizontal course are more firmly held together than in the conical dome.

133. The stone lanthern on the top of St. Paul's Cathedral weighs several hundred tons, and is supported by a brick cone, which is concealed between the outer and inner domes. The lowest course of this cone is above the stone gallery at the bottom of the outer dome, and is held from bursting outwards by an iron chain.

The pyramids, which form the steeples of Gothic architecture, are for the same reasons as the cone stable in their equilibrium. The enormous weight of these steeples is supported by very pointed arches, which spring from the square tower at a considerable distance below the top, and are of such a slight curvature that a straight line can be drawn from the key-stone through all the stones of each leg (Art. 129, 131.): and the thrust down these arches is counteracted by the massive masonry of the tower and the buttresses (as explained in Art. 122.)

In the walls and tower of Salisbury Cathedral are to be seen some fearful cracks which seem to indicate a want of sufficient support for the stupendous steeple which forms so striking a feature of this edifice. The foundation remains firm. Sir Christopher Wren examined these defects, and found that the steeple was braced in different parts with iron bars: and added more for greater security. A little less than a century ago Price, the author of the *British Carpenter*, narrowly inspected the whole, and seems to have proved, that the cathedral was erected under two architects, one completing the work that the other commenced; but that the first architect never contemplated the erection of the steeple nor so lofty a tower as the second architect was bold enough to add to the low tower built by his predecessor; and in consequence of the insufficiency of the supports the cracks now to be seen warned the architect to resort to the expedient of bracers to hold the base of the steeple from spreading. For a very interesting account of this building we refer the reader to Price's *Series of Observations upon the Cathedral-church of Salisbury.* We have mentioned these particulars because this building is a very remarkable illustration of the necessity of attending to the connexion between the weights and pressures in a building, and the walls arches and buttresses by which they are to be sustained.

134. As an example of the support of a stone vaulting, we shall explain the manner in which the roof of King's College Chapel, Cambridge, is supported.

Figure 55 represents a projection upon a horizontal plane of one compartment of the roof included between the four buttresses *f*, *g*, *h*, *k*; and figure 56 represents the projection of half this compartment upon the vertical plane of one of the windows on the south side: the same letters in the two figures refer to the same points.

The rib *bc* runs from the east to the west end of the Chapel, the stones which form it lie in the same horizontal line and at a greater elevation from the ground than any other part of the roof: *K* is the central stone of the compartment, and is the upper part of one of the ornamented drops seen hanging from the roof in the interior. The stones in *aKd* lie in an arch of which *K* is the key-stone: it is clear that the *tendency* of this arch is to sink at the crown *K*, and thrust down the walls at *a* and *d*. We shall proceed, then, to explain how the stones in this arch are supported; and also the stones in the rib *bc*: and in the course of the explanation it will be seen that we shew how every stone in the compartment *fghk* is supported.

On examining the roof carefully it will be found that the stones are placed in semi-arches in vertical planes through the buttresses; the spring of all the semi-arches in the space *ba* being at *f*, and their crowns or key-stones in the courses *bK* or *Ka*: this is best seen in figure 56. Now any stone *s* in the arch *aKd* is the key-stone of the two semi-arches *sf* and *sg*: and the thrust of the stones in *Ks* is propagated down the semi-arches *sf* and *sg*, and ultimately acts upon the buttresses at *f* and *g*; the same is true of every stone in *Ka*: likewise on the other side of *bc* the stones in *Kd* are supported by the semi-arches, of which they are the key-stones, and which spring from the buttresses h and k. Again, any stone r in Kb is the key-stone of two semi-arches rk and rf, and is held in its place by the thrust of the stones in Kr; and this thrust is propagated down the semi-arches rk and rf, and acts ultimately upon the buttresses k and f: the masonry of the rib bc is sufficiently heavy to prevent these semi-arches from sinking by their key-stone rising. It will be clearly seen, then, how every stone in bc and aKd is supported: it will also be seen that every other stone in the roof is sustained by being a member of a semi-arch springing from one of the buttresses, and having its key-stone in bc or aKd. The pressure of the compartment $fghk$ upon the buttresses acts obliquely: for instance, that on f will act downwards in a line whose projection on the horizontal plane will lie towards the south-east. But the compartment east of $fghk$ will press upon the buttress f in a line whose horizontal projection lies towards the south-west: and consequently the resultant of these pressures will act in a line whose horizontal projection runs due south: let fF be this line (fig. 57.); this figure represents one of the buttresses. The dimensions of the buttress are so arranged that fF shall lie within the masonry and pass into the foundation within the foot of the buttress.

The resultant pressure of the roof on the walls at each of the four angles acts obliquely; consequently instead of buttresses of the ordinary form at the four angles of the building, towers crowned with lofty turrets are erected of such a weight as to deflect the line of pressure of the roof, and cause it to pass into the ground through the masonry.

R 135. We proceed now to find the position of equilibrium of a chain suspended from two fixed points, and briefly to explain the construction of *Suspension Bridges*.

A chain is an assemblage of rigid pieces of iron linked together, or connected by pivots, as in the chains of suspension bridges. We may therefore apply the principles of Chapter III. to determine the position of equilibrium of the chain. The length of the chain is generally so great in comparison with the length of each link, that we shall suppose the polygonal figure in which the chain hangs to be a continuous curve. Also we suppose that the motion of the links about their points of connexion is perfectly free; or, in other words, that the mutual action of any two links acts in a tangent line to the curve in which the chain hangs. The curve in which the chain hangs when in equilibrium is called the *Catenary*.

PROP. *A chain of uniform density and thickness is suspended from two given points: required to find the equation to the curve in which the chain hangs when it is in equilibrium**.

136. Let A and B (in the plane of the paper, which is supposed to be vertical) be the two points of support: fig. 58. After the chain has ceased oscillating and has attained its position of permanent rest, suppose ACB is the curve which it forms, C being the lowest point: take this as the origin of co-ordinates, CM vertical $= x$; MP horizontal $= y$; $CP = s$; P being any point in the curve.

Now the equilibrium of any portion CP will not be disturbed if we suppose this part of the chain to become rigid: this appears from Art. 71. Let c and t be the lengths of portions of the chain of which the weights equal the tensions

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* We may calculate the form of the curve in the following manner.

Let us suppose the chain to consist of an infinitely great number of rigid and straight portions, each equal to δs in length: and let τ of these portions lie between C and P: fig. 58. then $s = r\delta s$: also α_r and α_{r-1} being the angles which these portions make with the axis of x, we have by Art. 123.

$$\tan\alpha_r = \frac{r+\frac{3}{2}}{r+\frac{1}{2}}\tan\alpha_{r-1}.$$

Hence $\delta\,.\tan PTM = \tan\alpha_r - \tan\alpha_{r-1}$

$$= \frac{1}{r+\frac{3}{2}}\tan\alpha_r,$$

$$\text{or } \delta\,.\frac{dy}{dx} = \frac{\frac{dy}{dx}\delta s}{s+\frac{3}{2}\delta s};$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{s}\frac{dy}{dx}\frac{ds}{dx}; \quad \therefore \log_e\frac{dy}{dx} = \log_e\frac{s}{c};$$

$$\therefore \frac{dy}{dx} = \frac{s}{c}, \text{ as in the text.}$$

at C and P. Then CP is a rigid body acted on by three forces which are proportional to c, t, s, and act respectively in the directions Cc, Pt, Gs.

Draw PT the tangent at P cutting the axis of x in T. Then the forces holding CP in equilibrium have their directions parallel to the sides of the triangle PMT, and therefore bear the same proportion one to another that these sides do; (see Art. 18.)

$$\therefore \frac{PM}{MT} = \frac{\text{tension at lowest point}}{\text{weight of the portion } CP}$$

$$\text{or } \frac{dy}{dx} = \frac{c}{s};$$

$$\therefore \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{\sqrt{c^2 + s^2}}{s},$$

$$\frac{dx}{ds} = \frac{s}{\sqrt{c^2 + s^2}};$$

$$\therefore x + c = \sqrt{c^2 + s^2} \text{ .... } (1),$$

the constant added being such that when $x = 0$ then $s = 0$, since the origin of co-ordinates is taken on the curve at C;

$$\therefore s^2 = x^2 + 2cx \text{ .... } (2).$$

$$\text{Also } \frac{dy}{dx} = \frac{c}{s} = \frac{c}{\sqrt{x^2 + 2cx}};$$

$$\therefore y = c \log_e \left\{\frac{x + c + \sqrt{x^2 + 2cx}}{c}\right\} \text{ .... } (3),$$

the constant being so chosen that x and y vanish together.

This last equation may be put under another form

$$e^{\frac{y}{c}} = \frac{x + c}{c} + \sqrt{\left(\frac{x + c}{c}\right)^2 - 1},$$

then transposing $\frac{x + c}{c}$ and squaring both sides of the equation

$$e^{\frac{2y}{c}} - 2e^{\frac{y}{c}} \cdot \frac{x+c}{c} = -1;$$

$$\therefore\ x + c = \frac{c}{2}\left\{e^{\frac{y}{c}} + e^{-\frac{y}{c}}\right\} \text{ .... } (4).$$

Also $s = \sqrt{(x+c)^2 - c^2}$ by equation (2),

$$= \frac{c}{2}\left\{e^{\frac{y}{c}} - e^{-\frac{y}{c}}\right\} \text{ .... } (5).$$

Any one of these five equations may be taken as the equation to the curve.

When the chain is uniform in density and thickness, (as in the present instance) the curve is called the *Common Catenary*.

137. Cor. 1. Of all curves of a given length drawn between two fixed points in a horizontal line, the common catenary is that which has its centre of gravity furthest from the line joining the points.

For since the chain is in equilibrium the depth of its centre of gravity from the horizontal line is a maximum or minimum (Art. 78.) and it is clear that it is a maximum and not a minimum, because if you displace the chain slightly it will return to its position of equilibrium, or its equilibrium is *stable* (Art. 79). Hence in any other position of the chain than that of equilibrium the centre of gravity will be nearer the given horizontal line. But the chain which hangs in the common catenary is of uniform density and thickness, and therefore its centre of gravity coincides with that of the curve: and consequently the common catenary is the curve of the nature described.

Cor. 2. By means of the formulæ of Art. 87. Ex. 2. we shall find that the co-ordinates to the centre of gravity from the lowest point are

$$\bar{x} = \frac{cy}{2s} + \frac{x-c}{2}, \qquad \bar{y} = y - \frac{cx}{s}.$$

138. Cor. 3. We might have taken the origin of co-ordinates at any other point than the lowest; as C', fig. 59.

Let the tangent at C' make an angle α with the vertical. We shall then readily get, if c' be used instead of c in the Proposition,

$$\frac{s}{c'} = \frac{C'T}{C'R} = \frac{\sin(\alpha - PTM)}{\sin PTM}$$

$$= \sin\alpha \cot PTM - \cos\alpha$$

$$= \sin\alpha \frac{dx}{dy} - \cos\alpha\,;$$

$$\therefore \frac{dx}{ds} = \frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} = \frac{s + c'\cos\alpha}{\sqrt{(s + c'\cos\alpha)^2 + (c'\sin\alpha)^2}},$$

$$x + c' = \sqrt{s^2 + 2sc'\cos\alpha + c'^2}.$$

We shall also find that

$$y = c'\sin\alpha \log_e \left\{ \frac{x + c' + \sqrt{x^2 + 2c'x + c'^2\cos^2\alpha}}{c'(1 + \cos\alpha)} \right\}.$$

Q

Prop. *To find the tension of the chain at any point.*

139. Let t be the tension at P acting in the direction of the tangent at P and estimated in terms of the length of chain of which the weight equals the tension: then, by what was mentioned in the last Proposition, (fig. 58.)

$$\frac{\text{tension at } P}{\text{weight of } CP} = \frac{PT}{MT}; \quad \therefore \frac{t}{s} = \frac{ds}{dx}.$$

But $s^2 = x^2 + 2cx$, by equation (2) of Art. 136;

$$\therefore t = x + c.$$

This shews that the lengths of chain of which the weights equal the tensions at the various points of the common catenary are such, that if they were suspended from those points their lower extremies would lie in a horizontal line.

For draw CE and PQ vertically downwards and equal to c and $x + c$ respectively: these then are the lengths of chain which measure the tensions at C and P. But $PQ = x + c = MC + CE$, and PM is horizontal: therefore Q and E are in the same horizontal line.

COR. 1. If a uniform chain hang freely over any two points, the extremities of the chain will lie in the same horizontal line when the chain is in equilibrium.

0

PROP. *A chain of variable thickness, but of the same material throughout, is suspended from two points: required to find the law of the thickness that the tension at different parts of the chain may vary as the strength of the chain at those parts.*

140. Let S be the length of a uniform chain of which the thickness equals that at the lowest point, and weight equals the weight of the length s of the chain to be suspended.

Let, as before, C be the lowest point (fig. 58.): $CM = x$, $MP = y$, $CP = s$: c the length of uniform chain of the thickness at C, of which the weight equals the tension at C. The portion CP when it has assumed its form of equilibrium may be supposed to become rigid. The forces which retain it in equilibrium are its weight and the tensions at C and P, and these are parallel to the sides of the triangle MTP;

$$\text{and } \because PT = \sqrt{PM^2 + MT^2};$$

$$\therefore \text{ tension at } P = \frac{\sqrt{c^2 + S^2}}{c} \text{ tension at } C.$$

But the thickness of the chain at P varies ultimately as the quantity of material in a given short length δs of the chain, since the density is constant: it therefore varies as $\dfrac{dS}{ds}$. But by the hypothesis the tension must vary as the thickness of the chain;

$$\therefore \frac{dS}{ds} \text{ varies as } \sqrt{c^2 + S^2} \text{ or } \frac{dS}{ds} = \frac{\sqrt{c^2 + S^2}}{c}$$

since S and s are ultimately equal;

$$\therefore s = c \log_e \left(\frac{S + \sqrt{S^2 + c^2}}{c}\right) \text{ .... } (6).$$

$$\text{Also } \frac{c}{S} = \frac{MP}{MT} = \frac{dy}{dx}; \quad \therefore \frac{ds}{dx} = \frac{\sqrt{c^2+S^2}}{S}.$$

$$\text{But } \frac{dS}{ds} = \frac{\sqrt{c^2+S^2}}{c};$$

$$\therefore \frac{dS}{dx} = \frac{c^2+S^2}{cS} \text{ or } \frac{dx}{dS} = \frac{cS}{c^2+S^2};$$

$$\therefore x = c \log_e \frac{\sqrt{c^2+S^2}}{c} \text{ .... (7).}$$

$$\text{Also } \frac{dy}{dS} = \frac{dy}{dx}\,\frac{dx}{dS} = \frac{c^2}{c^2+S^2};$$

$$\therefore y = c \tan^{-1}\frac{S}{c} \text{ or } S = c \tan\frac{y}{c} \text{ .... (8).}$$

141. These formulæ have been reduced to Tables by Sir Davies Gilbert in the Philosophical Transactions for 1826. We give the following extracts from them to elucidate the application of the equations to the construction of Suspension Bridges.

TABLE I. The Common Catenary.

$y = 100$.

c	x	s	t	Angle.
1000	5.004	100.166	1005.004	84° 16′ 48″
980	5.106	100.173	985.106	84 9 49
...	....	... ...	... ...	.. .. ..
420	11.961	100.947	431.961	76 29 6
400	12.565	101.045	412.565	75 49 22
380	13.234	101.158	393.234	75 5 35
...	.. ...	... ...	... ...	.. .. ..

TABLE II. The Common Catenary.

$c = 100$.

y	x	s	t	Angle.
1	.005	1.000	100.005	$89^{0}\ 25'\ 39''$
2	.020	2.000	100.020	88 51 15
.	...	.. ...	... ...	.. .. ..
20	2.007	20.134	102.007	78 36 59
21	2.213	21.155	102.213	78 3 19
..	....	.. ...	... ...	.. .. ..

TABLE III. The Catenary of equal strength.

$y = 100$.

c	x	s	S	t	Angle.
1000	5.008	100.167	100.334	1005.021	$84^{0}\ 16'\ 13''$
980	5.111	100.174	100.348	985.124	84 9 12
...	....	......	......	......	.. .. ..
420	12.019	100.958	101.933	432.193	76 21 29
400	12.631	101.057	102.137	412.832	75 40 33
...	.....	......	......	......	.. .. ..

TABLE IV. The Catenary of equal strength.

$c = 100$.

y	x	s	S	t	Angle.
1	.005	1.000	1.000	100.005	89° 25′ 37″
2	.020	2.000	2.000	100.020	88 51 14
.	...	....	....	... ...	.. .. ..
20	2.013	20.135	20.271	102.034	78 32 23
21	2.221	21.156	21.314	102.246	77 58 4
.	....	.....	.....	... ...	.. .. ..

To explain the use of these Tables we shall take an example of each species of Catenary.

Ex. 1. *Let the span proposed for a Suspension Bridge be* 800 *feet, and let the adjunct weight of suspension rods, road-way .... be taken at one half of the weight of the chains: and let it be determined to load the chains at the point of their greatest strain, that is at the points of suspension, with one-sixth part of the weight they are theoretically capable of sustaining.*

The modulus which measures the full tenacity of iron is shewn by numerous experiments to be 14800 feet: this being the greatest length of iron bar which another iron bar of equal transverse dimensions will support without sensibly stretching.

Now this modulus must be reduced in the ratio 3 : 2, since we have supposed the weight of the rods road-way ... to be equal to half the weight of the chains, and consequently we add to the weight of the chains without adding to their strength. The virtual modulus is therefore 9867 feet: and the tension of the chain at the points of support is by hypothesis to $= 9867 \div 6$ feet $= 1644.5$ feet.

The semi-span is 400 feet. In Table I. y is taken = 100 measures; therefore each of these measures is 4 feet: and the tension at the points of support expressed in these measures $= 1644.5 \div 4 = 411.124$. But by Table I. when $t = 412$,

$c =$ 400 measures = 1600 feet,

$x =$ 12.565 .... = 50.260 ....

$s =$ 101.045 .... = 404.180 ....

The angle of suspension $= 75^{0}\ 49'$.

