The mathematical principles of mechanical philosophy, and their application to the theory of universal gravitation

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Dynamics.

DYNAMICS.

CHAPTER I.

DEFINITIONS. LAWS OF MOTION.

187. In this part of our Work we are engaged with the laws which regulate the motion of bodies. We shall proceed therefore to explain the means we use for measuring the motion of a body algebraically.

The *position* of a body in space, considering the body as a material particle, is determined at any instant by its distances from three fixed planes at right angles to each other: these distances are called the *co-ordinates* of the particle; and the position of a rigid body in space is determined at any instant by the co-ordinates of a given point of the body and the angles which three fixed lines in the body make with three fixed lines in space.

If the body be in motion the co-ordinates will be continually changing in magnitude: and one of the chief objects of the Science of Dynamics is to find the analytical relation between each co-ordinate and the time of motion.

188. We shall pause, however, a little to make a few remarks which cannot be too carefully remembered.

All our ideas of the magnitude of quantities (such as space, time, and so on) are ideas of *comparative* and not *absolute* magnitude: for a quantity may be great when compared with one standard, and small when compared with another. In consequence of this it is necessary, in order to avoid ambiguity, to choose for quantities of the same kind a certain standard to which they may be referred. This standard is called the *unit* of these quantities. Thus we speak of the unit of time, and the unit of space; by which we mean the duration of time and the extent of space which we choose as standards to which all other quantities of these species are to be severally referred.

It is by this means that quantities are made the subjects of numerical calculation. For instance, when we say that a body describes a space x in a time t, we mean, that x and t represent the ratios which the space described and the time of describing it bear to their respective units: and so of all other quantities. We forbear choosing these units at once because it generally happens, as we shall see, that by a judicious selection our formulæ may be materially simplified. Before closing these remarks we will observe, that though in the same calculation we must have only one standard of quantities of the same kind, yet in different calculations we need not retain the same unit, so long as we bear in mind what unit is chosen in each calculation. Thus in one calculation we might take the length of the mean day as the unit to which we should refer all portions of time; while in another calculation we might take a year as the unit of time. We return now to the consideration of the means of measuring the motion of a body.

189. *Velocity* is a term used to indicate the degree of quickness or slowness with which a body moves. Velocity may be uniform or variable.

190. *Uniform Velocity.* Velocity is said to be uniform when the body passes through equal spaces in equal times.

It appears, then, that the magnitude of the velocity of a body moving uniformly depends conjointly upon the space described and the time of describing the space; and is greater or less exactly in the proportion in which the space described in any given time is greater or less, and the time of describing any given space is less or greater.

Consequently when bodies move with different uniform velocities, these velocities are in the proportion of the ratios which the spaces described in any times bear respectively to the times of describing them.

Suppose, then, that a body moving uniformly with the velocity v describes a space s in the time t: also suppose that a body moving uniformly with the *unit* of velocity describes a space S in the time T: then by what precedes

$$v : 1 :: \frac{s}{t} : \frac{S}{T};$$

$$\therefore v = \frac{T}{S}\frac{s}{t}.$$

In this formula the only arbitrary quantities are S and T; we shall choose them so as to simplify the formula as much as possible: in choosing their values we fix the unit of velocity.

We shall take $S = 1$ and $T = 1$; we then have

$$v = \frac{s}{t},$$

the unit of uniform velocity being the velocity of a body moving uniformly through the unit of space in the unit of time.

It will be seen that the units of space and time are as yet quite arbitrary.

191. *Variable Velocity.* Velocity is said to be variable when the body in motion does not describe equal spaces in equal times.

Suppose a body moves uniformly and at the time t we wish to estimate its velocity. Let s' be the space described in *any* portion of time t', this time either terminating or commencing with the instant of expiration of the time t. Then, by what precedes, the velocity will equal $\frac{s'}{t'}$, however large or small t' be taken.

But when the velocity is not uniform the ratio $\frac{s'}{t'}$ is not the same for different values of t', and therefore cannot be taken as a measure of the velocity at the time t: unless we select some particular value of t' always to be used.

Now if the time t' terminate with the time t, then the space $s - s'$ is described by the body in the time $t - t'$; and therefore by Taylor's Theorem, since s is a function of t,

$$s - s' = s - \frac{ds}{dt}t' + \frac{d^2s}{dt^2}\frac{t'^2}{1.2} - \ldots\ldots\ldots$$

$$\therefore \frac{s'}{t'} = \frac{ds}{dt} - \frac{d^2s}{dt^2}\frac{t'}{1.2} + \ldots\ldots\ldots$$

If t' commence with the expiration of t, then $s + s'$ is the space described in the time $t + t'$, and

$$s + s' = s + \frac{ds}{dt}t' + \frac{d^2s}{dt^2}\frac{t'^2}{1.2} + \ldots\ldots\ldots$$

$$\therefore \frac{s'}{t'} = \frac{ds}{dt} + \frac{d^2s}{dt^2}\frac{t'}{1.2} + \ldots\ldots\ldots$$

We gather from these expressions that when t' is taken indefinitely small the values of the ratio $\frac{s'}{t'}$ are the same, and each equal to $\frac{ds}{dt}$. We shall therefore select this particular value as the measure of variable velocity.

It will be observed that in selecting this as the measure of variable velocity we do not violate any conditions previously established in reference to uniform velocity: we only restrict those conditions, inasmuch as t' may be of any value in the case of uniform velocity, but we take it indefinitely small in that of variable velocity. But, notwithstanding this, the formula $v = \frac{ds}{dt}$ includes the case of uniform motion: for if v be constant we have by integration $vt = s$ (the constant of integration vanishes, since when $t = 0$, $s = 0$), and this is the formula already adopted for uniform motion.

Hence, then, *if* v *be the velocity of a body moving uniformly or not at the time* t *and* s *be the space described in that time, the quantities* v, s, t *are connected by the formula*

$$v = \frac{ds}{dt}.$$

192. Having thus explained the means of measuring algebraically the motion of a body we shall enter upon an

enquiry into the laws which regulate this motion. Since, as far as we know, it might have pleased the Author of the Universe to endue matter with laws and properties different from those which He has chosen to impress, it is evident that these laws can be discovered by no process of abstract reasoning, but solely by an appeal to experiment.