Having found the value of c we may make use of Table II. to find the lengths of the rods for the different ordinates of the curve. In this Table c is taken at 100 measures, consequently each measure equals 16 feet.

Each gradation of y in that Table will therefore be 16 feet; and the second column gives the number of measures by which the suspending rods corresponding to the respective values of y must exceed the length of the suspending rod at the apex or centre of the bridge.

Let the following Table be formed from Table II. by taking the successive differences of the values of s:

1st measure of y.	length of arc of catenary = 1.000 measures.
2nd	.... = 1.000 ....
....	....
....	....
21st	.... = 1.021 ....
....	....

The last column of numbers gives the proportional part of the adjunct weights which must be suspended from the successive portions of the catenary, in order to distribute them equally throughout.

In this example we have supposed the adjunct weights to be equally distributed along the chain, so as virtually merely to increase its uniform thickness. We shall now in

Ex. 2. *Suppose the catenary to be one of equal strength: i. e. the tension at every part proportional to the strength: the other data the same as before.*

In this case c represents the uniform tension on each portion of iron throughout the chains whose transverse section equals that at the lowest point. In the uniform catenary the greatest tension (that at the points of support) was found equal 411.125 measures of 4 feet each: we shall take this then for the value of c in the case of a catenary of equal strength.

Turning then to Table III. (in which, as before, each measure is 4 feet) and taking the proportional part between 400 and 420, we have

$x =$ 12.290 measures or 49.161 feet,

$s =$ 101.002 .... 404.008 ....

$S =$ 102.024 .... 408.096 ....

$t =$ 423.602 .... 1694.408 ....

angle = $76^\circ\ 3'\ 17''$.

We have taken c at 411.125 measures or 1644.5 feet, but Table IV. is calculated for $c = 100$: and therefore each measure of this table is 16.445 feet: and the second column determines the excess of length of the respective rods over that at the apex for every gradation of y.

Let us form a Table, as before, of the differences of s and S.

	Differences of *s*.	Differences of *S*.	Ratios of these Dif.
1st measure of y	1.000	1.000	1.000
2nd	1.001	1.000	·999
....	......	......	......
....	......	......	......
21st	1.021	1.043	1.002
....	......	......	......

The fourth column gives the quantity of matter of which the chain must be composed at the various ordinates of which the values are in the first column. Also the adjunct weights of rods, road-way...should be distributed in portions proportional to the numbers of the third column.

CHAPTER VII.

PROBLEMS.

142. In the last two Chapters we have illustrated the principles of equilibrium by applying them to the solution of various questions. Our object in the present Chapter is to make some general remarks upon the solution of Statical Problems, and to give a few more applications.

143. The conditions of equilibrium of a single particle acted upon by forces which act in any directions, are three in number,

$$\Sigma . X = 0, \quad \Sigma . Y = 0, \quad \Sigma . Z = 0,$$

X, Y, Z being the resolved parts parallel to three rectangular co-ordinate axes of any one of the forces: Art. 23.

If the directions of these forces all lie in the same plane, and this plane be taken for that of xy, then the third equation becomes identical and there are only two conditions. If the forces all act in the same line and this line be taken for the axis of x, then the last two equations are identical and there is only one condition.

The conditions of equilibrium of a rigid body, or of a system of rigid bodies, acted on by forces which act in any directions, are six in number,

$$\Sigma . X = 0, \quad \Sigma . Y = 0, \quad \Sigma . Z = 0,$$

$$\Sigma . (Zy - Yz) = 0, \quad \Sigma . (Xz - Zx) = 0, \quad \Sigma . (Yx - Xy) = 0,$$

X, Y, Z being the resolved parts parallel to three rectangular co-ordinate axes of any one of the forces, and xyz the co-ordinates to the point of application of that force.

If the forces all act in the same plane and this plane be taken for the plane of xy, the third, fifth and sixth equations become identical, and there are only three equations of condition. Art. 65, 53.

144. When we wish to solve a statical problem we must consider what forces act upon the body that is to be in equilibrium: for unknown pressures and reactions we must substitute unknown forces, which we shall call *mechanical quantities*: also for unknown distances, angles of position, and so on, we must use unknown quantities; these we shall term *geometrical quantities*. After this we must write down the equations of equilibrium, the number of which will depend upon the nature of the problem, as mentioned in the last article. We must next write down the equations (if there be any) which connect the geometrical quantities. Lastly, we must count the unknown quantities involved in the equations; and if their number exceed the number of equations, it shews that the problem is indeterminate, or else that we have not written down all the equations of condition: we must therefore search for more; they must be equations connecting the geometrical quantities, since we know, by the principles of equilibrium, that there cannot be any more mechanical equations.

If in the end the number of equations be less than the number of unknown quantities, then equilibrium will subsist under several circumstances, and is said to be indeterminate; it does not follow that *all* the unknown quantities are indeterminate. If the number of unknown quantities equal the number of equations, then equilibrium will subsist in one way only. If it be found that there are more equations than unknown quantities, then the equilibrium will not subsist unless the known quantities fulfil the conditions at which we arrive by eliminating the unknown quantities from the equations.

145. It will often happen that we can materially diminish the labour of solving the equations by properly choosing the centre of moments, and the lines parallel to which we resolve the forces. Also by having regard to the object of the problem, whether it be to find the position of equilibrium of a

body, the magnitude and direction of an unknown pressure, and so on, we may frequently set aside some of the equations as having no reference to the particular point of enquiry. Thus in Art. 121. the object is to find T, the tension of the tie-beam. Upon examining the four equations we see immediately that (2) may be set aside, because it contains an unknown quantity R, which does not enter any of the other equations, and therefore (2) is of use solely to determine R, a quantity which it is not the immediate object of the problem to discover. Equation (1) gives T when P and θ are known, and these are found from (3) and (4). Again, Art. 122. gives a good illustration of an indeterminate problem. For (1) (2) (3) are the only mechanical equations that can possibly exist, and these contain only one unknown geometrical quantity x, and consequently a fourth equation does not exist, or the problem is indeterminate: as we might easily have foreseen from the nature of the case. It does not follow that every unknown quantity in the equations is indeterminate, as we see in this instance.

146. We shall now add a few Problems.

Prob. 1. A given weight W is held at rest on a known curve AP lying in a vertical plane by means of a given weight Q acting over the pully B: required the position of rest: fig. 60.

The vertical BM through B is the axis of x, B the origin, $BM = x$, $MP = y$, P being the position of the weight; angle $B = \theta$. Now the weight is held in equilibrium by Q acting in PB, W in PW, and the reaction of the curve, or R, acting in GR a normal to the curve at P: hence, resolving these forces vertically and horizontally, Art. 23. gives

$$W - Q\cos\theta - R\cos PGB = 0,$$

$$Q\sin\theta - R\sin PGB = 0;$$

$$\text{or, since } \tan PGB = \frac{dx}{dy},$$

$$W - Q\cos\theta - R\frac{dy}{ds} = 0 \ldots\ldots (1),$$

$$Q \sin\theta - R\frac{dx}{ds} = 0 \ldots\ldots (2),$$

two equations and five unknown quantities R, θ, x, y, s: since equations (1) (2) are the only conditions of equilibrium, the other three equations must be among the quantities θ, x, y, s: they are

$$\tan\theta = \frac{y}{x} \ldots\ldots (3), \quad \frac{ds}{dx} = \sqrt{1 + \frac{dy^2}{dx^2}} \ldots\ldots (4),$$

$$\text{and } \phi(x, y) = 0 \ldots\ldots (5)$$

the equation to the curve. These five equations will solve the problem when we select any particular curve.

The elimination of R from (1) and (2) gives

$$W - Q\left(\cos\theta + \frac{dy}{dx}\sin\theta\right) = 0,$$

$$\text{or } W - Q\left(\frac{x}{r} + \frac{y}{r}\frac{dy}{dx}\right) = 0 \text{ by (3)}: \; r^2 = x^2 + y^2,$$

$$\text{or } W dx - Q dr = 0$$

the equation of virtual velocities which we should have obtained from Art. 24.

Suppose the curve is a circle the centre being at a vertical distance c from the point B: then a being the radius

$$y^2 + (x - c)^2 = a^2;$$

$$\therefore r^2 = x^2 + y^2 = a^2 - c^2 + 2cx; \quad \therefore \frac{dr}{dx} = \frac{c}{r};$$

$$\therefore \frac{r}{c} = \frac{Q}{W}, \quad \frac{r^2}{c^2} = \frac{Q^2}{W^2};$$

$$\therefore x = \frac{r^2 - a^2 + c^2}{2c} = \frac{(Q^2 + W^2)c^2 - W^2a^2}{2cW^2},$$

and the position of W is known.

Prob. 2. A cord $AA_1A_2\ldots a$ is held at rest by forces acting at its extremities and at the knots $A_1A_2A_3\ldots$in given directions: having given the form of the polygonal figure of the cord required to find the relations of the forces; also to find the tensions of the portions of cord: fig. 61.

The portions of cord need not be in the same plane; but the force which acts at any knot, as P_1 at A_1, must have its direction in the plane of the portions of cord which join in A_1. Let $P_1 P_2\ldots$be the forces acting at the knots $A_1A_2\ldots$: $T\,T_1\,T_2\ldots T_n$ the tensions of the portions of cord: $\alpha_1\beta_1$, $\alpha_2\beta_2$,...the angles which the directions of $P_1P_2\ldots$make respectively with the portions of cord at the knots.

Then A_1 is held at rest by the three forces $P_1\,T_1\,T$; hence, resolving these forces in the direction of P_1 and at right angles to this, we have by Art. 23.

$$P_1 - T\cos\alpha_1 - T_1\cos\beta_1 = 0 \;\ldots\ldots\ldots\ldots\ldots\; (1),$$

$$T\sin\alpha_1 - T_1\sin\beta_1 = 0 \;\ldots\ldots\ldots\ldots\ldots\; (2).$$

Again, A_2 is held at rest by $T_1\,P_2\,T_2$; hence

$$P_2 - T_1\cos\alpha_2 - T_2\cos\beta_2 = 0 \;\ldots\ldots\ldots\ldots\; (3),$$

$$T_1\sin\alpha_2 - T_2\sin\beta_2 = 0 \;\ldots\ldots\ldots\ldots\; (4),$$

and so on: if there be n knots we shall have $2n$ equations, involving $2n+1$ unknown forces $P_1\,P_2\ldots P_n\,T\,T_1\ldots\ldots T_n$: we shall therefore have an equation of condition connecting these forces, we shall suppose T to be known.

By equations (2) (4) ...... we have

$$\frac{T_1}{T} = \frac{\sin\alpha_1}{\sin\beta_1},\quad \frac{T_2}{T_1} = \frac{\sin\alpha_2}{\sin\beta_2},\;\ldots\ldots\ldots\; \frac{T_n}{T_{n-1}} = \frac{\sin\alpha_n}{\sin\beta_n};$$

$$\therefore\; T_1 = \frac{\sin\alpha_1}{\sin\beta_1}T,\quad T_2 = \frac{\sin\alpha_1\sin\alpha_2}{\sin\beta_1\sin\beta_2}T, \text{ and so on,}$$

and the tensions are all known in terms of T.

Also by (1) (2) eliminating T_1,

$$P_1 = \frac{T\sin\beta_1}{\sin(\alpha_1+\beta_1)}, \text{ and in like manner}$$

$$P_2 = \frac{T_1 \sin \beta_2}{\sin (\alpha_2 + \beta_2)} = \frac{T \sin \alpha_1 \sin \beta_2}{\sin \beta_1 \sin (\alpha_2 + \beta_2)}, \text{ and so on.}$$

Hence all the forces $P_1 P_2 \ldots$ are known in terms of T.

We shall now solve a few problems of forces acting on a rigid body in the same plane: see Art. 52, 53. When the system consists of more than one rigid body, we shall consider each body separately.

Prob. 3. A uniform beam passing freely through a hole H in a wall rests with one end on an inclined plane: find the position of equilibrium: fig. 62.

AH horizontal $=h$, $\angle A=\alpha$, $PH=x$, $PG=a$: $\angle AHP=\theta$, pressure at $P=R$ perpendicular to the plane, pressure at H perpendicular to beam and $=Q$: resolving the forces vertically and horizontally

$$W - R \cos \alpha - Q \cos \theta = 0 \;\ldots\ldots\ldots\ldots\; (1),$$

$$R \sin \alpha - Q \sin \theta = 0 \;\ldots\ldots\ldots\ldots\; (2),$$

taking the centre of moments at P,

$$Wa \cos \theta - Qx = 0 \;\ldots\ldots\ldots\ldots\; (3),$$

these equations involve four unknown quantities R, Q, θ, x, we must search for a relation between x and θ: this is

$$\frac{x}{h} = \frac{\sin \alpha}{\sin (\alpha + \theta)} \;\ldots\ldots\ldots\ldots\ldots\ldots\; (4).$$

Our object is to determine the position of equilibrium: that is, to find x and θ: we have one equation (4), we must therefore obtain another between x and θ by eliminating R and Q from (1) (2) (3).

By (1) (2), elim^g. R, $\dfrac{W}{Q} = \dfrac{\sin (\alpha + \theta)}{\sin \alpha}$: by (3) $\dfrac{W}{Q} = \dfrac{x}{a \cos \theta}$;

$$\therefore \frac{\sin (\alpha + \theta)}{\sin \alpha} = \frac{x}{a \cos \theta}.$$

Eliminating x from this by (4) we have

$$\cos\theta \sin^2(\alpha+\theta) = \frac{h}{a}\sin^2\alpha,$$

from which θ, and therefore the position of the beam, is to be determined.

$$\text{If } \alpha = 90^\circ, \quad \cos\theta = \sqrt[3]{\frac{h}{a}}.$$

PROB. 4. A sphere and cone in contact rest, as in fig. 63, on two inclined planes, the intersection of which is a horizontal line: required the angle of the cone and the position of equilibrium.

W, W' the weights of the sphere and cone: R the reaction at B: P the mutual action at E: the resultant of the reactions of the plane on the base of the cone must act at some point D, let Q be this resultant: $CD = x$: G the centre of gravity of the cone: rad. of sphere $= a$, $Ge = z$, e being the point where the normal at E cuts the axis of the cone: $2\theta =$ the angle of the cone: α, β the angles the planes make with the horizon.

$$\text{For the sphere,} \quad W - R\cos\beta + P\sin(\alpha-\theta) = 0 \;\ldots\ldots\; (1),$$

$$R\sin\beta - P\cos(\alpha-\theta) = 0 \;\ldots\ldots\; (2).$$

The equation of moments is an identical equation.

$$\text{For the cone,} \quad W' - Q\cos\alpha - P\sin(\alpha-\theta) = 0 \;\ldots\ldots\; (3),$$

$$Q\sin\alpha - P\cos(\alpha-\theta) = 0 \;\ldots\ldots\; (4),$$

$$\text{moments about } G, \quad Qx - Pz\cos\theta = 0 \;\ldots\ldots\; (5).$$

These five equations involve six unknown quantities: if there be a sixth equation it must be a relation connecting the *geometrical* quantities involved in these five equations: but a little consideration will shew us that no necessary connexion exists between any two of x, θ, z: hence the problem is indeterminate. By examining the equations we perceive that the first four involve only the four unknown quantities P, R, Q, θ: hence these are determinate; but x and z are indeterminate since they are connected only by (5): for any given position, however, of the bodies z is known by geometry, and consequently x becomes known by (5).

We learn from this that if θ be chosen so as to satisfy equations (1) (2) (3) (4), the bodies will remain at rest in whatever position they are placed, their centres of gravity remaining in the plane of the paper: and as we give the bodies different positions z varies, consequently x and therefore the point of application of Q changes.

$$\text{By (1) (2) } \frac{P}{W} = \frac{\sin\beta}{\cos(\alpha+\beta-\theta)}; \quad \text{by (3) (4) } \frac{P}{W'} = \frac{\sin\alpha}{\cos\theta};$$

$$\therefore \frac{W}{W'} = \frac{\sin\alpha\cos(\alpha+\beta-\theta)}{\sin\beta\cos\theta} = \frac{\sin\alpha}{\sin\beta}\{\cos(\alpha+\beta)+\sin(\alpha+\beta)\tan\theta\};$$

$$\therefore \tan\theta = \frac{W\sin\beta - W'\sin\alpha\cos(\alpha+\beta)}{W'\sin\alpha\sin(\alpha+\beta)}$$

$$= \frac{(W+W')\sin\beta}{W'\sin\alpha\sin(\alpha+\beta)} - \frac{\sin(2\alpha+\beta)}{\sin\alpha\sin(\alpha+\beta)}.$$

$$\text{By (4) (5) } \frac{x}{z} = \frac{\sin\alpha\cos\theta}{\cos(\alpha-\theta)} = \frac{\sin\alpha}{\cos\alpha+\sin\alpha\tan\theta}$$

$$= \frac{W'\sin\alpha\sin(\alpha+\beta)}{W'\{\cos\alpha\sin(\alpha+\beta)-\sin\alpha\cos(\alpha+\beta)\}+W\sin\beta}$$

$$= \frac{W'\sin\alpha\sin(\alpha+\beta)}{(W+W')\sin\beta}.$$

The value of $\tan\theta$ gives the angle of the cone necessary for equilibrium, and the value of x gives the point of application of Q for any given position of the bodies.

Prob. 5. A person suspended in a balance of which the arms are equal thrusts his centre of gravity out of the vertical by means of a rod fixed to the furthest extremity of the beam of the balance, the direction of the rod passing through his centre of gravity: given that the rod and the line from the nearer end of the beam of the balance to his centre of gravity make angles α, β with the vertical, shew that his apparent and true weights are in the ratio $\sin(\alpha+\beta) : \sin(\alpha-\beta)$.

PROB. 6. A uniform beam placed in a hemispherical bowl is in equilibrium, find its position.

PROB. 7. A cylinder with its axis horizontal is supported on an inclined plane by a beam which rests upon it and has its lower extremity fastened to the plane by a hinge: find the conditions of equilibrium.

PROB. 8. Two uniform beams of equal length are loosely connected, each by one extremity, to the extremities of another uniform beam, they are then placed on a sphere; find the pressures on the sphere at the three points of contact, the length of the middle beam being less than the diameter of the sphere.