193. As the simplest case we shall first consider the motion of a body uninfluenced by external forces. We have already defined *force* to be any cause which produces or tends to produce motion in a body; see Art. 5.

Throughout the whole universe it is impossible to find a single spot free from the action of force. It is consequently beyond our power to determine by direct experiment the nature of the motion of a body uninfluenced by external causes. But by combining the results of various experiments we shall be able to eliminate, so to speak, the principles which are foreign to our enquiry, and in that way ascertain the laws we are seeking.

Experience teaches us that the more external causes are removed the more nearly uniform is the motion of a body.

A bowl thrown along a bowling green is observed to move slower and slower till it finally stops: but the smoother the green is made, the longer does the motion continue. If the bowl be thrown with the same velocity along a pavement the motion is of longer duration; and still longer when the motion takes place on a sheet of ice. One cause of the diminution of velocity is the friction of the body on the plane: this is inferred from the fact, that the retardation is less the smoother the plane on which the motion takes place. Also any change in the uniformity of the decrease of the velocity can always be attributed to some disturbing cause; as the greater roughness of the surface and the deficiency in perfect horizontality.

The experiment shews likewise that the motion is in a straight line, unless some assignable cause produce a deviation.

Steam-carriages moving on horizontal rail-roads, when once in motion, require a constant power of the engine to maintain a uniform velocity: and since, when the motion is uniform, the retarding effect of friction and the resistance of the air may be assumed to be constant, we infer (after what we have said in the case of the bowl) that the constant power of the engine exactly counterbalances the constant retarding force, and that therefore supposing them both removed the result would be a uniform motion.

The reader is referred to Desaguliers' *Course of Experimental Philosophy*, 4to. 1734, Vol. I. Lecture V. for more experiments upon the motion of bodies.

194. Philosophers have assumed, then, as a fundamental principle of the motion of matter that

A body in motion, not acted on by any external force, will move uniformly and in a straight line.

This is called the *First Law of Motion.*

195. It must not be imagined that these experiments *prove* the truth of the law here enunciated: for the law embraces an infinite variety of cases, and many in which it would be impracticable to make experiments. Also the roughness of the experiments prevents our supposing it proved even for the cases we have mentioned. The truth is, that the law is only *suggested* by the facts we have detailed; and it remains to be seen whether or no this, in conjunction with other laws (which we shall soon consider), satisfies the tests we shall hereafter have to submit them to; whether, combined in endless variety, they will account for the numerous phenomena continually coming under our observation. It is found that they do lead to results which precisely accord with observation. Of the more obvious phenomena, the explanation of which depends on the truth of these laws, we may mention the prediction of the time of an eclipse and the certainty of its fulfilment. Results of this nature are the only satisfactory proofs.

196. It appears from the First Law of Motion that a body has no internal forces residing in it that influence its motion; for when all external forces are removed the velocity is uniform and in a straight line. In other words, matter has no inherent property of changing its state of motion. It is equally a result of experiment and observation that matter has no inherent property of changing its state of rest (Art. 4). This property of matter, that when not acted on by any external force it continues in the same state whether of rest or uniform rectilinear motion, is called its *Inertia*.

197. We proceed now to discover the laws which regulate the motion of a body when acted on by external forces.

198. But previous to this we must explain the means we use for measuring forces in terms of the magnitude of the motion they generate in a body subjected to their influence. We infer from the First Law of Motion, that when a body moves with a variable velocity force is acting on the body: and, conversely, when force acts upon a body its velocity is continually changing. Now we take the magnitude of the change of velocity during a given time as the measure of the magnitude of the force which acts upon the body: and, for the sake of distinction, when force is measured in this manner it is termed *Accelerating Force.** We have already mentioned that when force is measured statically, it is called *Pressure* (Art. 7).

199. Although the sources of force are very various yet its effect in accelerating the motion is always measured in Dynamics by the change in velocity in a given time. Thus when a body is dropped from the hand, the accelerating force of the Earth's attraction at any instant is estimated dynamically by the velocity generated in a given time after that instant. Suppose a body placed on a smooth horizontal table is drawn along by means of a thread passing over the edge of the table and attached to a falling body. The magnitude of the accelerating force which causes the body to move on the table, is measured by the change in velocity in a given time. When a body is moved along a smooth horizontal table by means of a constrained spring, the accelerating force which causes the body to move, though differing in its source from the force mentioned in the last case, is measured in the same way. If a body resting on a smooth horizontal table be set in motion by the sudden blow of another body upon it, the accelerating force which causes the motion is measured as before. When

* When a force retards the velocity of a body, it is called a *retarding force*; but still it is of exactly the same nature as an accelerating force since it is measured by the decrements instead of the increments of velocity in a given time. In short, a retarding force is an accelerating force when estimated in the direction of its action, and if the body were moving in the direction in which the force acts instead of the opposite direction, the force would become an accelerating force. Thus it will be seen that *retarding force* is merely a relative term and is included in the term accelerating force.

a ball is fired from a cannon the accelerating force which causes the motion is still measured by the velocity generated.

It will be seen in the first two of these cases (especially in the second if the descending body be small), that the motion is *gradually* communicated, the velocity increasing continuously. But in the last two cases it may perhaps be thought that the motion is *instantaneously* communicated: this is not, however, true: for the time occupied in generating the velocity is of finite duration, although, to our senses, it is of inappreciable magnitude. That it is of finite duration appears in the case of the collision of the bodies from the fact, that if a small spot of ink be put upon the point of contact of either of the bodies before the motion takes place, then after the collision the ink is found spread over a larger surface than it occupied before, and on both bodies; shewing that the bodies suffered mutual compression and then separated, and this must have occupied time. In the case of the cannon ball, the expansive force of the ignited powder acts during the time that the ball takes to move along the bore of the cannon. In both these instances, as well as in the others, the velocity of the body commences from zero and passes through successive and continuous gradations of magnitude, the only difference being that the intensity of the force originating from the collision and from the explosion is very far greater than the intensity of the force arising from the Earth's attraction; and consequently the velocity which is generated in a falling body, in a few seconds by the attraction of the Earth may be generated by impact, or other such means, in an extremely short portion of time.