PROB. 9. To determine the conditions of equilibrium on Roberval's Balance; see Art. 99. and fig. 35.

This machine consists of five rigid bodies; and since the forces all act in the same (the vertical) plane we shall have fifteen equations: the figure will point out the meaning of the various unknown quantities, the description of which we omit here to save room.

The equilibrium of the part supporting Q gives

$$Q - R\cos\theta - R'\cos\theta' = 0 \quad\ldots\ldots\ldots\ldots (1),$$

$$R\sin\theta - R'\sin\theta' = 0 \quad\ldots\ldots\ldots\ldots (2),$$

$$Qr - Rb\sin\theta = 0 \quad\ldots\ldots\ldots\ldots (3).$$

The equilibrium of the bar CC' gives

$$V\cos\psi - R\cos\theta - S\cos\phi = 0 \quad\ldots\ldots\ldots\ldots (4),$$

$$V\sin\psi + R\sin\theta - S\sin\phi = 0 \quad\ldots\ldots\ldots\ldots (5),$$

$$Ra\sin(\alpha+\theta) - Sa'\sin(\alpha-\phi) = 0 \quad\ldots\ldots\ldots\ldots (6).$$

The equilibrium of the bar DD' gives

$$V'\cos\psi' - R'\cos\theta' - S'\cos\phi' = 0 \quad\ldots\ldots\ldots\ldots (7),$$

$$V'\sin\psi' + R'\sin\theta' - S'\sin\phi' = 0 \quad\ldots\ldots\ldots\ldots (8),$$

$$R'a\sin(\alpha-\theta') - S'a'\sin(\alpha+\phi') = 0 \quad\ldots\ldots\ldots\ldots (9).$$

The equilibrium of the part supporting P gives

$$P - S\cos\phi - S'\cos\phi' = 0 \text{ .... } (10),$$

$$S\sin\phi - S'\sin\phi' = 0 \text{ .... } (11),$$

$$Ps - Sb\sin\phi = 0 \text{ .... } (12).$$

The equilibrium of the stem and stand gives

$$W - T + V\cos\psi + V'\cos\psi' = 0 \text{ .... } (13),$$

$$V\sin\psi - V'\sin\psi' = 0 \text{ .... } (14),$$

$$Tx - V'(h+b)\sin\psi' + Vh\sin\psi \text{ .... } (15).$$

These equations contain 15 unknown quantities, namely, $R\,R'$ $V\,V'\,S\,S'\,T\,\theta\,\theta'\,\psi\,\psi'\,\phi\,\phi'\,x$ and the ratio of Q to P. Some of these must be indeterminate since (as we might have foreseen) (14) is a consequence of (2) (5) (8) (11).

To obtain the ratio $\dfrac{Q}{P}$.

By (1) (2) $\dfrac{Q}{R} = \dfrac{\sin(\theta+\theta')}{\sin\theta'}$, by (10) (11) $\dfrac{P}{S} = \dfrac{\sin(\phi+\phi')}{\sin\phi'}$;

$$\therefore \frac{Q\sin(\phi+\phi')}{P\sin(\theta+\theta')} = \frac{R\sin\phi'}{S\sin\theta'} = \frac{a'\sin(a-\phi)\sin\phi'}{a\sin(a+\theta)\sin\theta'} \text{ by (6).}$$

If we had eliminated R and S first and then $R'\,S'$,

$$\frac{Q\sin(\phi+\phi')}{P\sin(\theta+\theta')} = \frac{R'\sin\phi}{S'\sin\theta} = \frac{a'\sin(a+\phi')\sin\phi}{a\sin(a-\theta')\sin\theta} \text{ by (9).}$$

Adding these equations after multiplying them respectively by the denominators of the right-hand sides we have

$$\{\sin(a+\theta)\sin\theta' + \sin(a-\theta')\sin\theta\}\,\frac{Q\sin(\phi+\phi')}{P\sin(\theta+\theta')}$$

$$= \frac{a'}{a}\,\{\sin(a-\phi)\sin\phi' + \sin(a+\phi')\sin\phi\};$$

$$\therefore \sin a\sin(\theta+\theta')\,.\,\frac{Q\sin(\phi+\phi')}{P\sin(\theta+\theta')} = \frac{a'}{a}\sin a\sin(\phi+\phi');$$

$$\therefore\ \frac{Q}{P} = \frac{a'}{a},$$

that is, the weights must always be inversely as the arms DE', $D'E'$, and do not depend on r and s.

To find T.

Add together (1) (4) (7) (10) (13) after changing the signs in (4) (7), we have

$$T = W + P + Q.$$

To find x.

By (14) (15) $T x = Vb \sin\psi$

$$= Ps - Qr \text{ by (5) (3) (12)} = P\left(s - \frac{a'}{a} r\right);$$

$$\therefore\ x = \frac{P}{W + P + Q}\left(s - \frac{a'}{a} r\right).$$

This shews that as we shift the weights P and Q the point B, at which the reaction and consequently the resultant downward-pressure acts, shifts also. If the ratio of r and s be such that B is at C, then if P be shifted outwards or Q inwards the balance will fall moving about the point C. If the stem be fixed of course the balance will not fall; but then the *strain* upon the stem will change as we shift P and Q. The strains at the pivots are indeterminate, nevertheless they alter as P and Q are shifted.

In this way the paradoxical character of the balance is explained.

We shall illustrate the Principle of Virtual Velocities in the solution of the following problem.

PROB. 10. A beam in a vertical plane rests on a post B and against a wall at A, as represented in fig. 64: required the circumstances of equilibrium.

Distance of B from the wall $= b$: $AG = a$: $\angle\ GAD = \theta$. The reaction (P) of the post at B is perpendicular to the surfaces in contact, and therefore to the beam: the reaction (R) of the wall is perpendicular to the wall for the same reason: W the weight of the beam. We may consider the beam in equilibrium under the action of P, R, W, and suppose the post and wall removed.

Now the object of the problem might be solely to determine the position of equilibrium, or also to determine P and not R, or R and not P, or to determine both P and R and also the position of equilibrium. We shall solve the problem by the Principle of Virtual Velocities under these four suppositions in order to explain the method of proceeding so as to avoid as much trouble as possible according to the nature of the question.

1. Suppose the position of equilibrium only required. We must then give the beam a small arbitrary geometric motion such that the unknown pressures P and R shall not occur in the equation of virtual velocities: the beam must therefore remain in contact with the wall and the post: as in fig. 64.

Let $\delta\theta$ be the increase of θ owing to the displacement. Then height of G above the horizontal through B (or h)

$$= GB \cos\theta = (a - b \operatorname{cosec}\theta) \cos\theta = a \cos\theta - b \cot\theta;$$

$$\therefore \text{ vertical space described by } G = \delta h = \left(\frac{b}{\sin^2\theta} - a \sin\theta\right) d\theta,$$

and by virtual velocities $W\delta h = 0$;

$$\therefore\ b - a \sin^3\theta = 0, \quad \sin\theta = \sqrt[3]{\frac{b}{a}},$$

and this determines the *position of equilibrium*.

2. But suppose we wished to find the pressure P as well as the position of equilibrium.

We ought in this case to have moved the beam off the post, as in fig. 65, in order that the virtual velocity of B with respect to P may not vanish, and consequently P not disappear as in case (1).

Let $AA' = c$, and let, as before, $\delta\theta$ be the change of θ.

Then the space described by B in direction of P's action, (since BP is perpendicular to AB) equals the difference of the resolved parts of AA' and $A'B'$ in the direction of P

$$= AA' \sin\theta - A'B' \cos(90^\circ - \delta\theta), \quad A'B' = AB = b \operatorname{cosec}\theta$$

$$= c \sin\theta - b \operatorname{cosec}\theta\, \delta\theta.$$

Also space described by G in direction of W

$$= AG \cos\theta - AA' - A'G' \cos(\theta + \delta\theta)$$

$$= a\cos\theta - c - a\cos\theta + a\sin\theta\,\delta\theta = a\sin\theta\,\delta\theta - c;$$

therefore by the equation of virtual velocities

$$W(a\sin\theta\,\delta\theta - c) + P(c\sin\theta - b\,\text{cosec}\,\theta\,\delta\theta) = 0;$$

$$\therefore\ \delta\theta\,(Wa\sin\theta - Pb\,\text{cosec}\,\theta) - c\,(W - P\sin\theta) = 0;$$

and since c and $\delta\theta$ may be any independent small quantities

$$Wa\sin\theta - Pb\,\text{cosec}\,\theta = 0, \quad W - P\sin\theta = 0;$$

$$\therefore\ \sin\theta = \sqrt[3]{\frac{b}{a}} \text{ and } \frac{P}{W} = \sqrt[3]{\frac{a}{b}}.$$

3. Suppose we wished to know R and the position of equilibrium, and not P.

Then we should give the beam such an arbitrary motion (fig. 66.) as to give A a virtual velocity with respect to R, but not one to B with respect to P. Let $AA' = c$, $BAA' = \alpha$;

$$\therefore\ \delta\theta = \frac{c\sin\alpha}{AB - c\cos\alpha} = \frac{c}{b}\sin\alpha\sin\theta;$$ and the virtual vel. of G

$$= AG\cos\theta - c\cos(\theta - \alpha) - A'G'\cos(\theta + \delta\theta)$$

$$= \left(\frac{a}{b}\sin^2\theta - \sin\theta\right) c\sin\alpha - c\cos\alpha\cos\theta;$$

and virtual velocity of $A = c\sin(\theta - \alpha)$;

$$\therefore\ W\left\{\left(\frac{a}{b}\sin^2\theta - \sin\theta\right) c\sin\alpha - c\cos\alpha\cos\theta\right\}$$

$$+ R(c\cos\alpha\sin\theta - c\sin\alpha\cos\theta) = 0;$$

$$\therefore\ W\left(\frac{a}{b}\sin^2\theta - \sin\theta\right) - R\cos\theta = 0, \quad W\cos\theta - R\sin\theta = 0;$$

$$\therefore\ \sin\theta = \sqrt[3]{\frac{b}{a}} \text{ and } \frac{R}{W} = \frac{\sqrt{a^{\frac{2}{3}} - b^{\frac{2}{3}}}}{b^{\frac{1}{3}}}.$$

4. Lastly, suppose we wished to determine P and R and the position of equilibrium.

Then we must give the beam the most general disturbance possible in the plane of the forces: fig. 67.

$AA' = c$: $BAA' = \alpha$: and $\delta\theta$ the increase of θ;

$\therefore$ vir. vel. of A with respect to $R = c\sin(\theta - \alpha)$,

$$\dots\dots\dots\; B \;\dots\dots\dots\dots\dots\; P = c\sin\alpha - \frac{b\,\delta\theta}{\sin\theta}$$

$$\dots\dots\dots\; G \;\dots\dots\dots\dots\dots\; W = a\sin\theta\,.\,\delta\theta - c\cos(\theta - \alpha);$$

$$\therefore\; W\{a\sin\theta\,.\,\delta\theta - c\cos(\theta-\alpha)\}$$

$$+ P\left(c\sin\alpha - \frac{b\,\delta\theta}{\sin\theta}\right) + Rc\sin(\theta - \alpha) = 0;$$

$$\therefore\; c\sin\alpha\,(P - W\sin\theta - R\cos\theta) - c\cos\alpha\,(W\cos\theta - R\sin\theta)$$

$$- \delta\theta\left(Wa\sin\theta - P\frac{b}{\sin\theta}\right) = 0;$$

and $c\sin\alpha$, $c\cos\alpha$, and $\delta\theta$ are independent;

$$\therefore\; P - W\sin\theta - R\cos\theta = 0 \dots\dots (1),$$

$$W\cos\theta - R\sin\theta \quad = 0 \dots\dots (2),$$

$$Wa\sin\theta - Pb\,\text{cosec}\,\theta = 0 \dots\dots (3).$$

These three equations are the equations which we should have obtained by the principles of Art 53. they give by elimination

$$\sin\theta = \sqrt[3]{\frac{b}{a}}\,; \quad \frac{P}{W} = \left(\frac{a}{b}\right)^{\frac{1}{3}}, \quad \frac{R}{W} = \frac{\sqrt{a^{\frac{2}{3}} - b^{\frac{2}{3}}}}{b^{\frac{1}{3}}}.$$

We have thus illustrated the method of application of this principle: and we observe, in general, that when the object of the problem does not require certain unknown forces we must give the body the most arbitrary geometrical motion possible without giving the points of application of these forces any motion in their direction.

The first case of the four just solved is an application of the principle proved in Art. 78. and which was deduced from the principle of virtual velocities. We may determine whether the equilibrium be stable or unstable (Art. 79.) by differentiating h a second time:

$$\frac{dh}{d\theta} = \frac{b}{\sin^2\theta} - a\sin\theta\,; \quad \therefore\ \frac{d^2h}{d\theta^2} = -\left(\frac{2b}{\sin^3\theta} + a\right)\cos\theta,$$

which is negative when $\frac{dh}{d\theta} = 0$: hence h is a maximum and the equilibrium is unstable.

We may frequently make use of this method to discover the nature of the equilibrium.

Prob. 11. A body with a convex surface rests on a fixed body with a convex surface: required whether the equilibrium is stable or unstable: fig. 68.

Let CAO be a normal to the two surfaces at the point of contact A of the two bodies when the upper body is at rest: then the centre of gravity of the upper body is in that line: let C be its distance from O the centre of curvature at A: let a and b be the radii of curvature at A of the curves in which the plane of the paper (supposed vertical) cuts the bodies: displace the upper body through a very small angle as in the figure: angle $C = \theta$:

$\therefore\ h =$ dist. of cen. of grav. from horizontal through C,

$$= (a+b)\cos\theta - c\cos(\theta + A'O'B), \quad A'O'B = \frac{A'B}{b} = \frac{a\theta}{b}$$

$$= (a+b)\cos\theta - c\cos\left(1 + \frac{a}{b}\right)\theta$$

$$= (a+b)\left(1 - \frac{c}{b}\right) - \left\{(a+b) - \frac{c}{b^2}(a+b)^2\right\}\frac{\theta^2}{2}.$$

Hence h is a maximum or minimum, or the equilibrium is unstable or stable, according as c is $<$ or $> \frac{b^2}{a+b}$

$$\text{or as } AG \text{ is} > \text{or} < (b - c \text{ or}) \frac{ab}{a+b}.$$

We shall close this Chapter with a few examples of Problems in which Friction is considered. The only change will be that we must substitute some unknown force for the friction acting at right angles to the pressure; if we suppose the parts acted on by friction to be on the point of slipping, this force $= \mu . P$, where P is pressure of the rough surfaces and μ a constant known by experiment: see Art. 118.

Prob. 12. A cylinder with its axis horizontal is held at rest on an inclined plane by a string coiled round its middle and then fastened on the plane; fig. 69: find the conditions of equilibrium friction being considered. The forces act as drawn in the figure.

The conditions of equilibrium are

$$W - R\sin\alpha - F\cos\alpha - T\cos(\theta+\alpha) = 0 \ldots\ldots (1),$$

$$R\cos\alpha - F\sin\alpha - T\sin(\theta+\alpha) = 0 \ldots\ldots (2),$$

$$\text{moments about the axis, } Ta - Fa = 0 \ldots\ldots (3),$$

these are the only equations; and they contain four unknown quantities R, T, F, x: but we know that F cannot be greater than $\mu . R$: this limits the indeterminateness of the problem.

$$\text{Eliminate } T \text{ from (2) (3)}; \quad \therefore \frac{F}{R} = \frac{\cos\alpha}{\sin\alpha + \sin(\theta+\alpha)};$$

$$\therefore \sin\alpha + \sin(\theta+\alpha) \text{ cannot be less than } \frac{\cos\alpha}{\mu},$$

$$\theta + \alpha \text{ cannot be less than } \sin^{-1}\left(\frac{\cos\alpha - \mu\sin\alpha}{\mu}\right),$$

but it may be greater.

Prob. 13. A cylinder lies upon two equal cylinders all in contact and having their axes parallel: and the lower cylinders rest on a horizontal plane: $\mu\, \mu'$ the coefficients of friction respectively between the cylinders and each cylinder and the plane: find the conditions of equilibrium, and the relation of μ and μ' that all the points of contact may begin to slip at the same instant: fig. 70.

The forces as in the figure.

$$\text{The upper cylinder, } W - 2R\cos\alpha - 2F\sin\alpha = 0 \ldots\ldots (1),$$

the other two equations of this cylinder are identical.

One of the lower cylinders,

$$W' - R' + R\cos\alpha + F\sin\alpha = 0 \ldots\ldots (2),$$

$$F' - R\sin\alpha + F\cos\alpha = 0 \ldots\ldots (3),$$

$$F' - F = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

these are all the equations.

By (3) (4) $\dfrac{F}{R} = \dfrac{\sin\alpha}{1+\cos\alpha} = \tan\dfrac{\alpha}{2}$, not greater than μ.

By (1) (2) $2R' = 2W' + W$,

by (1) (3) (4) $F' = \dfrac{W\sin\alpha}{2(1+\cos\alpha)} = \dfrac{W}{2}\tan\dfrac{\alpha}{2}$;

$$\therefore \frac{F'}{R'} = \frac{W\tan\dfrac{\alpha}{2}}{2W' + W}, \text{ not greater than } \mu'.$$

If $\mu = \mu'$ then since W is less than $W + 2W'$ the lower cylinders will slip first as we continually increase the weight of the upper cylinder. In order that the points of contact may all slip together, we must have

$$\tan\frac{\alpha}{2} = \mu \text{ and } \frac{W\tan\dfrac{\alpha}{2}}{2W' + W} = \mu';$$

$$\therefore \frac{W'}{W} = \frac{\mu - \mu'}{2\mu'}.$$

Prob. 14. Three equal rough rods are loosely connected together by one extremity of each, and placed on a rough horizontal plane. Shew how to graduate one of the rods so that by noting the position of a smooth ring resting in a horizontal position on the rods and *just* in equilibrium we may know the coefficient of friction between the rods and the plane.

CHAPTER VIII.

ATTRACTIONS.