200. When a body moves under the action of a force a continual change of velocity takes place; and if the force cease to act the body will move uniformly in a straight line with its last acquired velocity, as the First Law of Motion teaches us. If the force act for a finite time, then our object is to discover such laws of nature and to establish such conventional rules as shall enable us to determine the velocity acquired and the space described by the body during any portion of the time that the force is in action. If, however, the force act for only an indefinitely short time, we are concerned only with the velo-

city and position after the action of the force ceases, since the changes that take place during the action of the force are so rapid that the whole process of the action appears to our senses to be instantaneous.

We have a popular illustration of the effects of forces which act for a finite time and for an indefinitely short time in the game of cricket. The bowler rotates his arm in order to give the ball velocity, he opens his hand and the ball flies from him with the velocity acquired, and (supposing he delivers the ball *full pitch*) after moving in a curve slightly deflected downwards by the Earth's attraction is received upon the bat. Now this velocity was generated by the muscular effort of the bowler's arm acting on the ball during the finite time that he retained it in his grasp. While this is going on the batter swings his bat that it may acquire a great velocity; and the ball and bat come in collision: and what is the consequence? the ball flies back; not only is its original motion destroyed, but new motion is given to it, as if instantaneously, in an opposite direction.

We explain the phenomenon of this sudden recoil in the following manner. When the ball and bat come in contact their particles are moving in opposite directions, and tend to penetrate each other: but the molecular forces by which the particles of each of the bodies are bound together are too powerful to allow of this separation; nevertheless the relative positions of the particles are slightly changed by the yielding of the bodies, and in consequence of their unnatural restraint a mutual resultant pressure is exerted by the bat on the ball and by the ball on the bat, till their relative motion is destroyed: but the particles of the two bodies are still under restraint when the motion is destroyed, and the mutual pressure of the bodies now acts to effect their separation, and new velocity is generated: this process, which we conceive represents the actual process in nature, goes on with inconceivable rapidity in consequence of the great intensity of the molecular forces which bind the particles of each body together. If the bat split or the ball burst, then the molecular forces which held together those particles which separate were not powerful enough to resist the separation. It is evident that we are concerned not with the changes which take place during the collision but the whole change produced.

201. We have made these remarks in this place in order to shew that it is necessary in explaining the means of measuring force dynamically to consider two cases: first when the force acts for a finite time; and, secondly, when the force acts for an indefinitely short time.

In the second case the accelerating force is measured by the whole velocity generated during the action of the force. And such forces we shall term, for the sake of distinction, *Impulsive Accelerating Forces:* and in contradistinction accelerating forces which require an appreciable duration of time to manifest their effects may be termed *Finite Accelerating Forces.* We shall, however, generally drop the term *Finite:* and it must therefore be remembered that when we speak of accelerating forces we mean finite accelerating forces, and never impulsive accelerating forces unless the term *impulsive* be prefixed.

202. We proceed now to explain more fully how accelerating forces which require an appreciable duration of time to manifest their effects are measured.

Accelerating force may be uniform or variable.

203. *Uniform Accelerating Force.* When equal velocities are generated in equal times the force is said to be uniform.

It appears, then, that the magnitude of the force depends conjointly upon the velocity generated by the action of the force and the time in which this velocity is generated: and is greater or less exactly in the proportion in which the velocity generated in a given time is greater or less, and the time in which a given velocity is generated is less or greater.

Consequently when bodies are acted upon by different uniform accelerating forces, these forces are in the proportion of the ratios which the velocities generated in any times bear respectively to the times in which they are generated.

Suppose, then, that a body acted on by the constant accelerating force f has the velocity v generated in the time t: also suppose that a body acted on by the unit of uniform accelerating force has a velocity V generated in the time T: then by what precedes

$$f : 1 :: \frac{v}{t} : \frac{V}{T}; \quad \therefore f = \frac{T}{V}\frac{v}{t}.$$

In this formula the only arbitrary quantities are V and T: we shall choose them so as to simplify the formula as much as possible: in choosing their values we fix the unit of uniform accelerating force.

We shall take $V = 1$ and $T = 1$, we then have $f = \frac{v}{t}$, *the unit of uniform accelerating force being the force which generates in a body a unit of velocity in a unit of time.*

We have already chosen the unit of velocity (Art. 190); we may consequently say that the unit of uniform accelerating force is the force that causes a body during each successive unit of time in its motion to describe a space greater by the unit of space than it did during the unit of time immediately preceding.

204. Hence, in uniformly accelerated motion, s the space described from rest, t the time of describing it, v the velocity acquired during that time, and f the constant force are connected by the equations

$$v = \frac{ds}{dt} \text{ and } f = \frac{v}{t};$$

the units of v and f being given in Arts. 190. and 203. By means of these equations we can obtain four equations differing from each other, and each containing three of the quantities s, t, v, f. Thus, if we eliminate v we have

$$\frac{ds}{dt} = ft; \quad \therefore s = \frac{ft^2}{2} \text{ .... (1), } f \text{ is constant.}$$

By eliminating t we have

$$\frac{ds}{dv} = \frac{v}{f}; \quad \therefore s = \frac{v^2}{2f} \text{ .... (2).}$$

Also $v = ft$ ...... (3), $2s = vt$ ...... (4), by (2) (3).

205. *Variable Accelerating Force.* Accelerating force is said to be variable when equal degrees of velocity are not generated in equal times.

Suppose a body is moving under the action of a uniform accelerating force and at the time t we wish to estimate the magnitude of the force. Let v' be the velocity generated in *any* portion of time t', this time either terminating or commencing with the instant of expiration of the time t. Then, by what precedes, the uniform force will equal $\frac{v'}{t'}$, however large or small t' be taken.

But when the force is not uniform the ratio $\frac{v'}{t'}$ is not the same for all values of t', and therefore cannot be taken as a measure of the force at the time t, unless we select some particular value of t' always to be taken.

Now if the time t' terminate with the time t, then the velocity $v - v'$ is generated in the time $t - t'$, and therefore by Taylor's Theorem, since v is a function of t,

$$v - v' = v - \frac{dv}{dt}t' + \frac{d^2v}{dt^2}\frac{t'^2}{1.2} - \ldots\ldots$$

$$\therefore \frac{v'}{t'} = \frac{dv}{dt} - \frac{d^2v}{dt^2}\frac{t'}{2} + \ldots\ldots$$

If t' commence with the expiration of t then $v + v'$ is the velocity generated in the time $t + t'$;

$$\therefore v + v' = v + \frac{dv}{dt}t' + \frac{d^2v}{dt^2}\frac{t'^2}{1.2} + \ldots\ldots$$

$$\therefore \frac{v'}{t'} = \frac{dv}{dt} + \frac{d^2v}{dt^2}\frac{t'}{2} + \ldots\ldots$$

We gather from these expressions that when t' is taken indefinitely small the values of the ratio $\frac{v'}{t'}$ are the same and each equal to $\frac{dv}{dt}$. We shall therefore select this particular value as the measure of variable accelerating force.