147. THE phenomena of the motion of the heavenly bodies lead us to conjecture, as we shall hereafter perceive, that the various particles of matter in the universe attract each other with a force which varies directly as the mass of the attracting particle and inversely as the square of the distance of the attracted from the attracting particle. Now in anticipation of this it will be an interesting and useful enquiry to calculate the resultant attraction of an assemblage of molecules which constitute a mass such as the Earth, the Sun, or any of the heavenly bodies. We shall commence with the calculations of the attraction of homogeneous bodies bounded by surfaces of the second order, and then of any homogeneous bodies differing but little in figure from a sphere, and lastly of heterogeneous bodies consisting of homogeneous strata all differing but little from spherical shells in their form. Also in the course of these calculations we shall introduce a few Propositions which we shall find of use hereafter.

PROP. *To find the resultant attraction of an assemblage of particles constituting a homogeneous spherical shell of very small thickness upon a particle outside the shell: the law of attraction of the particles being that of the inverse square of the distance.*

148. Let O be the centre of the shell (fig. 71), P any particle of it, dr its thickness: C the attracted particle $OC = c$, $\angle POC = \theta$. $OP = r$: $mPMn$ a plane perpendicular to OC, $\angle mMP = \phi$, $PC = y$.

The attraction of the whole shell C acts in CO.

Let OP revolve about O through a small angle $d\theta$ in the plane MOP: then $rd\theta$ is the space described by P. Again, let OPM revolve about OC through a small angle $d\phi$, then $r\sin\theta d\phi$ is the space described by P. Likewise the thickness of the shell equals dr. Hence the volume of the elementary solid at P equals $dr\,rd\theta\,r\sin\theta d\phi$ ultimately, since its sides are ultimately at right angles to each other.

Then, if the unit of attraction be chosen to be the attraction of a unit of mass at a unit of distance, the attraction of the elementary mass at P on C in the direction CP

$$=\frac{\rho r^2\sin\theta drd\theta d\phi}{y^2},\ \rho \text{ the density of the shell};$$

$$\therefore \text{ attraction of } P \text{ on } C \text{ in } CO=\frac{\rho r^2\sin\theta drd\theta d\phi}{y^2}\,\frac{c-r\cos\theta}{y}.$$

We shall eliminate θ from this equation by means of

$$y^2=c^2+r^2-2cr\cos\theta,$$

$$\therefore\ \sin\theta\frac{d\theta}{dy}=\frac{y}{cr},\quad c-r\cos\theta=\frac{y^2+c^2-r^2}{2c};$$

$$\therefore \text{ attrac. of } P \text{ on } C \text{ in direct. } CO=\frac{\rho rdr}{2c^2}\left(1+\frac{c^2-r^2}{y^2}\right)dyd\phi.$$

To obtain the attraction of all the particles of the shell we integrate this with respect to ϕ and y, the limits of ϕ being 0 and 2π, those of y being $c-r$ and $c+r$;

$$\therefore \text{ att}^{\text{n}}\text{. of shell on } C \text{ in } CO=\frac{\rho rdr}{2c^2}\int_{c-r}^{c+r}\int_0^{2\pi}\left(1+\frac{c^2-r^2}{y^2}\right)dyd\phi$$

$$=\frac{\pi\rho rdr}{c^2}\int_{c-r}^{c+r}\left(1+\frac{c^2-r^2}{y^2}\right)dy$$

$$=\frac{\pi\rho rdr}{c^2}(2r+2r)=\frac{4\pi\rho r^2dr}{c^2}$$

$$=\frac{\text{mass of the shell}}{c^2}.$$

This result shews that the shell attracts the particle at C in the same manner as if the mass of the shell were condensed into its centre.

149. It follows also that a sphere which is either homogeneous or consists of concentric spherical shells of uniform density will attract the particle at C in the same manner as if the whole mass were collected at its centre.

PROP. *To find the attraction of a homogeneous spherical shell of small thickness on a particle placed within it.*

150. We must proceed as in the last Proposition: but the limits of y are in this case $r - c$ and $r + c$: hence

$$\text{attraction of shell} = \frac{\pi \rho r dr}{c^2} \int_{r+c}^{r-c} \left(1 - \frac{r^2 - c^2}{y^2}\right) dy$$

$$= \frac{\pi \rho r dr}{c^2} (2c - 2c) = 0;$$

therefore a particle within the shell is equally attracted in every direction.

PROP. *To find the attraction of a homogeneous spherical shell on a particle without it; the law of attraction being represented by* $\phi(y)$, y *being the distance.*

151. The calculation is exactly analogous to that of Art. 148: we have only to alter the law of attraction: then attraction on C in CO

$$= \frac{\pi \rho r dr}{c^2} \int_{c-r}^{c+r} (y^2 + c^2 - r^2)\, \phi(y)\, dy, \text{ (integrated by parts)}$$

$$= \frac{\pi \rho r dr}{c^2} \left\{ (y^2 + c^2 - r^2) \int \phi(y)\, dy - 2 \int [y \int \phi(y)\, dy]\, dy \right\}$$

$$= \frac{\pi \rho r dr}{c^2} \left\{ (y^2 + c^2 - r^2)\, \phi_1(y) - 2\psi(y) + \text{const.} \right\} \text{ suppose}$$

between the specified limits

$$= 2\pi \rho r dr \left\{ \frac{c+r}{c} \phi_1(c+r) - \frac{1}{c^2} \psi(c+r) - \frac{c-r}{c} \phi_1(c-r) + \frac{1}{c^2} \psi(c-r) \right\}$$

$$= 2\pi \rho r dr \frac{d}{dc} \left\{ \frac{\psi(c+r) - \psi(c-r)}{c} \right\},$$

this latter form being introduced merely as an analytical artifice to simplify the expression.

PROP. *To find the attraction of the shell on an internal particle.*

152. The calculation is the same as in the last article except that the limits of y are $r - c$ and $r + c$;

$$\therefore \text{ attraction} = 2\pi\rho r dr \left\{\frac{r+c}{c}\phi_1(r+c) - \frac{1}{c^2}\psi(r+c) + \frac{r-c}{c}\phi_1(r-c) + \frac{1}{c^2}\psi(r-c)\right\}$$

$$= 2\pi\rho r dr \frac{d}{dc}\left\{\frac{\psi(r+c) - \psi(r-c)}{c}\right\}.$$

The formulæ of these two Articles will give the attraction when the law of attraction is known.

Ex. 1. Let $\phi(r) = \frac{1}{r^2}$;

$$\therefore \phi_1(r) = -\frac{1}{r} + A$$

$\psi(r) = -r + \frac{A}{2}r^2 + B$: A and B arbitrary constants;

therefore attraction on an external particle

$$= 2\pi\rho r dr \frac{d}{dc}\left\{\frac{-4r + A\{(c+r)^2 - (c-r)^2\}}{2c}\right\}$$

$$= 2\pi\rho r dr \frac{d}{dc}\left\{\frac{-2r}{c} + 2Ar\right\}$$

$$= \frac{4\pi\rho r^2 dr}{c^2}, \text{ (see Art. 148.)}$$

Attraction on an internal particle

$$= 2\pi\rho r dr \frac{d}{dc}\left\{\frac{-4c + A\{(r+c)^2 - (r-c)^2\}}{2c}\right\}$$

$$= 2\pi\rho r dr \frac{d}{dc}\{-2 + 2Ar\}$$

$$= 0, \text{ (see Art. 150.)}$$

Ex. 2. Let $\phi(r) = r$;

$$\therefore\ \phi_1(r) = \frac{r^2}{2} + A, \quad \psi(r) = \frac{r^4}{8} + \frac{A}{2}r^2 + B.$$

Attraction on an external particle

$$= 2\pi\rho r dr \frac{d}{dc}\left\{\frac{(c+r)^4 - (c-r)^4 + 4A\{(c+r)^2 - (c-r)^2\}}{8c}\right\}$$

$$= 2\pi\rho r dr \frac{d}{dc}\{c^2 r + r^3 + 2Ar\}$$

$$= 4\pi\rho r^2 dr c = \text{mass of shell} \times c.$$

The attraction is the same as if the shell were collected at its centre. This property we discovered for the law of the inverse square. We shall now ascertain whether there are any other laws which give the same property.

Prop. *To find what laws of attraction allow us to suppose a spherical shell condensed into its centre when attracting an external particle.*

153. Let $\phi(r)$ be the law of force: then if c be the distance of the centre of the shell from the attracted point and r the radius of the shell, and $\psi(r) = \int\{r\int\phi(r)\,dr\}\,dr$, then the attraction of the shell

$$= 2\pi\rho r dr \frac{d}{dc}\left\{\frac{\psi(c+r) - \psi(c-r)}{c}\right\}.$$

But if the shell be condensed into its centre this attraction

$$= 4\pi r^2 dr \rho\phi(c);$$

$$\therefore\ 2r\phi(c) = \frac{d}{dc}\left\{\frac{\psi(c+r) - \psi(c-r)}{c}\right\}$$

$$\therefore 2r\phi(c) = 2\frac{d}{dc}\left\{\frac{d\psi c}{dc}\frac{r}{c} + \frac{d^3\psi c}{dc^3}\frac{r^3}{c}\frac{1}{1.2.3} + \dots\right\}$$

$$= 2r\phi(c) + 2\frac{d}{dc}\left\{\frac{d^3\psi(c)}{dc^3}\frac{r^3}{c}\frac{1}{1.2.3} + \dots\right\};$$

$$\therefore \frac{d}{dc}\left\{\frac{1}{c}\frac{d^3\psi(c)}{dc^3} + \dots\right\} = 0 \text{ whatever } r \text{ be};$$

$$\therefore \frac{d}{dc}\left\{\frac{1}{c}\frac{d^3\psi c}{dc^3}\right\} = 0, \quad \frac{d}{dc}\left\{\frac{1}{c}\frac{d^5\psi c}{dc^5}\right\} = 0\dots$$

But $\frac{d\psi c}{dc} = c\int\phi(c)\,dc$, $\frac{d^2\psi c}{dc^2} = \int\phi(c)\,dc + c\phi c$

$$\frac{d^3\psi c}{dc^3} = 2\phi c + c\frac{d\phi c}{dc};$$

therefore by the first of the above equations of condition for $\psi(c)$

$$\frac{2}{c}\phi c + \frac{d\phi c}{dc} = 3A,$$

and multiplying by c^2 and integrating

$c^2\phi(c) = Ac^3 + B$: A and B being independent of c

$$\phi(c) = Ac + \frac{B}{c^2},$$

and this satisfies all the other equations of condition for $\psi(c)$; therefore the required laws of attraction are those of the direct distance, the inverse square, and a law compounded of these.

Prop. *To find for what laws the shell attracts an internal point equally in every direction.*

154. When this is the case

$$\frac{d}{dc}\left\{\frac{\psi(r+c) - \psi(r-c)}{c}\right\} = 0$$

$$\frac{d\psi(r)}{dr} + \frac{d^3\psi(r)}{dr^3}\cdot\frac{c^2}{1.2.3} + \dots = -A$$

whatever c is, A being a constant independent of c;

$$\therefore \frac{d\psi(r)}{dr} = -A, \quad \frac{d^3\psi(r)}{dr^3} = 0 \ldots\ldots\ldots$$

These conditions are all satisfied if the first is: this gives

$$r\int\phi(r)\,dr = -A, \quad \phi(r) = \frac{A}{r^2},$$

and therefore the inverse square is the only law which satisfies the condition.

Prop. *To find the attraction of a homogeneous oblate spheroid on a particle at its pole: the law being the inverse square of the distance.*

155. Let $APBp$, $AQBq$ be sections of the spheroid and the sphere touching it made by a plane through the axis of the spheroid: fig. 72. $AM = x$, $MP = y$, $AC = c$, $CD = a$, $c = a(1-\epsilon)$, ϵ very small. The mass of the annulus Pp between the sphere and spheroid and of thickness dx

$$= \pi\rho y^2 dx \left(1 - \frac{c^2}{a^2}\right) : \text{ also } AQ = \sqrt{2cx}\,; \; y^2 = \frac{a^2}{c^2}(2cx - x^2),$$

and if we consider every particle of the annulus Pp equidistant from A, the attraction of this annulus on A in direction AB

$$= \pi\rho y^2 dx \left(1 - \frac{c^2}{a^2}\right) \frac{1}{2cx} \frac{x}{\sqrt{2cx}} = \frac{2\pi\rho\epsilon}{(2c)^{\frac{3}{2}}} (2cx^{\frac{1}{2}} - x^{\frac{3}{2}})\,dx\,;$$

therefore attraction of whole difference of sphere and spheroid

$$= \frac{2\pi\rho\epsilon}{(2c)^{\frac{3}{2}}} \int_0^{2c} (2cx^{\frac{1}{2}} - x^{\frac{3}{2}})\,dx$$

$$= 2\pi\rho\epsilon \left\{\frac{4c}{3} - \frac{4c}{5}\right\} = \frac{16\pi\rho\epsilon c}{15}$$

the attraction of the sphere on $A = \dfrac{4\pi\rho c}{3}$ (Art. 149.)

therefore attraction of spheroid on $A = \dfrac{4\pi\rho}{3}\left(1 + \dfrac{4\epsilon}{5}\right) c.$

Prop. *To find the attraction on a particle at the equator.*

156. Let DC be the axis of revolution (fig. 73), $APBp$ and $AQBq$ sections of the spheroid and circumscribing sphere by the plane of the paper passing through the axis of revolution: $AM = x$, $MP = y$, $AC = a$, $CD = c$, $y^2 = \frac{c^2}{a^2}(2ax - x^2)$.

Let an elementary slice of the spheroid and sphere be made by planes perpendicular to the axis of x, one passing through P and the other at a distance dx from it; therefore mass of the part of this slice between the sphere and spheroid

$$= \pi\rho\,(QM^2 - MN\,.\,PM)\,dx$$

$$= \pi\rho\left\{\frac{a^2}{c^2}\,y^2 - \frac{a}{c}\,y^2\right\}dx\text{; because } MN = QM$$

$$= \pi\rho\left(1 - \frac{c}{a}\right)(2ax - x^2)\,dx.$$

Now the distance of each portion of this from A nearly $= AQ = \sqrt{2ax}$; therefore attraction of the part between the sphere and spheroid in the direction AC

$$= \pi\rho\left(1 - \frac{c}{a}\right)\int_0^{2a}(2ax - x^2)\,\frac{x\,dx}{(2ax)^{\frac{3}{2}}}$$

$$= \pi\rho\left(1 - \frac{c}{a}\right)\int_0^{2a}\frac{1}{(2a)^{\frac{3}{2}}}\left(2ax^{\frac{1}{2}} - x^{\frac{3}{2}}\right)dx$$

$$= \pi\rho\left(1 - \frac{c}{a}\right)\left(\frac{4a}{3} - \frac{4a}{5}\right) = \frac{8\pi\rho a\epsilon}{15},$$

and the attraction of the sphere $= \frac{4\pi\rho}{3}\,a$;

therefore attraction of the spheroid $= \frac{4\pi\rho}{3}\left(1 - \frac{2\epsilon}{5}\right)a$

$$= \frac{4\pi\rho}{3}\left(1 + \frac{3\epsilon}{5}\right)c.$$

157. In the same manner it might be shewn that the attractions of a homogeneous prolate spheroid of small ellipticity on particles at the pole and equator are respectively

$$\frac{4\pi\rho}{3}\left(1-\frac{4\epsilon}{5}\right)c \text{ and } \frac{4\pi\rho}{3}\left(1-\frac{3\epsilon}{5}\right)c,$$

$2c$ being the axis of revolution of the spheroid.

Prop. *To find the attraction of a homogeneous oblate spheroid upon a particle within its mass: the law of attraction being that of the inverse square of the distance.*

158. Let a, c be the semi-axes, the minor-axis of $2c$ coinciding with the axis of z: then the equation to the spheroid from the centre is

$$\frac{x^2+y^2}{a^2}+\frac{z^2}{c^2}=1,$$

fgh the co-ordinates to the attracted particle: we shall take this as the origin of polar co-ordinates, A in fig. 30.

r = radius vector of any particle of the attracting mass:

θ = angle which r makes with a line parallel to z:

ϕ = angle which the plane rz makes with the plane xz:

$\therefore\ x=f+r\sin\theta\cos\phi,\quad y=g+r\sin\theta\sin\phi,\quad z=h+r\cos\theta,$

and the equation to the spheroid becomes

$$\frac{(f+r\sin\theta\cos\phi)^2+(g+r\sin\theta\sin\phi)^2}{a^2}+\frac{(h+r\cos\theta)^2}{c^2}=1,$$

$$\text{or } r^2\left\{\frac{\sin^2\theta}{a^2}+\frac{\cos^2\theta}{c^2}\right\}+2r\left\{\frac{f\sin\theta\cos\phi+g\sin\theta\sin\phi}{a^2}+\frac{h\cos\theta}{c^2}\right\}$$

$$=1-\frac{f^2+g^2}{a^2}-\frac{h^2}{c^2}$$

$$\text{put } \frac{\sin^2\theta}{a^2}+\frac{\cos^2\theta}{c^2}=K,\quad \frac{f\sin\theta\cos\phi+g\sin\theta\sin\phi}{a^2}+\frac{h\cos\theta}{c^2}=F$$

and $F^2 + K\left(1 - \frac{f^2+g^2}{a^2} - \frac{h^2}{c^2}\right) = H$, then

$$K^2r^2 + 2KFr + F^2 = H$$

and the values of r are

$$r' = \frac{-F+\sqrt{H}}{K} \text{ and } r'' = \frac{-F-\sqrt{H}}{K}.$$

Volume of element at $P = r^2 \sin\theta\, dr\, d\theta\, d\phi$ as in p. 68. let ρ be the density of the spheroid: then the attraction of this element on the attracted particle is $\rho \sin\theta\, dr\, d\theta\, d\phi$; and the resolved parts of this parallel to the axes of xyz are

$$\rho \sin^2\theta \cos\phi\, dr\, d\theta\, d\phi, \quad \rho \sin^2\theta \sin\phi\, dr\, d\theta\, d\phi$$
$$\text{and } \rho \sin\theta \cos\theta\, dr\, d\theta\, d\phi.$$

Let A, B, C be the attractions of the whole spheroid in the directions of the axes estimated positive towards the centre of the spheroid: then these equal the integrals of the attractions of the element; the limits of r being $-r'$ and r'', of θ being 0 and π, and of ϕ being 0 and π: hence

$$A = -\int_{-r'}^{r''}\int_0^{\pi}\int_0^{\pi} \rho \sin^2\theta \cos\phi\, dr\, d\theta\, d\phi,$$

$$B = -\int_{-r'}^{r''}\int_0^{\pi}\int_0^{\pi} \rho \sin^2\theta \sin\phi\, dr\, d\theta\, d\phi$$

$$\text{and } C = -\int_{-r'}^{r''}\int_0^{\pi}\int_0^{\pi} \rho \sin\theta \cos\theta\, dr\, d\theta\, d\phi.$$

$$\text{Then } A = -\rho\int_0^{\pi}\int_0^{\pi} (r''+r') \sin^2\theta \cos\phi\, d\theta\, d\phi$$

$$= 2\rho\int_0^{\pi}\int_0^{\pi} \frac{F}{K} \sin^2\theta \cos\phi\, d\theta\, d\phi.$$

Now it is easily seen that if $R(\sin\alpha, \cos^2\alpha)$ be a rational function of $\sin\alpha$ and $\cos^2\alpha$ then $\int_0^{\pi} R(\sin\alpha, \cos^2\alpha)\cos\alpha\, d\alpha = 0$. Wherefore by substituting for F and K we have

$$A = 2f\rho c^2 \int_0^\pi \int_0^\pi \frac{\sin^3\theta \cos^2\phi \, d\theta \, d\phi}{c^2 \sin^2\theta + a^2 \cos^2\theta}$$

$$= \pi f \rho c^2 \int_0^\pi \frac{\sin^3\theta \, d\theta}{c^2 \sin^2\theta + a^2 \cos^2\theta}$$

$$= \pi f \rho c^2 \int_0^\pi \frac{(1 - \cos^2\theta) \sin\theta \, d\theta}{c^2 + (a^2 - c^2) \cos^2\theta} \, *$$

$$= \pi f \rho \frac{c^2}{a^2 - c^2} \int_0^\pi \left\{ \frac{a^2 \sin\theta}{c^2 + (a^2 - c^2) \cos^2\theta} - \sin\theta \right\} d\theta$$

$$= \pi f \rho \frac{c^2}{a^2 - c^2} \left\{ - \frac{a^2}{c\sqrt{a^2 - c^2}} \tan^{-1} \left(\frac{\sqrt{a^2 - c^2}}{c} \cos\theta \right) + \cos\theta + C \right\}$$

between specified limits,

$$= 2\pi f \rho \frac{c^2}{a^2 - c^2} \left\{ \frac{a^2}{c\sqrt{a^2 - c^2}} \tan^{-1} \frac{\sqrt{a^2 - c^2}}{c} - 1 \right\}, \quad \frac{c^2}{a^2} = 1 - e^2$$

$$= 2\pi f \rho \left\{ \frac{\sqrt{1 - e^2}}{e^3} \tan^{-1} \frac{e}{\sqrt{1 - e^2}} - \frac{1 - e^2}{e^2} \right\}$$

$$= 2\pi f \rho \left\{ \frac{\sqrt{1 - e^2}}{e^3} \sin^{-1} e - \frac{1 - e^2}{e^2} \right\}.$$

In the same manner we should find that

$$B = 2\pi g \rho \left\{ \frac{\sqrt{1 - e^2}}{e^3} \sin^{-1} e - \frac{1 - e^2}{e^2} \right\}.$$

Also $C = 2\rho \int_0^\pi \int_0^\pi \frac{F}{K} \sin\theta \cos\theta \, d\theta \, d\phi$

$$= 2\rho h a^2 \int_0^\pi \int_0^\pi \frac{\sin\theta \cos^2\theta \, d\theta \, d\phi}{c^2 \sin^2\theta + a^2 \cos^2\theta}$$

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* If the spheroid be *prolate* c is $> a$ and the denominator of this must be written $c^2 - (c^2 - a^2) \cos^2\theta$, and the integral would involve logarithms instead of circular arcs.

$$= 2\pi\rho h \frac{a^2}{a^2 - c^2} \int_0^\pi \left\{ \sin\theta - \frac{c^2 \sin\theta}{c^2 + (a^2 - c^2)\cos^2\theta} \right\} d\theta$$

$$= 4\pi\rho h \frac{a^2}{a^2 - c^2} \left\{ 1 - \frac{c}{\sqrt{a^2 - c^2}} \tan^{-1} \frac{\sqrt{a^2 - c^2}}{c} \right\}$$

$$= 4\pi\rho h \left\{ \frac{1}{e^2} - \frac{\sqrt{1 - e^2}}{e^3} \sin^{-1} e \right\}.$$

159. Cor. 1. We see from these expressions that the attraction is independent of the magnitude of the spheroid, and depends solely upon the eccentricity.

Hence the attraction of the spheroid similar to the given one and passing through the attracted particle is the same as of any other similar concentric spheroid comprising the attracted particle in its mass. Hence a spheroidal shell the surfaces of which are similar and concentric, attracts a point within it equally in all directions.

160. Cor. 2. If we put the ellipticity of the spheroid $= \epsilon$ and suppose ϵ very small so that we may neglect its square, we have $e^2 = 1 - \frac{c^2}{a^2} = 1 - (1 - \epsilon)^2 = 2\epsilon$;

$$\therefore\ A = \frac{4\pi\rho}{3}\left(1 - \frac{2\epsilon}{5}\right) f, \quad B = \frac{4\pi\rho}{3}\left(1 - \frac{2\epsilon}{5}\right) g,$$

$$C = \frac{4\pi\rho}{3}\left(1 + \frac{4\epsilon}{5}\right) h.$$

161. Cor. 3. By the values of A, B, C after integrating with respect to r we have

$$\frac{A}{f} + \frac{B}{g} + \frac{C}{h} = 2\rho \int_0^\pi \int_0^\pi \frac{(c^2 \sin^3\theta + a^2 \sin\theta \cos^2\theta)\, d\theta\, d\phi}{c^2 \sin^2\theta + a^2 \cos^2\theta}$$

$$= 2\rho \int_0^\pi \int_0^\pi \sin\theta\, d\theta\, d\phi = 2\pi\rho \int_0^\pi \sin\theta\, d\theta = 4\pi\rho.$$

But if $V = \int \frac{\text{element of mass}}{\text{distance from attracted point}}$

$$= \int \frac{dm}{\{(x - f)^2 + (y - g)^2 + (z - h)^2\}^{\frac{1}{2}}};$$

$$\therefore -\frac{dV}{df} = -\int \frac{dm\,(x-f)}{\{(x-f)^2+(y-g)^2+(z-h)^2\}^{\frac{3}{2}}} = A;$$

$$\therefore -\frac{d^2V}{df^2} = \frac{dA}{df} = \frac{A}{f} \text{ by the form of } A.$$

In the same manner $-\dfrac{d^2V}{dg^2} = \dfrac{B}{g}, \quad -\dfrac{d^2V}{dh^2} = \dfrac{C}{h}.$

Hence for an internal particle

$$\frac{d^2V}{df^2} + \frac{d^2V}{dg^2} + \frac{d^2V}{dh^2} = -4\pi\rho.$$

162. Cor. 4. If we had taken an ellipsoid instead of a spheroid we should have had

$$A = \frac{3fM}{a^3}L, \quad B = \frac{3gM}{a^3}\,\frac{d(\lambda L)}{d\lambda}, \quad C = \frac{3hM}{a^3}\,\frac{d(\lambda' L)}{d\lambda'},$$

where M is the mass of the ellipsoid, $\lambda^2 = \dfrac{a^2-b^2}{a^2}$, $\lambda'^2 = \dfrac{a^2-c^2}{a^2}$,

$$\text{and } L = \int_0^1 \frac{x^2\,dx}{\sqrt{1-\lambda^2x^2}\,\sqrt{1-\lambda'^2x^2}},$$

the integration of this depends upon the properties of elliptic transcendants: see Legendre's *Traité des Fonctions Elliptiques*, Tome I, p. 545.

163. Cor. 5. If we wished to find the attraction on an external particle we should have the same integrals for A, B, C as in the Proposition, but the limits of r would be r' and r'' (and not $-r'$ and r''), since the point from which r is measured, the attracted particle, is outside the spheroid;

$$\therefore A = -\rho\int_{r'}^{r''}\int_0^\pi\int_0^\pi \sin^2\theta\cos\phi\,dr\,d\theta\,d\phi$$

$$= \rho\int_0^\pi\int_0^\pi (r'-r'')\sin^2\theta\cos\phi\,d\theta\,d\phi$$

$$= 2\rho\int_0^\pi\int_0^\pi \frac{\sqrt{H}}{K}\sin^2\theta\cos\phi\,d\theta\,d\phi,$$

and this cannot be integrated by any known method.

Mr Ivory has, however, discovered a relation between the attractions of ellipsoids on external and internal particles: so that by means of this relation we can calculate the attraction on an external particle.

PROP. *To enunciate and prove Ivory's Theorem.*

164. Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1$

be the equations to the surfaces bounding two homogeneous ellipsoids having the same centre and foci: then

$$a^2 - b^2 = \alpha^2 - \beta^2, \quad a^2 - c^2 = \alpha^2 - \gamma^2 \text{ .... } (1).$$

Let fgh, $f'g'h'$ be the co-ordinates to two particles so situated on the surfaces of these ellipsoids that

$$\frac{f}{f'} = \frac{a}{\alpha}, \quad \frac{g}{g'} = \frac{b}{\beta}, \quad \frac{h}{h'} = \frac{c}{\gamma} \text{ .... } (2).$$

Also since (fgh) and $(f'g'h')$ are points in the surfaces of the first and second ellipsoids respectively, we have

$$\frac{f^2}{a^2} + \frac{g^2}{b^2} + \frac{h^2}{c^2} = 1, \quad \frac{f'^2}{\alpha^2} + \frac{g'^2}{\beta^2} + \frac{h'^2}{\gamma^2} = 1 \text{ ...... } (3).$$

Then *the attraction of the first ellipsoid parallel to the axis of* z *on the particle situated at the point* (f′g′h′) *on the surface of the second is to the attraction of the second ellipsoid on the particle situated at the point* (fgh) *on the surface of the first in the same direction as* ab: $\alpha\beta$ *the law of attraction being any function of the distance: and similarly with respect to the axes of* y *and* z. This is Ivory's Theorem. We shall, for convenience, represent the law of attraction by the function $r\phi(r^2)$, r being the distance.

The attraction of the first ellipsoid on the particle $(f'g'h')$ parallel to the axis of z

$$= \rho \iiint (h' - z)\, \phi \left\{(f' - x)^2 + (g' - y)^2 + (h' - z)^2\right\} dx\, dy\, dz,$$

the limits of z are $-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$ and $c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$, the limits of y are $-b\sqrt{1-\frac{x^2}{a^2}}$ and $b\sqrt{1-\frac{x^2}{a^2}}$, the limits of x are $-a$ and a

$$=\rho\iint\{\psi[(f'-x)^2+(g'-y)^2+(h'+z)^2]$$
$$-\psi[(f'-x)^2+(g'-y)^2+(h'-z)^2]\}\,dx\,dy$$

between the specified limits: $\psi(r)=2\int\phi(r)\,dr$: it must be remembered that in this expression $z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$, we do not substitute this value merely for preserving the functions under as simple a form as possible. Now put $x=ar$, $y=bs$, $z=ct$, then the attraction

$$=\rho ab\iint\{\psi[(f'-ar)^2+(g'-bs)^2+(h'+ct)^2]$$
$$-\psi[(f'-ar)^2+(g'-bs)^2+(h'-ct)^2]\}\,dr\,ds,$$

the limits of s being $-\sqrt{1-r^2}$ and $\sqrt{1-r^2}$, and those of r being -1 and 1: also $t=\sqrt{1-r^2-s^2}$.

$$\text{Now }(f'-ar)^2+(g'-bs)^2+(h'\pm ct)^2$$
$$=f'^2+g'^2+h'^2-2(f'ar+g'bs\pm h'ct)+a^2r^2+b^2s^2+c^2t^2,$$

substituting for h'^2 by (3) and putting $1-r^2-s^2$ for t^2

$$=f'^2\left(1-\frac{\gamma^2}{\alpha^2}\right)+g'^2\left(1-\frac{\gamma^2}{\beta^2}\right)+\gamma^2-2(f'ar+g'bs\pm h'ct)$$
$$+(a^2-c^2)r^2+(b^2-c^2)s^2+c^2;$$

eliminating $f'g'h'$ by (2) and making use of (1)

$$=\frac{f^2}{a^2}(a^2-c^2)+\frac{g^2}{b^2}(b^2-c^2)+c^2-2(f\alpha r+g\beta s\pm h\gamma t)$$
$$+(\alpha^2-\gamma^2)r^2+(\beta^2-\gamma^2)s^2+\gamma^2$$

$$= f^2 + g^2 + h^2 - 2\,(f\alpha r + g\beta s \pm h\gamma t) + \alpha^2 r^2 + \beta^2 s^2 + \gamma^2 t^2 \text{ by (3)}$$
$$= (f - \alpha r)^2 + (g - \beta s)^2 + (h \pm \gamma t)^2.$$

Hence the attraction of the first ellipsoid on $(f'g'h')$ parallel to z

$$= \rho a b \iint \{\psi\,[(f - \alpha r)^2 + (g - \beta s)^2 + (h + \gamma t)^2]$$
$$- \psi\,[(f - \alpha r)^2 + (g - \beta s)^2 + (h - \gamma t)^2]\}\; dr ds,$$

the limits of s being $-\sqrt{1 - r^2}$, $\sqrt{1 - r^2}$; of r being -1, 1

$$= \frac{ab}{\alpha\beta} \times \text{attraction of second ellipsoid on } (fgh) \text{ parallel to } z:$$

the same may be proved for the attractions parallel to the other axes: and consequently the Theorem, as enunciated, is true.

We observe that one of these ellipsoids lies wholly within the other: for if not the points in which they cut each other lie in the line of which the equations are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ and } \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1.$$

We shall suppose a less than α: the points of intersection must therefore satisfy the equation

$$x^2\left(\frac{1}{a^2} - \frac{1}{\alpha^2}\right) + y^2\left(\frac{1}{b^2} - \frac{1}{\beta^2}\right) + z^2\left(\frac{1}{c^2} - \frac{1}{\gamma^2}\right) = 0,$$

and this by (1) becomes $\left(\frac{x}{a\alpha}\right)^2 + \left(\frac{y}{b\beta}\right)^2 + \left(\frac{z}{c\gamma}\right)^2 = 0$, an equation which can be satisfied solely by $x = 0$, $y = 0$, $z = 0$: but these do not satisfy the equations above, and therefore the surfaces do not intersect in any point.

Hence to find the attraction of an ellipsoid of which the semi-axes are a, b, c on an external particle of which the co-ordinates are $f'g'h'$, we must first calculate the attraction of an ellipsoid of which the semi-axes are α, β, γ parallel to the axes on an internal particle of which the co-ordinates are f, g, h, these six quantities being determined by the equations

$$\alpha^2 - \beta^2 = a^2 - b^2, \quad \alpha^2 - \gamma^2 = a^2 - c^2, \quad \frac{f'^2}{\alpha^2} + \frac{g'^2}{\beta^2} + \frac{h'^2}{\gamma^2} = 1,$$

$$f = \frac{af'}{a}, \quad g = \frac{bg'}{\beta}, \quad h = \frac{ch'}{\gamma},$$

and then the attractions required will be these three calculated attractions multiplied respectively by

$$\frac{bc}{\beta\gamma}, \quad \frac{ac}{a\gamma}, \quad \frac{ab}{a\beta}.$$

The following Proposition we shall find of use in a subsequent part of this work.

Prop. *To prove that the resultant attraction of the particles of a body of any figure upon a body of which the distance is very great in comparison with the greatest diameter of the attracting body, is very nearly the same, as if the particles were condensed into their centre of gravity and attracted according to the same law, whatever that law be.*

165. Let the origin of co-ordinates be taken at the centre of gravity of the attracting body, the axis of x through the attracted particle; let c be its abscissa and xyz the co-ordinates of any particle of the body, ρ the density of that particle.

Then the distance between these two particles, or r, $= \sqrt{(c-x)^2 + y^2 + z^2}$.

Let $r\phi(r^2)$ be the law of attraction: then the whole attraction parallel to the axis of x, or A

$$= \iiint \rho\,(c-x)\,\phi\,(c^2 - 2cx + x^2 + y^2 + z^2)\,dx\,dy\,dz,$$

the limits being obtained from the equation to the surface of the body

$$= \iiint \rho\,(c-x)\,\{\phi\,(c^2) - (2cx - x^2 - y^2 - z^2)\,\phi'(c^2) + \ldots\}\,dx\,dy\,dz$$

$$= c\phi(c^2)\iiint \rho\left\{1 - \frac{x}{c}\left(1 + \frac{2c^2\phi'(c^2)}{\phi(c^2)}\right) + (y^2 + z^2 - x^2)\frac{\phi'(c^2)}{\phi(c^2)} + \ldots\right\}dx\,dy\,dz$$

$$= Mc\phi\,(c^2) + c^3\phi'(c^2)\iiint \rho\,\frac{y^2 + z^2 - x^2}{c^2}\,dx\,dy\,dz + \ldots\ldots$$

M being the mass of the body: also $\iiint \rho\,x\,dx\,dy\,dz = 0$ since x is measured from the centre of gravity of the body (p. 67).

Now suppose xyz to be exceedingly small in comparison with c; then all the terms of A after the first are extremely small in comparison with that term, it being observed that $c^3\phi'(c^2)$ is of the same order as $c\phi(c^2)$ in terms of c. Hence the Proposition is true.

Cor. It appears also that to produce a given resultant law, the law of attraction of the constituent molecules must be the same.