It will be observed (as in the case of variable velocity) that in selecting this as the measure of variable accelerating force we do not violate any conditions previously established in reference to uniform accelerating force: we only restrict these conditions, inasmuch as t' may be of any value in the case of uniform force, but we take it indefinitely small in the case of variable force. But, notwithstanding this, the formula $f = \frac{dv}{dt}$ includes the case of uniform motion: for if f be constant we have by integration $ft = v$ (the constant of integration vanishes since when $t = 0$, then $v = 0$), and this is the formula already adopted for uniform accelerating force. Hence, *if* f *be the accelerating force, uniform or variable, which generates the velocity* v *in a body in the time* t, then f, v, t *are connected by the equation* $f = \frac{dv}{dt}$.

206. We have seen (Art. 191.) that $v = \frac{ds}{dt}$. Hence the equations connecting f, v, s, t are

$$v = \frac{ds}{dt}, \quad f = \frac{dv}{dt} \left(= \frac{d^2 s}{dt^2} \right),$$

in which it must be observed that the unit of velocity is the velocity of a body moving uniformly through a unit of space in a unit of time: and the unit of accelerating force is the uniform force which generates a unit of velocity in a unit of time.

We have thus explained the methods of estimating the magnitude of forces dynamically.

207. The next enquiry we shall make into the laws which regulate the motion of bodies is, how to calculate the combined effect of two or more causes acting simultaneously on a body.

We must, as before, appeal to experimental facts for the solution of this question. And first we will take the case of two causes acting upon the body, each of which would by itself make the body move uniformly: for instance, suppose the body projected at the same instant by impulsive forces acting in different directions.

A ball rolled along the horizontal deck of a vessel moving equably will move on the deck as it would if the vessel were at rest; this is proved by experiment. Suppose S is the deck of a boat moving uniformly on a sheet of water, fig. 74: and in a given time suppose it moves to S'. Let A be the place of the ball at the beginning of the time of motion: and B' its place in space at the end. Draw AA' in the direction of the boat's motion, and equal to the distance through which the boat has moved; and join $A'B'$. Suppose AB is the space the ball would have described if the vessel had not moved. Now, as we have already stated, experiment shews that $A'B'$, the space actually described on the deck, is the same in reference to the vessel as if the vessel had been stationary. Hence $A'B'$ is equal and parallel to AB. From this we gather, that if two causes act simultaneously on a body to produce uniform motions, each cause will have its full effect in its own direction; and the body will be found at the extremity of the diagonal of the parallelogram described on the linear spaces, which the body would have passed through under the action of the causes separately.

This principle is found to be true if one or both of the separate motions be not uniform. For a ball dropped from the top of the vertical mast of a vessel sailing uniformly, falls at the foot of the mast, although the vertical motion is not uniform.

The following experiment well illustrates this principle. Two balls are placed at the same height above the ground: one is projected horizontally, the other suffered to fall of itself: it is so contrived that the motions shall commence at the same instant. The result is that they are heard to strike the ground at the same time, although they describe very different paths, one ball having moved in a straight line, the other in a curve. This experiment shews that although one ball had a horizontal motion, still the attraction of the Earth produced the same effect on the two balls in a vertical direction.

The muscular efforts necessary to raise the arm, move the head, or raise the body are the same on board a vessel sailing equably, or in a steam-carriage moving uniformly on a rail-road, as when the ship or carriage is at rest.

It can be proved independently of any mechanical principles that the Earth revolves round its axis from east to west; but the effort of moving a body from one place to another does not depend, *cæteris paribus*, on the point of the compass towards which the motion is directed. To bring to our aid, however, more delicate tests, it is found that the motion of a pendulum is presisely the same in whatever vertical plane it vibrates, whether east and west, or north and south, or in any other direction.

For more facts and experiments we refer again to Lecture V. of Desaguliers' *Experimental Philosophy*.

208. These facts point out to us the following general principle:

When a force acts upon a body in motion, the change of motion in magnitude and direction is the same as if the force acted on the body at rest.

This is called the *Second Law of Motion.*

For the full elucidation and proof of this Law we ought to make experiments with forces of all degrees of magnitude and motions combined in all directions; since, however, this can never be accomplished, we must have recourse to the expedient spoken of in Art. 195, to satisfy ourselves of the truth of this as well as the First Law.

209. We shall now shew the importance of this Law in enabling us to refer the motion of a particle to three rectangular axes.

Since the various positions of a material particle in space are generally determined by means of co-ordinate axes, it becomes necessary to refer the motion to these lines. At any proposed instant of the motion the particle is moving with a definite velocity and in a definite direction. Now this motion may be supposed to be the result of three motions taking place simultaneously parallel to the three axes of co-ordinates. Imagine the particle, in the first place, to have only its motions parallel to the axes of y and z combined. Then, in the second place, by combining with these the motion parallel to the axis of x, we have the actual motion of the particle in space: and the *change* in the motion by this last step is, that the particle has moved to a distance x parallel to the axis of x in the time t.

But by the Second Law of Motion this change is the same as if the other motions did not exist. Hence the velocity and accelerating force of the particle parallel to the axis of x are the same as if the particle described the space x in the time t: and we have proved in Arts. 191, 205, that these are $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$: and in a similar way it may be shewn that those parallel to the axes of y and z are $\frac{dy}{dt}$, $\frac{d^2y}{dt^2}$ and $\frac{dz}{dt}$, $\frac{d^2z}{dt^2}$.

210. It follows then that when a particle is moving in space, and xyz are its co-ordinates at the expiration of the time t, the velocities of the particle parallel to the axes are

$$\frac{dx}{dt}, \quad \frac{dy}{dt}, \quad \frac{dz}{dt},$$

and the accelerating forces parallel to the three axes are

$$\frac{d^2x}{dt^2}, \quad \frac{d^2y}{dt^2}, \quad \frac{d^2z}{dt^2}.$$

Cor. By the Differential Calculus

$$\frac{ds^2}{dt^2} = \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2},$$

s being the length of the curve described. If we compare this with the formula $R^2 = X^2 + Y^2 + Z^2$ taken from Arts. 20, 21, it follows that velocities may be resolved and compounded in exactly the same way that we resolve and compound statical forces.