166. We shall now proceed to the calculation of the attraction of bodies differing but little from a sphere in figure. The object of these calculations will be seen when we come to the higher branches of Physical Astronomy. The reader may therefore, if he please, omit the remainder of this Chapter till he enters upon those investigations. We shall suppose that the law of attraction is that of the inverse square of the distance.

Prop. *To obtain formulæ for the calculation of the attraction of a heterogeneous mass upon any particle.*

167. Let ρ be the density of the body at the point (xyz): fgh the co-ordinates of the attracted particle: and, as before, suppose A, B, C are the attractions parallel to the axes of x, y, z. Then

$$A = \iiint \frac{\rho\,(f-x)\,dx\,dy\,dz}{\{(f-x)^2+(g-y)^2+(h-z)^2\}^{\frac{3}{2}}},$$

$$B = \iiint \frac{\rho\,(g-y)\,dx\,dy\,dz}{\{(f-x)^2+(g-y)^2+(h-z)^2\}^{\frac{3}{2}}},$$

$$C = \iiint \frac{\rho\,(h-z)\,dx\,dy\,dz}{\{(f-x)^2+(g-y)^2+(h-z)^2\}^{\frac{3}{2}}},$$

the limits being determined by the equation to the surface of the body.

$$\text{Let } V = \iiint \frac{\rho\,dx\,dy\,dz}{\{(f-x)^2+(g-y)^2+(h-z)^2\}^{\frac{1}{2}}};$$

$$\therefore\ A = -\frac{dV}{df}, \quad B = -\frac{dV}{dg}, \quad C = -\frac{dV}{dh}.$$

It follows, then, that the calculation of the attractions *A*, *B*, *C* depends upon that of *V*. This function cannot be calculated except when expanded into a series: it satisfies a differential equation, which leads to some remarkable properties of the coefficients of the terms of the series into which *V* is developed. We proceed to determine this equation.

PROP. *To prove that* $\frac{d^2V}{df^2}+\frac{d^2V}{dg^2}+\frac{d^2V}{dh^2}=0$, *or* $-4\pi\rho'$, *according as the attracted particle is not or is part of the mass itself: ρ' being the density of the attracted particle in the latter case.*

168. By differentiating *V* we have

$$\frac{dV}{df}=\iiint\frac{-\rho\,(f-x)\,dx\,dy\,dz}{\{(f-x)^2+(g-y)^2+(h-z)^2\}^{\frac{3}{2}}},$$

$$\frac{d^2V}{df^2}=\iiint\frac{\rho\,\{2\,(f-x)^2-(g-y)^2-(h-z)^2\}\,dx\,dy\,dz}{\{(f-x)^2+(g-y)^2+(h-z)^2\}^{\frac{5}{2}}}.$$

In the same manner we should have

$$\frac{d^2V}{dg^2}=\iiint\frac{\rho\,\{2\,(g-y)^2-(f-x)^2-(h-z)^2\}\,dx\,dy\,dz}{\{(f-x)^2+(g-y)^2+(h-z)^2\}^{\frac{5}{2}}},$$

$$\frac{d^2V}{dh^2}=\iiint\frac{\rho\,\{2\,(h-z)^2-(f-x)^2-(g-y)^2\}\,dx\,dy\,dz}{\{(f-x)^2+(g-y)^2+(h-z)^2\}^{\frac{5}{2}}};$$

$$\therefore\ \frac{d^2V}{df^2}+\frac{d^2V}{dg^2}+\frac{d^2V}{dh^2}=\iiint\frac{0\times dx\,dy\,dz}{\{(f-x)^2+(g-y)^2+(h-z)^2\}^{\frac{5}{2}}}.$$

When the attracted particle is not a portion of the attracting mass itself then *xyz* will never equal *fgh* respectively: and consequently the expression under the signs of integration vanishes for every particle of the mass:

$$\therefore\ \frac{d^2V}{df^2}+\frac{d^2V}{dg^2}+\frac{d^2V}{dh^2}=0.$$

This equation was first given by Laplace: and Poisson was the first who shewed that it was not true when the attracted particle is part of the attracting mass. The error arises in consequence of the expression under the signs of integration not vanishing for all values of xyz; since it equals $\frac{0}{0}$ when $x=f$, $y=g$, $z=h$.

To determine the value of $\frac{d^2V}{df^2}+\frac{d^2V}{dg^2}+\frac{d^2V}{dh^2}$ in this case, suppose a sphere described in the body so that it shall include the attracted particle: and let $V=U+U'$, U referring to the sphere, and U' to the excess of the body over the sphere. Then, by what is already proved,

$$\frac{d^2U'}{df^2}+\frac{d^2U'}{dg^2}+\frac{d^2U'}{dh^2}=0.$$

$$\text{Hence } \frac{d^2V}{df^2}+\frac{d^2V}{dg^2}+\frac{d^2V}{dh^2}=\frac{d^2U}{df^2}+\frac{d^2U}{dg^2}+\frac{d^2U}{dh^2}.$$

The centre of the sphere may be chosen as near the attracted particle as we please, and therefore the radius of the sphere may be taken so small that its density may be considered ultimately uniform and equal to that at the particle (fgh), which we shall call ρ'.

Let $f'g'h'$ be the co-ordinates to the centre of the sphere; then the attractions of the sphere on the attracted particle parallel to the axes are, by Art. 149, 150,

$$\frac{4\pi\rho'}{3}(f-f'),\quad \frac{4\pi\rho'}{3}(g-g'),\quad \frac{4\pi\rho'}{3}(h-h'),$$

$$\text{or } -\frac{dU}{df},\quad -\frac{dU}{dg},\quad -\frac{dU}{dh} \text{ by Art. 167.;}$$

$$\therefore \frac{d^2U}{df^2}+\frac{d^2U}{dg^2}+\frac{d^2U}{dh^2}=-4\pi\rho';$$

$$\therefore \frac{d^2V}{df^2}+\frac{d^2V}{dg^2}+\frac{d^2V}{dh^2}=-4\pi\rho',$$

when the attracted particle is within the attracting mass.

Since the attracting body is supposed to be nearly spherical we shall find it most convenient to transform our rectangular to polar co-ordinates, the origin of the radius vector of the surface being near the centre.

PROP. *To transform the partial differential equation in* V *to polar co-ordinates.*

169. Let r, θ, ω be the co-ordinates to the point (fgh),

$$r', \theta', \omega' \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (xyz),$$

the angles θ and θ' being measured from the axis of z: ω and ω' being the angles which the planes on which θ and θ' are measured make with the plane zx; as in fig. 30. and at p. 67, ϕ being replaced by ω;

$$\therefore\ f = r\sin\theta\cos\omega, \quad g = r\sin\theta\sin\omega, \quad h = r\cos\theta,$$

$$x = r'\sin\theta'\cos\omega', \quad y = r'\sin\theta'\sin\omega', \quad z = r'\cos\theta'.$$

Also the volume of an element of the mass$=dr'\,r'\,d\theta'\,r'\sin\theta'\,d\omega'$; therefore $V=$ the sum of the elements of the mass divided respectively by their distances from the attracted particle

$$=\int_0^{\mathrm{r}}\int_0^{\pi}\int_0^{2\pi}\frac{\rho r'^2\sin\theta'\,dr'\,d\theta'\,d\omega'}{\{r^2+r'^2-2rr'\,[\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\omega-\omega')]\}^{\frac{1}{2}}},$$

r being the value of r' at the surface of the body.

$$\text{Now } r^2=f^2+g^2+h^2,\ \ \cos\theta=\frac{h}{\sqrt{f^2+g^2+h^2}},\ \ \tan\omega=\frac{g}{f}\ \ldots.\ (1);$$

$$\therefore\ \frac{dV}{df}=\frac{dV}{dr}\frac{dr}{df}+\frac{dV}{d\theta}\frac{d\theta}{df}+\frac{dV}{d\omega}\frac{d\omega}{df},$$

$$\frac{d^2V}{df^2}=\frac{d}{df}\frac{dV}{dr}\frac{dr}{df}+\frac{d}{df}\frac{dV}{d\theta}\frac{d\theta}{df}+\frac{d}{df}\frac{dV}{d\omega}\frac{d\omega}{df}$$

$$+\frac{dV}{dr}\frac{d^2r}{df^2}+\frac{dV}{d\theta}\frac{d^2\theta}{df^2}+\frac{dV}{d\omega}\frac{d^2\omega}{df^2}$$

$$=\frac{d^2V}{dr^2}\frac{dr^2}{df^2}+\frac{d^2V}{d\theta^2}\frac{d\theta^2}{df^2}+\frac{d^2V}{d\omega^2}\frac{d\omega^2}{df^2}$$

$$+2\frac{d^2V}{drd\theta}\frac{dr}{df}\frac{d\theta}{df}+2\frac{d^2V}{drd\omega}\frac{dr}{df}\frac{d\omega}{df}+2\frac{d^2V}{d\theta d\omega}\frac{d\theta}{df}\frac{d\omega}{df}$$

$$+\frac{dV}{dr}\frac{d^2r}{df^2}+\frac{dV}{d\theta}\frac{d^2\theta}{df^2}+\frac{dV}{d\omega}\frac{d^2\omega}{df^2}.$$

The expresssions for $\frac{d^2V}{dg^2}$ and $\frac{d^2V}{dh^2}$ are of the same form. These must all be added together and equated to zero or $-4\pi\rho'$. When this is effected the formulæ (1) make

the coefficient of $\frac{d^2V}{dr^2}=\frac{dr^2}{df^2}+\frac{dr^2}{dg^2}+\frac{dr^2}{dh^2}=1,$

the coefficient of $\frac{d^2V}{d\theta^2}=\frac{d\theta^2}{df^2}+\frac{d\theta^2}{dg^2}+\frac{d\theta^2}{dh^2}=\frac{1}{r^2},$

the coefficient of $\frac{d^2V}{d\omega^2}=\frac{d\omega^2}{df^2}+\frac{d\omega^2}{dg^2}+\frac{d\omega^2}{dh^2}=\frac{1}{r^2\sin^2\theta},$

the coefficient of $\frac{d^2V}{drd\theta}=2\frac{dr}{df}\frac{d\theta}{df}+2\frac{dr}{dg}\frac{d\theta}{dg}+2\frac{dr}{dh}\frac{d\theta}{dh}=0,$

the coefficient of $\frac{d^2V}{drd\omega}=2\frac{dr}{df}\frac{d\omega}{df}+2\frac{dr}{dg}\frac{d\omega}{dg}+2\frac{dr}{dh}\frac{d\omega}{dh}=0,$

the coefficient of $\frac{d^2V}{d\theta d\omega}=2\frac{d\theta}{df}\frac{d\omega}{df}+2\frac{d\theta}{dg}\frac{d\omega}{dg}+2\frac{d\theta}{dh}\frac{d\omega}{dh}=0,$

the coefficient of $\frac{dV}{dr}=\frac{d^2r}{df^2}+\frac{d^2r}{dg^2}+\frac{d^2r}{dh^2}=\frac{2}{r},$

the coefficient of $\frac{dV}{d\theta}=\frac{d^2\theta}{df^2}+\frac{d^2\theta}{dg^2}+\frac{d^2\theta}{dh^2}=\frac{\cos\theta}{r^2\sin\theta},$

the coefficient of $\frac{dV}{d\omega}=\frac{d^2\omega}{df^2}+\frac{d^2\omega}{dg^2}+\frac{d^2\omega}{dh^2}=0.$

Hence the equation in V becomes

$$\frac{d^2V}{dr^2}+\frac{2}{r}\frac{dV}{dr}+\frac{1}{r^2}\frac{d^2V}{d\theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{dV}{d\theta}+\frac{1}{r^2\sin^2\theta}\frac{d^2V}{d\omega^2}=0 \text{ or } -4\pi\rho';$$

$$\therefore\ r\,\frac{d^2rV}{dr^2}+\frac{d^2V}{d\theta^2}+\frac{\cos\theta}{\sin\theta}\,\frac{dV}{d\theta}+\frac{1}{\sin^2\theta}\,\frac{d^2V}{d\omega^2}=0 \text{ or } -4\pi\rho'r^2.$$

Put $\cos\theta=\mu$ and $\cos\theta'=\mu'$: then

$$V=\int_0^{\mathrm{r}}\int_{-1}^{1}\int_0^{2\pi}\frac{\rho r'^2dr'\,d\mu'\,d\omega'}{\{r^2+r'^2-2rr'[\mu\mu'+\sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\omega-\omega')]\}^{\frac{1}{2}}},$$

and $$r\,\frac{d^2rV}{dr^2}+\frac{d}{d\mu}\left\{(1-\mu^2)\,\frac{dV}{d\mu}\right\}+\frac{1}{1-\mu^2}\,\frac{d^2V}{d\omega^2}=0 \text{ or } -4\pi\rho'r^2.$$

By integrating this equation we should determine the value of V. But this has never been accomplished, and we are consequently obliged to resort to approximation by series.

PROP. *To explain the method of expanding* V *in a series.*

170. The expression

$$\{r^2+r'^2-2rr'\,[\mu\mu'+\sqrt{1-\mu^2}\,\sqrt{1-\mu'^2}\cos(\omega-\omega')]\}^{-\frac{1}{2}},$$

may be expanded into either of the series

$$\left.\begin{array}{l} P_0\dfrac{1}{r'}+P_1\dfrac{r}{r'^2}+\ldots\ldots\ldots\ldots+P_i\dfrac{r^i}{r'^{i+1}}+\ldots\ldots \\ \text{or } P_0\dfrac{1}{r}+P_1\dfrac{r'}{r^2}+\ldots\ldots\ldots\ldots+P_i\dfrac{r'^i}{r^{i+1}}+\ldots\ldots \end{array}\right\}\ldots\ldots (1),$$

where $P_0, P_1, \ldots\ldots\ldots P_i, \ldots\ldots\ldots$ are all rational and entire functions of μ, $\sqrt{1-\mu^2}\cos\omega$, and $\sqrt{1-\mu^2}\sin\omega$, and the same functions of μ', $\sqrt{1-\mu'^2}\cos\omega'$, and $\sqrt{1-\mu'^2}\sin\omega'$: the general coefficient P_i is of i dimensions in μ, $\sqrt{1-\mu^2}\cos\omega$ and $\sqrt{1-\mu^2}\sin\omega$.

The greatest value of P_i (disregarding its sign) is unity. For if we put

$$\mu\mu'+\sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\omega-\omega')=\cos\phi=\tfrac{1}{2}\left(z+\frac{1}{z}\right),$$

then P_i = coefficient of c^i in

$$(1 + c^2 - 2c \cos \phi)^{-\frac{1}{2}} \text{ or } (1 - cz)^{-\frac{1}{2}} \left(1 - \frac{c}{z}\right)^{-\frac{1}{2}}$$

$$= \text{coefficient of } c^i \text{ in}$$

$$\left\{1 + \tfrac{1}{2} cz + \frac{1 \cdot 3}{2 \cdot 4} c^2 z^2 + \ldots\ldots\right\} \left\{1 + \tfrac{1}{2} \frac{c}{z} + \frac{1 \cdot 3}{2 \cdot 4} \frac{c^2}{z^2} + \ldots\ldots\right\}$$

$$= A \left(z^i + \frac{1}{z^i}\right) + B \left(z^{i-2} + \frac{1}{z^{i-2}}\right) + \ldots\ldots\ldots$$

$$= 2A \cos i\phi \quad + 2B \cos (i-2)\phi + \ldots\ldots\ldots$$

A, B ...... being all positive and finite: the greatest value of this is when $\phi = 0$: hence P_i is greatest when $\phi = 0$.

But then P_i = coefficient of c^i in $(1 + c^2 - 2c)^{-\frac{1}{2}}$ or $(1 - c)^{-1}$

$$= \ldots\ldots\ldots\ldots\ldots\ldots \quad (1 + c + c^2 + \ldots + c^i \ldots)$$

$$= 1.$$

Hence 1 is the greatest value of P_i. It follows that the first or second of the series (1) will be convergent according as r is less than or greater than r'.

Using the first series we have

$$V = \int_0^{\mathrm{r}} \int_{-1}^{1} \int_0^{2\pi} \rho r' \left\{P_0 + P_1 \frac{r}{r'} + P_2 \frac{r^2}{r'^2} + \ldots\ldots + P_i \frac{r_i}{r'^i} + \ldots\right\} dr' d\mu' d\omega'.$$

We substitute this in the equation in V of Art. 169, and then remove the powers of r outside the signs of integration and equating the coefficients of the same powers of r on each side of the equation we have a series of equations of which the general one is

$$\int_0^{\mathrm{r}} \int_{-1}^{1} \int_0^{2\pi} \rho \left\{\frac{d}{d\mu} \left((1-\mu^2) \frac{dP_i}{d\mu}\right) + \frac{1}{1-\mu^2} \frac{d^2 P_i}{d\omega^2} + i(i+1) P_i\right\} \frac{dr' d\mu' d\omega'}{r'^{i-1}} = 0,$$

excepting the single case when $i = 2$ and the particle is internal, in which case the second side is $-4\pi\rho'$.

It follows, then, that in every case

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dP_i}{d\mu}\right\}+\frac{1}{1-\mu^2}\frac{d^2P_i}{d\omega^2}+i(i+1)P_i=0.$$

For this is evidently true in every case but the excepted one mentioned above; and in that case it is equally true, for if the multiplier of $\frac{1}{r'}$ did not equal zero, the definite integral would be infinitely great (instead of equalling $-4\pi\rho'$) since r' can become infinitely small. When P_i is determined from this equation then V will be known.

171. This equation we meet with very frequently in the higher branches of Physical Science. It has never yet been integrated except by series. Laplace has demonstrated various properties of the integral, and Mr Murphy has effected the same by a new analysis: *Treatise on Electricity*. We shall call the functions which satisfy this equation *Laplace's Coefficients* of the first, second,...orders according as $i = 1, 2, \ldots$: and the equation itself *the Equation of Laplace's Coefficients*.

Prop. *To calculate the value of* V *for a homogeneous sphere.*

172. Let the sphere be referred to polar co-ordinates the centre being the pole (fig. 71): C the attracted particle; $OC = r$; P a particle in a shell of the sphere of which the radius $OP = r_1$, $\angle POC = \theta$, $\angle PMm = \omega$; a the radius of the sphere: then $PC = \sqrt{r^2 + r_1^2 - 2rr_1\cos\theta}$; and the mass of the element at $P = \rho r_1^2 \sin\theta\, dr_1\, d\theta\, d\omega$, the limits of ω are 0 and 2π; of θ are 0 and π; of r_1 are 0 and a;

$$\therefore\ V = \int_0^a\int_0^\pi\int_0^{2\pi}\frac{\rho r_1^2\sin\theta\, dr_1\, d\theta\, d\omega}{\sqrt{r^2+r_1^2-2rr_1\cos\theta}}$$

$$= 2\pi\rho\int_0^a\int_0^\pi\frac{r_1^2\sin\theta\, dr_1\, d\theta}{\sqrt{r^2+r_1^2-2rr_1\cos\theta}}$$

$$= 2\pi\rho\int_0^a\frac{r_1}{r}\left\{\sqrt{r^2+r_1^2-2rr_1\cos\theta}+\text{const.}\right\}dr_1$$

$$= 2\pi\rho \int_0^a \frac{r_1}{r} \{(r + r_1) \mp (r - r_1)\} \, dr_1$$

– when C is without, and + when C is within the shell,

$$= \frac{4\pi\rho}{r} \int_0^a r_1^2 dr_1 = \frac{4\pi\rho a^3}{3r},$$

when C is without the sphere.

And when C is within the sphere, the part of V for the shells which enclose $C = 2\pi\rho \int_r^a 2r_1 dr_1 = 2\pi\rho (a^2 - r^2)$: and the part of V for the other shells of the sphere

$$= \frac{4\pi\rho}{r} \int_0^r r_1^2 dr_1 = \frac{4\pi\rho r^2}{3}.$$

Hence $V = \dfrac{4\pi\rho a^3}{3r}$ for an *external* particle

$V = 2\pi\rho a^2 - \dfrac{2\pi\rho r^2}{3}$ for an *internal* particle.

We shall find the use of these in the next two Propositions.

Prop. *To find the attraction of a homogeneous body, differing little from a sphere in form, upon a particle without it.*

173. Since the attracted particle is without the attracting mass we must expand V in a descending series of powers of r:

$$\text{let } V = \frac{v_0}{r} + \frac{v_1}{r^2} + \frac{v_2}{r^3} + \ldots\ldots + \frac{v_i}{r^{i+1}} + \ldots\ldots$$

But by Art. 170. taking the *second* of expansions (1),

$$V = \int_0^{\mathrm{r}} \int_{-1}^1 \int_0^{2\pi} \rho \left\{ P_0 \frac{r'^2}{r} + P_1 \frac{r'^3}{r^2} + \ldots + P_i \frac{r'^{i+2}}{r^{i+1}} + \ldots \right\} dr' \, d\mu' \, d\omega';$$

$$\therefore\ v_i = \int_0^{\mathrm{r}} \int_{-1}^1 \int_0^{2\pi} \rho P_i r'^{i+2}.$$

Let the mean radius of the body $=\mathrm{a}$: and let $\mathrm{a}(1+\alpha y')$ be the variable radius, y' being a function of μ' and ω', and α being a very small numerical quantity whose square and higher powers are to be neglected.

Then, for the excess of the attracting mass over the sphere of which the radius $=\mathrm{a}$, the value of ν_i

$$=\alpha\mathrm{a}^{i+3}\rho\int_0^{y'}\int_{-1}^{1}\int_0^{2\pi} P_i\, dy'\, d\mu'\, d\omega' = \alpha\mathrm{a}^{i+3}\rho\int_{-1}^{1}\int_0^{2\pi} P_i y'\, d\mu'\, d\omega'.$$

$$\text{Let } \int_{-1}^{1}\int_0^{2\pi} P_i y'\, d\mu'\, d\omega' = U_i,$$

hence for the excess over the sphere we have

$$V=\alpha\rho\left\{\frac{\mathrm{a}^3}{r}U_0+\frac{\mathrm{a}^4}{r^2}U_1+\ldots\ldots+\frac{\mathrm{a}^{i+3}}{r^{i+1}}U_i+\ldots\right\}.$$

But for the sphere of which the radius is a, $V=\dfrac{4\pi\rho\mathrm{a}^3}{3r}$ by Art. 172.

Hence for the whole mass

$$V=\frac{4\pi\rho\mathrm{a}^3}{3r}+\frac{\alpha\rho\mathrm{a}^3}{r}\left\{U_0+\frac{\mathrm{a}}{r}U_1+\ldots\ldots+\frac{\mathrm{a}^i}{r^i}U_i+\ldots\right\}$$

and the attraction $=-\dfrac{dV}{dr}$, (see Art. 167.)

$$=\frac{4\pi\rho\mathrm{a}^3}{3r^2}+\frac{\alpha\rho\mathrm{a}^3}{r^2}\left\{U_0+\frac{2\mathrm{a}}{r}U_1+\ldots\ldots+\frac{(i+1)\mathrm{a}^i}{r^i}U_i+\ldots\right\}.$$

Prop. *To find the attraction of a homogeneous body, differing but little from a sphere in form, upon an internal particle.*

174. We must in this case expand V in an ascending series of powers of r: let

$$V=\nu_0+\nu_1 r+\nu_2 r^2+\ldots\ldots+\nu_i r^i+\ldots$$

But by Art. 170.

$$V=\int_0^{\mathrm{r}}\int_{-1}^{1}\int_0^{2\pi}\rho\left\{P_0 r'+P_1 r+P_2\frac{r^2}{r'}+\ldots+P_i\frac{r^i}{r'^{i-1}}+\ldots\right\}dr'\, d\mu'\, d\omega';$$

$$\therefore\ \nu_i = \int_0^{\mathrm{r}}\int_{-1}^{1}\int_0^{2\pi}\frac{\rho P_i}{r'^{\,i-1}}\,dr'\,d\mu'\,d\omega',$$

and as in the preceding Proposition the value of ν_i for the excess of the attracting mass over the sphere of which the radius is (a)

$$= \frac{\alpha\rho}{\mathrm{a}^{i-2}}\int_{-1}^{1}\int_0^{2\pi} P_i y'\,d\mu'\,d\omega' = \frac{\alpha\rho}{\mathrm{a}^{i-2}}\,U_i.$$

Also for the sphere of which the radius $=\mathrm{a}$ the value of V is $2\pi\rho\mathrm{a}^2 - \dfrac{2\pi\rho r^2}{3}$ (Art. 172). Hence for the whole mass

$$V = 2\pi\rho\mathrm{a}^2 - \frac{2\pi\rho r^2}{3} + \alpha\rho\mathrm{a}^2\left\{U_0 + \frac{r}{\mathrm{a}}U_1 + \ldots + \frac{r^i}{\mathrm{a}^i}U_i + \ldots\right\}$$

and the attraction $= -\dfrac{dV}{dr}$

$$= \frac{4\pi\rho r}{3} - \alpha\rho\mathrm{a}\left\{U_1 + \frac{2r}{\mathrm{a}}U_2 + \ldots + \frac{ir^{i-1}}{\mathrm{a}^{i-1}}U_i + \ldots\right\}.$$

175. The calculation of the functions $U_0, U_1, \ldots U_i, \ldots$ can be effected without integration when we know the equation to the surface of the body. We proceed to demonstrate this in the three following Propositions.

Prop. *To prove that a function of* μ, $\sqrt{1-\mu^2}\cos\omega$ *and* $\sqrt{1-\mu^2}\sin\omega$, *as* $F(\mu,\omega)$, *can be expanded in a series of Laplace's Coefficients: provided that* $F(\mu,\omega)$ *do not become infinite between the values* -1 *and* 1 *of* μ, *and* 0 *and* 2π *of* ω.

176. Let $\mu'\mu + \sqrt{1-\mu'^2}\sqrt{1-\mu^2}\cos(\omega'-\omega) = p$: then by Art. 170.

$$(1 + c^2 - 2cp)^{-\frac{1}{2}} = P_0 + P_1 c + P_2 c^2 + \ldots + P_i c^i + \ldots$$

c being any quantity not greater than unity.

Differentiating with respect to c,

$$\frac{p - c}{(1 + c^2 - 2cp)^{\frac{3}{2}}} = P_1 + 2P_2 c + \ldots + iP_i c^{i-1} + \ldots$$

Multiply the latter equation by $2c$ and add it to the former,

$$\therefore \frac{1-c^2}{(1+c^2-2cp)^{\frac{3}{2}}} = P_0 + 3P_1c + 5P_2c^2 + \ldots + (2i+1)P_ic^i + \ldots$$

$$\therefore \int_{-1}^{1}\int_{0}^{2\pi} \frac{(1-c^2)F(\mu', \omega')\,d\mu'\,d\omega'}{(1+c^2-2cp)^{\frac{3}{2}}} = \int_{-1}^{1}\int_{0}^{2\pi} \{P_0 + 3P_1c + \ldots$$
$$+ (2i+1)P_ic^i + \ldots\}\, F(\mu', \omega')\,d\mu'\,d\omega'.$$

Now c being arbitrary we may put it $= 1$. Then the fraction under the symbols of integration on the left-hand side of this equation vanishes, except when $p = 1$, in which case the fraction equals $\frac{0}{0}$. We proceed, then, to determine the value of the integral, we shall call it X. When $p = 1$ then

$$\cos(\omega' - \omega) = \frac{1 - \mu'\mu}{\sqrt{(1-\mu'^2)(1-\mu^2)}} = \sqrt{\frac{1 - 2\mu'\mu + \mu'^2\mu^2}{1 - \mu'^2 - \mu^2 + \mu'^2\mu^2}},$$

and that this may not be greater than unity we must take $\mu'^2 + \mu^2$ not greater than $2\mu'\mu$, or $(\mu' \sim \mu)^2$ not greater than zero: hence $\mu' = \mu$, and therefore $\cos(\omega' - \omega) = 1$ and $\omega' = \omega$. We shall therefore put $\mu' = \mu + \nu$ and $\omega' = \omega + z$ and $1 - c = g$: ν, z being small quantities which vanish when $p = 1$, and g a small quantity which vanishes when $c = 1$. Then we have

$$1 - c^2 = 2g, \quad p = 1 - (1-\mu^2)\frac{z^2}{2} - \frac{\nu^2}{2(1-\mu^2)},$$

in which we retain only the lowest powers of g, z, ν which occur, since the higher powers must ultimately vanish in comparison with the lower:

$$\therefore \frac{1-c^2}{(1+c^2-2cp)^{\frac{3}{2}}} = \frac{1-c^2}{\{(1-c)^2 + 2c(1-p)\}^{\frac{3}{2}}}$$
$$= \frac{2g}{\left\{g^2 + \frac{\nu^2}{1-\mu^2} + (1-\mu^2)z^2\right\}^{\frac{3}{2}}}.$$

Also since, by hypothesis, $F(\mu', \omega')$ does not become infinite between the specified limits of μ' and ω', then

$$F(\mu', \omega') = F(\mu + \nu, \omega + z) = F(\mu, \omega) + \xi$$

where ξ is a very small quantity which vanishes with ν and z: then after the substitutions

$$X = 2F(\mu, \omega) \iint \frac{g\,d\nu\,dz}{\left\{g^2 + \frac{\nu^2}{1-\mu^2} + z^2(1-\mu^2)\right\}^{\frac{3}{2}}}$$

$$+ 2 \iint \frac{\xi g\,d\nu\,dz}{\left\{g^2 + \frac{\nu^2}{1-\mu^2} + z^2(1-\mu^2)\right\}^{\frac{3}{2}}}$$

between proper limits.

Now, since when $c = 1$ or $g = 0$ the fraction under the symbols of integration in X vanishes for all values of ν and z that are not indefinitely small, it follows, that we may choose limiting values for ν and z of any magnitude that we please, if we put $g = 0$ after the integration. We shall take $-h$ and h for the limits of ν and $-\infty$ and ∞ for the limits of z.

Put $z^2(1-\mu^2) = \left(g^2 + \frac{\nu^2}{1-\mu^2}\right) x^2$;

therefore the first term of X

$$= 2F(\mu, \omega) \int_{-h}^{h} \frac{g\sqrt{1-\mu^2}\,d\nu}{g^2(1-\mu^2) + \nu^2} \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{\frac{3}{2}}}$$

$$= 4F(\mu, \omega) \int_{-h}^{h} \frac{g\sqrt{1-\mu^2}\,d\nu}{g^2\ 1-\mu^2) + \nu^2} = 8F(\mu, \omega) \tan^{-1} \frac{h}{g\sqrt{1-\mu^2}}$$

$= 4\pi F(\mu, \omega)$, when $g = 0$, or $c = 1$.

Now ξ being a very small quantity, let β be its greatest value within given small values of ν and z: then the second term of X is less than

$$\beta \int_{-h}^{h} \int_{-\infty}^{\infty} \frac{g\,d\nu\,dz}{\left\{g^2 + \frac{\nu^2}{1-\mu^2} + z^2(1-\mu^2)\right\}}$$

or is less than $2\pi\beta$: but β ultimately vanishes and therefore $X = 4\pi F(\mu, \omega)$. Thus we have determined the value of the integral;

$$\therefore F(\mu, \omega) = \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}\{P_0 + 3P_1 + 5P_2 + \ldots\ldots$$
$$+ (2i+1)P_i + \ldots\}F(\mu', \omega')\,d\mu'\,d\omega'.$$

Now the general term $\frac{2i+1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} P_i F(\mu', \omega')\,d\mu'\,d\omega'$ is a function of μ and ω which satisfies the equation of Laplace's Coefficients (see Art. 171.)

Hence $F(\mu, \omega)$ can be expanded in a series of Laplace's Coefficients.

177. If ω or μ have one of its limiting values 0 and 2π or -1 and 1, we must not take the limits of integration which we have used in the last Article.

For it is only for the values of ω' and μ' which differ by indefinitely small quantities from ω and μ that the fraction under the symbols of integration in X does not vanish. Now if $\omega = 0$ then $\omega' = 0$ and 2π; and therefore it is for indefinitely small positive or negative values of ω' and for values a very little less and a very little greater than 2π that X does not vanish: but the negative values and the values greater than 2π are not included between the specified limits (0 and 2π) of ω' and must therefore be left out of consideration, though this was not necessary in the general case, because the fraction X then vanished for these values. Hence if $\omega = 0$ the value of X is found by putting $\omega' = 0$ and integrating with respect to z from 0 to ∞, and adding to it the result of putting $\omega' = 2\pi$ and integrating from $z = -\infty$ to $z = 0$: we then have

$$X = 2\pi\{F(\mu, 0) + F(\mu, 2\pi)\} = 4\pi F(\mu, 0) \text{ and } 4\pi F(\mu, 2\pi)$$
$$\text{for } F(\mu, 0) = F(\mu, 2\pi)$$

since $F(\mu, \omega)$ is a function of μ, $\sqrt{1-\mu^2}\cos\omega$, and $\sqrt{1-\mu^2}\sin\omega$.

The same will be the result if ω have its other limiting value 2π.

Again, suppose $\mu = -1$; then $\mu' = -1$, and we must not take the limits of ν negative since μ' cannot be less than -1: but since in this case ω' is quite indeterminate we shall refer to the original form of X. Since $F(\mu', \omega')$ is a function of μ', $\sqrt{1-\mu'^2}\cos\omega'$ and $\sqrt{1-\mu'^2}\sin\omega'$ which does not become infinite between the specified limits of μ' and ω', it follows, that $F(-1, \omega')$ is independent of ω': hence, since $p = -\mu'$ when $\mu = -1$, we have

$$X = \int_{-1}^{1}\int_{0}^{2\pi} \frac{(1-c^2)\,F(\mu', \omega')\,d\mu'\,d\omega'}{(1+c^2+2c\mu')^{\frac{3}{2}}}$$

$$= F(-1, \omega')\int_{-1}^{1}\int_{0}^{2\pi} \frac{(1-c^2)\,d\mu'\,d\omega'}{(1+c^2+2c\mu')^{\frac{3}{2}}}$$

$$= 2\pi F(-1, \omega')\int_{-1}^{1} \frac{(1-c^2)\,d\mu'}{(1+c^2+2c\mu')^{\frac{3}{2}}}$$

$$= 2\pi F(-1, \omega')\left\{\text{const.} - \frac{(1-c^2)}{c\,(1+c^2+2c\mu')^{\frac{1}{2}}}\right\}$$

$$= 2\pi F(-1, \omega')\,\frac{1}{c}\,\{(1+c)-(1-c)\} = 4\pi F(-1, \omega').$$

In the same way, when $\mu = 1$, $X = 4\pi F(1, \omega')$: and consequently the result of the last Article is correct even in these limiting cases.

178. The demonstration of this Proposition is the substance of that given by Poisson in several memoirs: the last place in which it appeared is in the *Théorie Mathématique de la Chaleur*, Chap. VIII. He intoduces this Theorem with these words, "La démonstration que j'en ai donnée dans plusieurs mémoires, et que je vais reproduire ici, me semble "propre à dissiper tous les doutes que l'on avait élevés sur "sa généralité." If the reader be inclined to enquire into the merits of the controversy hinted at in this passage, he may consult the *Mécanique Céleste*, Livre III. Chap. II., two Papers by Mr Ivory, in the Philosophical Transactions for 1812 and 1822, a Memoir by Lagrange in the Journal de l'Ecole Polytechnique, Cahier 15, Poisson's Papers, and a Paper in the Cambridge Philosphical Transactions, Vol. II. by Mr (Professor) Airy, Astronomer Royal.

It will be observed, that in the enunciation of the Proposition we have restricted the nature of the function $F(\mu, \omega)$ to such forms as fulfil the conditions of Art. 107. of the *Théorie de la Chaleur*, by supposing it a function of μ, $\sqrt{1-\mu^2}\cos\omega$, $\sqrt{1-\mu^2}\sin\omega$. It is the opinion of those mathematicians who most restrict the generality of this theorem, that it is demonstrated only in such cases as $F(\mu, \omega)$ is a *rational and entire* function of μ, $\sqrt{1-\mu^2}\cos\omega$, and $\sqrt{1-\mu^2}\sin\omega$. The doubt attending this question will not affect any results to which we come in this work, since it will be seen that we apply the theorem only to such functions as prove in the end to be rational and entire functions of μ, $\sqrt{1-\mu^2}\cos\omega$, and $\sqrt{1-\mu^2}\sin\omega$.