211. The grand Problem of Dynamics is to find the relation which exists between the motion of a system of bodies and the forces which act upon them: so that when the forces are known the motion may be determined, and vice versâ.

We have seen that if no forces act upon any part of the system, each part will move uniformly in a straight line, when once put in motion. This will also happen if the forces acting upon each particle of the system are in equilibrium with each other.

In the general case, however, each particle will move in a determinate curvilinear path, and the acceleration (or retardation) of its motion will take place whenever the forces acting on the particle are not in equilibrium, *i. e.* whenever they have a resultant. This resultant is measured at every instant by the change of velocity produced in a given time, as explained in Art. 198. Let xyz be the co-ordinates of position of any particle of the system at the expiration of the time t, then the resultant is measured dynamically by the accelerating forces $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$ acting parallel to the fixed axes of co-ordinates. These are termed the *effective accelerating forces* of the particle at the time t parallel to the axes of co-ordinates. The forces which act upon the particle to produce the motion, not including the molecular actions of the particles on each other (if there be any), are termed the *impressed forces* by way of distinction.

212. Now it is immediately evident that if at any instant of the motion we were to apply to each particle of the system forces equal in magnitude but opposite in direction to the effective forces of that particle, these would at that instant check the acceleration of the motion, or, in other words, would be in equilibrium with the impressed and molecular forces which act upon the system: and will therefore together with them satisfy the equations of condition we have deduced in the former part of this Work for the equilibrium of forces.

By this principle, the truth of which is self-evident, we shall obtain equations which connect together the forces that act upon the system and the analytical quantities $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$ and all similar quantities for the other particles. If the question be to determine the motion when the forces are given in terms of xyz and t, the solution is effected by integrating these equations. If, on the other hand, the question be to determine the forces which will cause the system of particles to move in given curves, we must differentiate the equations to the curves with respect to t, and substitute for $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}\cdots$

in the equations resulting from the application of the above principle: in this way the forces will become known.

213. But we have been supposing that the forces which act upon the system are of the nature described in Art. 201. as requiring time to manifest their effect. We shall now consider the case of impulsive forces.

Let $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ be the resolved parts parallel to the axes of the velocity of any particle of the system arising from the action of the impulsive forces. Then the effect of the impulsive forces is the same as three impulsive accelerating forces acting parallel to the axes and equal to $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, (Art. 201); these are termed the *effective* impulsive accelerating forces: and the original forces are termed the *impressed* impulsive forces.

Wherefore it is immediately evident that if, at the instant of the action of the impulsive forces on the system, we were to apply to each particle impulsive forces equal but opposite to the effective impulsive forces of that particle, these would check the effect of the impulsive forces actually impressed on the system and would consequently with them satisfy the equations of condition for the equilibrium of forces.

As before, then, we obtain equations by means of which the motion of the system may be calculated.

214. Now in the calculations of the conditions of equilibrium of forces acting upon a single particle, a rigid body, or any material system given in Chapters I, II, and III of Statics we have considered the magnitudes of the forces to be estimated statically; in other words, we have supposed them to be pressures. Wherefore before we can make use of the results of those Chapters for determining the equations of motion of a system, in the manner explained in Arts. 212, 213, we must discover the relation that connects the dynamical and statical measures of force; so that when we know the degree of acceleration of a force, we may be able to determine the magnitude of the pressure that the force causes the body on which it acts to exert; and *vice versâ*. It is manifest that

some relation between these two measures of force must exist, since the cause of the pressure and the cause of the motion and of the change of motion are the same. But since causing pressure and causing motion are two properties of force which, abstractedly speaking, have no common character, we cannot discover the relation they bear to each other by reasoning *à priori*; but must again appeal to experiment.

215. When two balls of the same size and substance are dropped at the same instant from the same altitude they move downwards in exactly the same manner; having the same velocity at every instant and having moved through the same spaces during any given time. If the bodies be connected the motion is the same. And the same would be the case whatever number of balls were connected. Hence it appears that although the weight of a body of homogeneous structure, or the statical measure of the Earth's attraction, varies as the mass of the body, yet the accelerating force, or the dynamical measure, is invariable when the experiments are made at the same place. Newton made a variety of experiments with gold, silver, lead, glass, sand, common salt, wood, water, and wheat, and arrived at the same result. *Principia*, Vol. III. Prop. VI. It is found by experiments made under a receiver exhausted of air that a guinea and a feather fall in exactly the same manner and strike the plate of the air pump at the same instant, if they are set at liberty together and from the same altitude. These experiments shew that the accelerating force of all falling bodies at the same place is the same: and, after what has been proved of a homogeneous mass of matter, these experiments lead us to conclude that bodies differ in weight at the same place in the same proportion as their masses differ. We therefore infer that at the same place on the Earth's surface the weight of a body varies as its mass. Let M be the mass of a body of which the weight is W; then $W = Mg$, g being some arbitrary quantity which is constant for the same place and depends upon the units of weight and mass. It remains to be determined what change the weight undergoes if the body be removed to a place where the accelerating force is different: or if by any contrivance the accelerating force of a body be changed without changing the

place of the experiment. We shall for this purpose describe a machine invented by Atwood: see Atwood on *Rectilinear Motion* for a full explanation.

216. Four wheels, two of which A and B are represented in figure 75, the other two being hid by these, are placed parallel to each other, their centres being fixed so as to allow of rotation with as little friction as possible: A and B are placed as near as possible without touching: and so are the other two wheels. Upon these four rests the axle of another wheel C placed midway between A and B and the other two wheels: a fine string as flexible and inextensible as possible is passed over the circumference of C and two weights P and Q are attached to its extremities. When P and Q are left to themselves the heavier will descend and draw up the lighter of the two. It will be readily understood that the object of the four wheels is to diminish the friction on the axle of C; which it does very considerably, since the friction of *rolling* is far less than that of *rubbing*. Suppose that P descends, then $P - Q$ is the weight or pressure which causes the motion and $P + Q$ is the weight put in motion. It is found by experiment that the inertia of the wheels produces the effect of adding to the weight moved without adding to the pressure producing motion. Atwood determines by experiment what this weight is, we shall call it W. Hence $P - Q$ is the weight causing the motion and $P + Q + W$ is the weight put in motion. A graduated scale of inches is placed behind the thread supporting P in order to mark the motion of P. The excellence of this machine consists in this, that we can have bodies falling with various degrees of acceleration and as slowly as we please by altering P and Q. The time of motion is marked by a seconds pendulum.