179. It results from this Proposition that in spheroids of which the radius vector can be expressed in terms of μ, $\sqrt{1-\mu^2}\cos\omega$, $\sqrt{1-\mu^2}\sin\omega$ we can expand y in a series of Laplace's Coefficients

$$Y_0 + Y_1 + Y_2 + \dots\dots\dots + Y_i + \dots\dots$$

We proceed to demonstrate in the course of the next two Propositions that y can be expanded in only one such series, a result of the utmost importance in the future calculations.

We must first prove the following property of Laplace's Coefficients.

PROP. *To prove that if* $\mathrm{Q_i}$ *and* $\mathrm{R_i}$ *be two of Laplace's Coefficients, then* $\int_{-1}^{1}\int_0^{2\pi} \mathrm{Q_i R_{i'}}\, d\mu\, d\omega = 0$, i *and* i′ *being different integers.*

180. By the equation of Laplace's Coefficients, (Art. 170.)

$$i(i+1)\,Q_i = -\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dQ_i}{d\mu}\right\} - \frac{1}{1-\mu^2}\frac{d^2Q_i}{d\omega^2};$$

$$\therefore \int_{-1}^{1}\int_0^{2\pi} Q_i R_{i'}\, d\mu\, d\omega$$

$$= -\frac{1}{i(i+1)}\int_{-1}^{1}\int_0^{2\pi}\left(\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dQ_i}{d\mu}\right\} + \frac{1}{1-\mu^2}\frac{d^2Q_i}{d\omega^2}\right) R_{i'}\, d\mu\, d\omega.$$

Now by a double integration by parts

$$\int \frac{d}{d\mu}\left\{(1-\mu^2)\frac{dQ_i}{d\mu}\right\} R_{i'}\,d\mu = (1-\mu^2)\frac{dQ_i}{d\mu}R_{i'} - (1-\mu^2)\frac{dR_{i'}}{d\mu}Q_i$$

$$+ \int \frac{d}{d\mu}\left\{(1-\mu^2)\frac{dR_{i'}}{d\mu}\right\} Q_i\,d\mu\,;$$

$$\therefore \int_{-1}^{1} \frac{d}{d\mu}\left\{(1-\mu^2)\frac{dQ_i}{d\mu}\right\} R_{i'}\,d\mu = \int_{-1}^{1} \frac{d}{d\mu}\left\{(1-\mu^2)\frac{dR_{i'}}{d\mu}\right\} Q_i\,d\mu.$$

Again, $\displaystyle\int R_{i'}\frac{d^2Q_i}{d\omega^2}\,d\omega = R_{i'}\frac{dQ_i}{d\omega} - Q_i\frac{dR_{i'}}{d\omega} + \int Q_i\frac{d^2R_{i'}}{d\omega^2}\,d\omega\,;$

$$\therefore \int_0^{2\pi} R_{i'}\frac{d^2Q_i}{d\omega^2}\,d\omega = \int_0^{2\pi} Q_i\frac{d^2R_{i'}}{d\omega^2}\,d\omega,$$

since when $\omega = 0$ and 2π, each of the functions Q_i, $R_{i'}$, $\dfrac{dQ_i}{d\omega}$, $\dfrac{dR_{i'}}{d\omega}$ has the same values, because they are functions of μ, $\sqrt{1-\mu^2}\cos\omega$, $\sqrt{1-\mu^2}\sin\omega$.

$$\text{Hence} \int_{-1}^{1}\int_0^{2\pi} Q_iR_{i'}\,d\mu\,d\omega$$

$$= -\frac{1}{i(i+1)}\int_{-1}^{1}\int_0^{2\pi}\left(\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dR_{i'}}{d\mu}\right\} + \frac{1}{1-\mu^2}\frac{d^2R_{i'}}{d\omega^2}\right)Q_i\,d\mu\,d\omega$$

$$= \frac{i'(i'+1)}{i(i+1)}\int_{-1}^{1}\int_0^{2\pi} Q_iR_{i'}\,d\mu\,d\omega$$

by the equation of Laplace's Coefficients.

Hence $\int_{-1}^{1}\int_0^{2\pi} Q_iR_{i'}\,d\mu\,d\omega = 0$, when i and i' are unequal.

If $i = i'$ the above equation becomes identical and therefore gives no condition.

We are enabled now to prove the following very important Proposition.

PROP. *To prove that* F (μ, ω) *can be expanded in only* one *series of Laplace's Coefficients.*

181. We have shewn in Art. 176. that

$$F(\mu, \omega) = \frac{1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} \{P_0 + 3P_1 + \dots\dots\dots$$

$$+ (2i+1) P_i + \dots\dots\} F(\mu', \omega')\, d\mu' d\omega' \dots\dots\dots (1)$$

$$= R_0 + R_1 + \dots\dots\dots + R_i + \dots\dots \text{ suppose,}$$

this being a *determinate* series of Laplace's Coefficients, since $P_0, P_1 \dots\dots P_i \dots.$ are determinate (Art. 170).

We have now to shew, that use what artifice of development we may instead of that used in Art. 176, it will be impossible to expand $F(\mu, \omega)$ in a series of Laplace's Coefficients differing from $R_0, R_1 \dots\dots R_i \dots$ respectively.

For, if possible, let $Q_0 + Q_1 + \dots\dots + Q_i + \dots$ be another development of $F(\mu, \omega)$ in which Q_i differs from R_i;

$$\therefore (Q_0 - R_0) + (Q_1 - R_1) + \dots\dots + (Q_i - R_i) + \dots. = 0.$$

Multiply by P_i, then by Art. 180.

$$\int_{-1}^{1}\int_{0}^{2\pi} P_i (Q_i - R_i)\, d\mu d\omega = 0 \dots\dots\dots\dots (2).$$

But, by hypothesis, $Q_i - R_i$ does not $= 0$, but equals some function of μ and ω which does not become infinite between the limits of μ and ω: hence interchanging μ and μ', ω and ω' in (1) and putting $Q_i - R_i$ for $F(\mu, \omega)$ and $Q_i' - R_i'$ for $F(\mu', \omega')$ and observing that $P_0 P_1 \dots.$ are not altered by this substitution (Art. 170), we have by the general principle expressed in the formula (1)

$$Q_i' - R_i' = \frac{1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} (P_0 + \dots + (2i+1) P_i + \dots)(Q_i - R_i)\, d\mu d\omega$$

$$= \frac{2i+1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} P_i (Q_i - R_i)\, d\mu d\omega \text{ (Art. 180.)}$$

$$= 0 \text{ by (2);}$$

$\therefore Q_i' = R_i'$ and therefore $Q_i = R_i$ and our hypothesis is false, and $F(\mu, \omega)$ admits of only one expansion.

182. We are now able to shew, as we promised in Art. 175, that the functions U_0 U_1 $U_2 \dots\dots$ can be calculated

without integration where the equation of the surface of the attracting body is known.

For y can be expanded in a series of Laplace's Coefficients

$$\frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}\{P_0+3P_1+\dots\dots+(2i+1)P_i+\dots\}\,y'd\mu'd\omega'$$
$$=Y_0+Y_1+\dots\dots+Y_i+\dots\dots$$

and this series admits of only one form by what has just been proved. Hence when the equation to the surface, and therefore y, is known the functions Y_0, Y_1, ... Y_i, ... are determinate, and we may equate terms of the same order in the two series for y written above:

$$\therefore\ \int_{-1}^{1}\int_{0}^{2\pi}(2i+1)\,P_i y'd\mu'd\omega' = 4\pi Y_i;$$

$$\therefore\ U_i=\int_{-1}^{1}\int_{0}^{2\pi}P_i y'd\mu'd\omega'\ (\text{Art. } 173.) = \frac{4\pi}{2i+1}Y_i,$$

and consequently when the equation to the surface is known $U_0\,U_1\,U_2$ ...... U_i, ... are also known, as was mentioned in Art. 175.

By substituting for $U_0\,U_1$ ... in the expressions of Arts. 173, 174. we have for an *external* particle

$$V=\frac{4\pi\rho\mathrm{a}^3}{3r}+\frac{4\pi\rho a\mathrm{a}^3}{r}\left\{Y_0+\frac{\mathrm{a}}{3r}Y_1+\dots\dots+\frac{\mathrm{a}^i}{(2i+1)r^i}Y_i+\dots\right\}:$$

and for an *internal* particle

$$V=2\pi\rho\mathrm{a}^2-\frac{2\pi\rho r^2}{3}+4\pi\rho a\mathrm{a}^2\left\{Y_0+\frac{r}{3\mathrm{a}}Y_1+\dots+\frac{r^i}{(2i+1)\mathrm{a}^i}Y_i+\dots\right\}.$$

The property of Laplace's Coefficients proved in Art. 180, enables us to prove that Y_0 and Y_1 may be made to disappear from the expression for y by properly choosing the value of (a) and the origin of the radius-vector of the surface.

PROP. *By choosing* a *equal to the radius of the sphere of which the mass equals that of the attracting body we cause* Y_0 *to vanish from the series* $Y_0+Y_1+\dots+Y_i+\dots$; *and by* *taking the centre of gravity of the body as the origin of the radius-vector we cause* Y_1 *to vanish.*

183. If r, θ, ω be the co-ordinates to any point in the body, an element of the mass

$$= \rho\, dr\, r\, d\theta\, r \sin\theta\, d\omega = -\rho r^2 dr\, d\mu\, d\omega\,;$$

therefore the mass of the body

$$= \rho \int_0^{\mathrm{r}} \int_{-1}^{1} \int_0^{2\pi} r^2 dr\, d\mu\, d\omega = \frac{\rho}{3} \int_{-1}^{1} \int_0^{2\pi} \mathrm{r}^3 d\mu\, d\omega,$$

then putting $\mathrm{r} = \mathrm{a}\,(1 + ay)$ we have

mass of body = mass of sphere (rad. = a) + $\rho\, \mathrm{a}^3 a \int_{-1}^{1}\int_0^{2\pi} y\, d\mu\, d\omega$

= mass of sphere + $\rho\, \mathrm{a}^3 a \int_{-1}^{1}\int_0^{2\pi} Y_0\, d\mu\, d\omega$ by Art. 180.

= mass of sphere + $4\pi\rho\, \mathrm{a}^3 a Y_0$, since Y_0 is constant.

If, then, a be taken equal to the radius of the sphere of which the mass equals the mass of the body $Y_0 = 0$, as was stated.

184. Again: let $\bar{x}\, \bar{y}\, \bar{z}$ be the co-ordinates to the centre of gravity of the body, M its mass: the co-ordinates to the element, of which the mass is $-\rho r^2 dr\, d\mu\, d\omega$, are

$$r\sqrt{1-\mu^2}\cos\omega,\quad r\sqrt{1-\mu^2}\sin\omega,\quad \text{and } r\mu\,;$$

$$\therefore\ M\bar{x} = \int_0^{\mathrm{r}}\int_{-1}^{1}\int_0^{2\pi} \rho r^3 \sqrt{1-\mu^2}\cos\omega\, dr\, d\mu\, d\omega$$

$$= \tfrac{1}{4}\int_{-1}^{1}\int_0^{2\pi} \rho \mathrm{r}^4 \sqrt{1-\mu^2}\cos\omega\, d\mu\, d\omega,$$

$$M\bar{y} = \int_0^{\mathrm{r}}\int_{-1}^{1}\int_0^{2\pi} \rho \mathrm{r}^3 \sqrt{1-\mu^2}\sin\omega\, dr\, d\mu\, d\omega$$

$$= \tfrac{1}{4}\int_{-1}^{1}\int_0^{2\pi} \rho r^4 \sqrt{1-\mu^2}\sin\omega\, d\mu\, d\omega,$$

$$M\bar{z} = \int_0^{\mathrm{r}}\int_{-1}^{1}\int_0^{2\pi} \rho r^3 \mu\, dr\, \mu\, d\omega = \tfrac{1}{4}\int_{-1}^{1}\int_0^{2\pi} \rho \mathrm{r}^4 \mu\, d\mu\, d\omega,$$

putting $\mathrm{r} = \mathrm{a}\,(1 + ay) = \mathrm{a}\,(1 + aY_0 + aY_1 + \ldots + aY_i + \ldots)$, and observing that $\sqrt{1-\mu^2}\cos\omega$, $\sqrt{1-\mu^2}\sin\omega$, μ satisfy Laplace's Equation (Art. 170), and are of the first order (Art. 171), we have by Art. 180,

$$M\bar{x} = \rho \mathrm{a}^4 \alpha \int_{-1}^{1}\int_{0}^{2\pi} Y_1 \sqrt{1-\mu^2} \cos \omega \, d\mu \, d\omega,$$

$$M\bar{y} = \rho \mathrm{a}^4 \alpha \int_{-1}^{1}\int_{0}^{2\pi} Y_1 \sqrt{1-\mu^2} \sin \omega \, d\mu \, d\omega,$$

$$M\bar{z} = \rho \mathrm{a}^4 \alpha \int_{-1}^{1}\int_{0}^{2\pi} Y_1 \mu \, d\mu \, d\omega.$$

But Y_1, being a function of μ, $\sqrt{1-\mu^2}\cos\omega$, $\sqrt{1-\mu^2}\sin\omega$ of the first order, is of the form

$$A\sqrt{1-\mu^2}\cos\omega + B\sqrt{1-\mu^2}\sin\omega + C\mu;$$

$$\therefore\ M\bar{x} = \frac{4\pi\rho \mathrm{a}^4 \alpha}{3} A, \quad M\bar{y} = \frac{4\pi\rho \mathrm{a}^4 \alpha}{3} B, \quad M\bar{z} = \frac{4\pi\rho \mathrm{a}^4 \alpha}{3} C.$$

Hence if we take the origin of co-ordinates at the centre of gravity $\bar{x} = 0$, $\bar{y} = 0$, $\bar{z} = 0$, and consequently $A = 0$, $B = 0$, $C = 0$ and therefore $Y_1 = 0$, as stated in the enunciation.

We shall in future parts of this work require to know the attraction of a body consisting of strata nearly spherical and varying in density according to any law. We shall therefore proceed to the calculation of these attractions.

Prop. *To find the attraction of a heterogeneous body upon a particle without it: the body consisting of thin strata nearly spherical, homogeneous in themselves, but differing one from another in density.*

185. Let $a'(1+\alpha y')$ be the radius of the external surface of any stratum, a' being chosen so that

$$y' = Y_1' + Y_2' + \ldots\ldots\ldots + Y_i' + \ldots \text{ (Art. 183).}$$

Since the strata are supposed not to be similar to each other y' is a function of a' as well as of μ' and ω'. Let ρ' be the density of the stratum of which the mean radius is a'. Now the value of V for this stratum equals the difference between the values of V for two homogeneous bodies of the density ρ' and mean radii a' and $a' - da'$. But for the body of which the mean radius is a' (Art. 182.)

$$V = \frac{4\pi\rho' a'^3}{3r} + \frac{4\pi\alpha\rho' a'^3}{r}\left\{\frac{a'}{3r} Y_1' + \ldots\ldots + \frac{a'^i}{(2i+1)r^i} Y_i' + \ldots\right\}$$

hence for the stratum of which the external mean radius is a'

$$V=\frac{4\pi\rho' a'^2}{r}\,da'+\frac{4\pi\alpha\rho'}{r}\,\frac{d}{da'}\left\{\frac{a'^4}{3r}Y_1'+\ldots\ldots+\frac{a'^{i+3}}{(2i+1)r^i}Y_i'+\ldots\right\}da',$$

and therefore for the whole body

$$V=\frac{4\pi}{r}\int_0^{\mathrm{a}}\rho'\left\{a'^2+\alpha\frac{d}{da'}\left(\frac{a'^4}{3r}Y_1'+\ldots\ldots+\frac{a'^{i+3}}{(2i+1)r^i}Y_i'+\ldots\right)\right\}da'.$$

The attraction $=-\dfrac{dV}{dr}$, and is easily found by differentiation.

Prop. *To find the attraction of the same body on an internal particle.*

186. Let $r=a(1+\alpha y)$ be the radius of the attracted particle. Then for the strata within the surface of which the radius is $a(1+\alpha y)$ we have (Art. 185.)

$$V=\frac{4\pi}{r}\int_0^{a}\rho'\left\{a'^2+\alpha\frac{d}{da'}\left(\frac{a'^4}{3r}Y_1'+\ldots\ldots+\frac{a'^{i+3}}{(2i+1)r^i}Y_i'+\ldots\right)\right\}da'.$$

But for a stratum external to the attracted particle we obtain by Art. 182.

$$V=4\pi\rho' a'da'+4\pi\rho'\alpha\frac{d}{da'}\left(\frac{ra'}{3}Y_1'+\ldots\ldots+\frac{r^i}{(2i+1)a'^{i-2}}Y_i'+\ldots\right)da',$$

and therefore for all the strata external to the particle

$$V=4\pi\int_a^{\mathrm{a}}\rho'\left\{a'+\alpha\frac{d}{da'}\left(\frac{ra'}{3}Y_1'+\ldots\ldots+\frac{r^i}{(2i+1)a'^{i-2}}Y_i'+\ldots\right)\right\}da',$$

and consequently for the whole body

$$V=\frac{4\pi}{r}\int_0^{a}\rho'\left\{a'^2+\alpha\frac{d}{da'}\left(\frac{a'^4}{3r}Y_1'+\ldots\ldots+\frac{a'^{i+3}}{(2i+1)r^i}Y_i'+\ldots\right)\right\}da'$$

$$+4\pi\int_a^{\mathrm{a}}\rho'\left\{a'+\alpha\frac{d}{da'}\left(\frac{a'r}{3}Y_1'+\ldots\ldots+\frac{r^i}{(2i+1)a'^{i-2}}Y_i'+\ldots\right)\right\}da'.$$

From this the attraction, or $-\dfrac{dV}{dr}$, is easily obtained.