Now suppose P is placed with its lowest surface level with the zero point of the scale and set at liberty at any tick of the pendulum: it is always found, however much P and Q are altered, that in each experiment the spaces described by P in successive seconds form an arithmetic progression, and therefore that the accelerating force in each case is uniform. Also it is found that the common differences of the series in the various experiments are proportional to the respective values of the ratio $\frac{P-Q}{P+Q+W}$. This is proved by numerous experiments, for the details of which we refer to the work of Atwood.

We gather, then, from this that the accelerating force varies as $\frac{P-Q}{P+Q+W}$ in the different experiments, and therefore the pressure producing motion (or $P-Q$) varies as the product of the accelerating force and the weight moved (or $P+Q+W$), and therefore as the product of the accelerating force and the mass moved, since the weight of a body at the same place varies as the mass (Art. 215), and these experiments were made at the same place.

217. The product of the mass of a body and the accelerating force is called by Newton the *Moving Force* of the body: and the product of the velocity and mass of a body he calls its *Momentum*, or quantity of motion. These experiments therefore shew that the pressure communicating the motion varies as the moving force, or as the momentum generated in the body: for moving force must be measured by the momentum generated in a given time, since accelerating force is measured by the velocity generated in a given time, Art. 198.

218. We shall now give the results of experiments with pendulums. It is found by numberless trials that the time of oscillation of a leaden ball suspended by a fine thread and moving through any very small angle is constant for the same length of thread, but for different lengths it varies as the square root of the length. Now let s be the small circular arc measuring the distance of the centre of the ball from its point of rest, fig. 76, t the time of describing s, l the length of the pendulum: then l varies as t^2, by experiment. M the mass of the ball, and Mg its weight, Art. 215. Pt a tangent to the arc s at P: then the weight Mg is employed partly in producing the motion and partly in stretching the thread: the part producing motion $= Mg \cos tPW = Mg \sin A$;

$$\therefore \text{ the pressure} = Mg \sin A = Mg \frac{s}{l} \text{ nearly} \propto M \frac{s}{t^2}.$$

But if f be a uniform accelerating force causing a body to describe a space s in the time t (and the acceleration in this case is ultimately uniform) then $2s = ft^2$;

$\therefore$ the pressure $\propto Mf \propto$ moving force, as before.

219. We must now enquire into the connexion between pressure and the motion generated or destroyed when impulsive forces act.

We have already explained the nature of impulsive forces; and have shewn that they differ from finite forces solely in intensity, and that we measure them dynamically by the velocity generated during the action, and not by the velocity generated in a unit of time*. The result of the last four Articles must therefore be true for impulsive forces; and we shall assume that impulsive pressure is proportional to the momentum generated or destroyed. In fact, the pressure is so enormous that to make its measure a matter of experiment would be very difficult. We can nevertheless mention some experiments which illustrate, and in part prove, that impulsive pressure varies as the momentum generated. We shall, however, first speak of the elasticity of bodies.

220. It is found that all rigid bodies rebound more or less when struck together: this property is termed their *elasticity*. Consequently no bodies are totally devoid of this property: yet some have it more eminently than others; balls of clay have little elasticity, but ivory balls and balls of glass are considerably elastic. The degree of elasticity is measured

* The following experimental fact seems to shew that impulsive forces are of the same nature as finite forces, generating or destroying velocity by continuous gradations.

Robins' experiments on the velocity of bullets and cannon balls lead to the following result. If bullets of the same diameter and density impinge on the same solid substance with different velocities they will penetrate that substance to different depths, which will be in the duplicate ratio of those velocities nearly; Robins' *Mathematical Tracts*, edited by Wilson, Vol. I. p. 152.

This was proved by various experiments. Now a property of uniformly accelerating (or retarding) forces is, that the squares of the velocities generated (or destroyed) are proportional to the spaces described: Art. 204. Hence the retarding force of the solid substance used in each experiment was a uniform force. But the duration of its action was so short and its intensity so great, that although the changes effected by the force were continuous, yet they were so rapid, that the force comes under the denomination of what we term impulsive forces.

by the ratio which the velocity of rebound bears to the velocity at the first contact. The elasticity is perfect when these two velocities are the same, but this is a limit which no bodies actually attain. The cause of this property of matter is of course conjectural, and our conclusions as to its laws are deduced solely from experiment.

Tables of the results of a series of experiments made by Mr Hodgkinson, of Manchester, on the elasticity of bodies will be found in Vol. III. p. 534. of the Reports of the British Association for the Advancement of Science. The following are the Conclusions deduced.

(1). All rigid bodies are possessed of some degree of elasticity: and among bodies of the same nature, the hardest are generally the most elastic.

(2). There are no perfectly hard inelastic bodies, as assumed by the earlier, and some modern writers on Mechanics.

(3). The elasticity as measured by the velocity of recoil divided by the velocity of impact is a ratio, which, though decreasing as the velocity increases, is nearly constant, when the same rigid bodies are struck together with considerably different velocities.

(4). The elasticity, as defined in (3), is the same whether the impinging bodies be great or small.

(5). The elasticity is the same, whatever be the relative weights of the impinging bodies.

(6). In impacts between bodies differing very much in hardness, the common elasticity is nearly that of the softer body.

(7). In impacts between bodies of which the hardness differs in any degree the resulting elasticity is made up of the elasticities of both; each body contributing a part of its own elasticity in proportion to its relative softness or compressibility.

221. Hence when one body impinges on another a mutual pressure takes place, which by incessantly acting as the compression of the bodies goes on finally checks their relative motion; after this new velocities are generated and the balls separate: and our object now is to enquire what the connexion is between the velocity destroyed during the compression of the balls and the pressure that destroyed it, and also between the velocity gene-

rated and the pressure by which it is generated. The difficulty of discovering this from experiment arises partly from the immediate action of the forces of restitution after the force of compression ceases to act: in consequence of which any experiment upon impinging balls is sure to involve the action of two sets of impulsive forces; first, those that act while the compression of the figures of the bodies is taking place, and secondly, those that act during the restitution of figure. We may call the first kind *impulsive forces of the nature of collision*, and the second kind *impulsive forces of the nature of explosion*: by the first velocity is destroyed, by the second velocity is generated. By certain artifices, however, we may overcome this difficulty.

222. We shall mention a few experiments which will lead us to a satisfactory conclusion.

Let A and B be two balls (fig. 77.) suspended by threads from two points C and D, so that they may just touch when at rest and have their centres in the same horizontal line: FAE, fBe circular arcs with centres C, D: now the velocities of a ball in falling through different arcs of a circle to the lowest point are in the proportion of the chords of those arcs, as is proved in the note*. Let therefore a scale be placed

* The pressure producing motion $= W \cos rPt = W \sin\theta$ (fig. 76): therefore the moving force (Art. 218), and also the accelerating force (since the mass of the body is invariable) varies as $\sin\theta$. Let α be the greatest value of θ: l the length of the thread: then $l(\alpha-\theta)$ is the space described by the body in the time t; and, supposing that at each instant the body is moving in the tangent line to the arc, the accelerating force $= l\dfrac{d^2(\alpha-\theta)}{dt^2} = -l\dfrac{d^2\theta}{dt^2}$;

$$\therefore -\frac{d^2\theta}{dt^2} \text{ varies as } \sin\theta, \text{ and } = 2c^2\sin\theta \text{ suppose;}$$

$$\therefore \left(\frac{d\theta}{dt}\right)^2 = 4c^2(\cos\theta + \text{const.})$$

$$\text{When } \theta = \alpha, \text{ velocity} = 0, \ \frac{d\theta}{dt} = 0;$$

$$\therefore \left(\frac{d\theta}{dt}\right)^2 = 4c^2(\cos\theta - \cos\alpha),$$

$$\text{angular velocity at lowest point} = 2c\sqrt{1-\cos\alpha} = 4c\sin\frac{\alpha}{2}$$

$$= 2c \text{ chord } \alpha.$$

Hence the velocity varies as the chord of the arc.

below A and B so graduated as to mark the velocities of the balls A and B when at the lowest positions by knowing the arcs through which they move.

Now suppose a small steel point is fixed in A so that when A and B come in contact separation is prevented. It is found that if A and B are drawn through arcs of which the chords are inversely as the masses of the bodies, and then left to themselves, they will impinge and exactly destroy each others velocity, a small allowance being made for the resistance of the air. If one of the balls be moved through a greater arc, then when the balls come in contact they will not be at rest, but move in the direction in which that ball was moving before impact. This shews, that when the bodies impinge on each other with equal momenta, their mutual pressures exactly balance the momenta; but if the momentum of one ball be greater than the momentum of the other the mutual pressure is not sufficient to overcome the momentum of the first, but not only overcomes the momentum of the second but generates new momentum. This is found to be true for masses and velocities of all finite magnitudes.

Desaguliers mentions an experiment (*Experimental Philosophy*, Vol. II. Lecture VI. p. 62.) in which he replaced A and B by two cylinders closed at the outer extremities; one was introduced a short way into the other, and the cavity filled with gunpowder: it was found that after the explosion the cylinders rose through arcs varying inversely as their masses. Consequently the momenta generated by the action of the impulsive force of the explosion were the same.

In these experiments suppose that the mass and velocity of the body A remains the same: then if we vary the mass or velocity of B we must change them so that the pressure on A shall be the same: and this condition is that their product shall be constant. Hence, then, a given impulsive pressure generates in different bodies the same momentum. This is all that these experiments prove: they do not shew that the pressure varies directly as the momentum generated. This, however, we infer as in Art. 219.

223. Wherefore the results of the last nine Articles lead us to the following Principle.

When pressure communicates motion to a body, the momentum generated in a given short time is proportional to the pressure.

This is called the *Third Law of Motion.*

Newton has given this Law under the more general form, that *Action and Reaction are equal and opposite.* If action and reaction in dynamics be measured by the quantity of motion gained and lost, this is an immediate deduction from our Third Law of Motion.

224. Leibnitz in the *Acta Eruditorum* 1695, p. 149, and after him Jean Bernoulli and others raised objections to Newton's measure of force, contending that it ought to be proportional to the product of the mass and the square of the velocity.

In their own words, "A force is said to be dead (*vis mortua*) which consists in nothing but the endeavour, or the tendency to motion. Such is gravity" it was said "as long as a heavy body hung by a thread endeavours to descend, but cannot actually descend. A force is said to be alive or quick (*vis viva*) which always accompanies actual motion, and tends to produce a local motion. There is such a force in a body falling by gravity when it has already acquired some degrees of velocity." Professor Wolfius, quoted by Desaguliers; *Exp. Phil.* Vol. II. p. 72, 80.

Our object in making this quotation is to shew the origin of the term *vis viva*, which, as a term only, is still in use among us. The incorrectness of the above notion appears from the fact that it implies that matter has some inherent power of exerting force when in motion which it has not when at rest.

The reasoning by which these philosophers were led to the idea that pressure should be measured by the product of mass and the square of the velocity generated appears from the nature of the experiments from which they argued. It was found that when balls of equal size and density impinged upon clay they penetrated the clay by spaces which are as the squares of the velocities of impact: as in the example of the note in page 197. It was reasoned (as in that note) that when balls are projected against different solid substances so as to penetrate to the same depth the forces will be as the squares of the velocities: and hence arises the mistake, for this supposes that we measure force by the velocity generated or destroyed in moving through a given space irrespective of the time of motion: but we measure force by the velocity generated in a given time irrespective of the space described. If then we retain our definition of force estimated dynamically by the velocity generated in a given time, the force must vary as the product of the mass and the velocity generated in a given time: but if we were to adopt the second measure of force estimated dynamically by the velocity generated in moving through a given space we should find that the force varies as the product of the mass and the square of the velocity generated.

The term *vis viva* is still used to express the product of the mass and square of the velocity.

225. We shall now choose the *units* of pressure, or statical force, and mass.

Let P be the pressure, f the accelerating force and M the mass, then P varies as Mf. Let the unit of pressure be that of a body of which the mass is M' and the accelerating force f': then

$$P : 1 :: Mf : M'f';$$

$$\therefore P = \frac{Mf}{M'f'};$$

we shall choose M' and f' so as to simplify this formula as much as possible: let $M' = 1$, $f' = 1$; then

$$P = Mf \text{ ....} (1),$$

the unit of pressure being the pressure of a body of a unit of mass and acted on by the unit of accelerating force.

When the pressure is impulsive *its unit is that of a body of mass unity moving with a unit of velocity:* if we, as above, suppose

$$P = Mv \text{ ....} (2).$$

Let W be the weight of a body of which the mass is M, and let the accelerating force of the Earth's attraction, or gravity, equal g: then

$$W = Mg \text{ .... } (3).$$

Also suppose that the body is homogeneous, of density ρ, and volume V: let ρ' and V' be the density and volume of a body of which the mass equals the unit of mass: then

$$M : 1 :: \rho V : \rho' V';$$

$$\therefore\ M = \frac{\rho V}{\rho' V'};$$

we shall choose ρ' and V' so as to simplify this formula as much as possible: let $\rho' = 1$, $V' = 1$: then

$$M = \rho V \text{ .... } (4),$$

the unit of mass being the mass of a body of a unit of volume and a unit of density.

By (3) (4) we have

$$W = \rho V g \text{ .... } (5).$$

Now by experiments made by Atwood's Machine described in Art. 216, it is found that the spaces described by a body falling freely from rest are 16.1, 3×16.1, 5×16.1, ... feet in the first, second, third, ... seconds of time. Hence gravity is a constant force and generates a velocity of 2×16.1 or 32.2 feet in a second of time. Wherefore if we take *a foot as the unit of length and a second as the unit of time we have*

$$g = 32.2 \text{ .... } (6),$$

$$W = 32.2\, V\rho \text{ .... } (7);$$

and when $\rho = 1$ and $W = 1$, $V = \frac{1}{32.2}$; hence the relation among the units chosen gives this result, *that the unit of weight is the weight of a body of the unit of density and volume equal the* 32.2^{th} *part of the unit of volume.* The density of distilled water is taken generally as the unit of density; and a cubic foot as the unit of volume.

226. Having discovered the relation between the statical and dynamical measures of force, (which was the desideratum in Art. 214), we may now enunciate the Principles mentioned in Arts. 212, 213.

The Third Law of Motion and the units of measure chosen in the last Article shew, that finite statical force equals the moving force of the body resulting from its action; and impulsive statical force equals the momentum of the body resulting from its action. We shall suppose in what follows that statical forces are all replaced by these dynamical measures.

Let m be the mass of any particle of a material system, xyz its rectangular co-ordinates, then,

I. *If the system be in motion under the action of* finite *forces, the forces*

$$-m\frac{d^2x}{dt^2}, \quad -m\frac{d^2y}{dt^2}, \quad -m\frac{d^2z}{dt^2}$$

acting on m *parallel to the axes of* x y z *respectively, and similar forces acting on each of the other particles of the system, must, together with the impressed moving forces (Art.* 211*), satisfy the conditions of equilibrium.*

II. *If the system be acted on by* impulsive *forces, the forces*

$$-m\frac{dx}{dt}, \quad -m\frac{dy}{dt}, \quad -m\frac{dz}{dt}$$

acting on m *parallel to the axes, and similar forces acting on each of the other particles of the system, must, together with the impressed impulsive forces or momenta, satisfy the conditions of equilibrium.*

227. These Principles are the interpretation of the Three Laws of Motion into mathematical language. The Laws themselves are the results solely of observation and experiment. But these Principles are the results not only of the Laws, but also of certain conventional rules for measuring the quantities treated of; without which indeed we could not make the phenomena resulting from the Laws subjects of calculation. We must therefore be careful to interpret all results to which they lead us in conformity to these conventional rules.

CHAPTER II.

THE MOTION OF A MATERIAL PARTICLE.

228. LET xyz be the co-ordinates to the particle at the end of the time t, and m its mass.

Suppose the accelerating forces acting on the particle are resolved parallel to the axes and compounded into three X, Y, Z in these directions.

Then by the first of the Principles enunciated in Art. 226, the moving forces

$$mX, \quad mY, \quad mZ$$

$$-m\frac{d^2x}{dt^2}, \quad -m\frac{d^2y}{dt^2}, \quad -m\frac{d^2z}{dt^2}$$

will be in equilibrium with each other at the time t.

Hence by the conditions of equilibrium of a particle acted on by any forces given in Art. 23, we have the equations

$$mX - m\frac{d^2x}{dt^2} = 0, \qquad mY - m\frac{d^2y}{dt^2} = 0,$$

$$mZ - m\frac{d^2z}{dt^2} = 0,$$

$$\text{or} \quad \frac{d^2x}{dt^2} = X, \quad \frac{d^2y}{dt^2} = Y, \quad \frac{d^2z}{dt^2} = Z.$$

These are called the equations of motion of the material particle: and by integration we shall have three equations involving x, y, z, t and constant quantities.

By eliminating t we have two equations involving x, y, z without t. These are the equations to the curve described by the particle.

229. In the course of the integration six arbitrary constants will be introduced: these are determined by the initial circumstances of the motion: by the term *initial* we mean at the epoch from which t is measured*. The general integrals determine the *nature* only and not the *dimensions* of the curve described. The dimensions depend upon the initial conditions. These are, first, the three co-ordinates which give the position of the particle at the commencement of the motion. By substituting these in the three integrals and putting $t = 0$ we have three equations involving the six arbitrary constants and known quantities. The other initial quantities are the velocity and direction of projection, or, which amounts to the same, the initial velocities* parallel to the three axes.

By differentiating the three integrals with respect to t, we shall have three equations involving x, y, z, $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ and the arbitrary constants: and giving the variable quantities their initial values we have three more equations involving the arbitrary constants and known quantities.

From these six equations, then, we can determine the six arbitrary constants and the problem is completely solved.

230. Suppose, on the other hand, the problem to be solved be the converse of the one already considered, namely, to determine the forces which will make a body describe a given curve.

* If any particle of the system commence its motion with a finite velocity, this is imparted to it by an impulsive force, which acts for so short a time as to produce its effect instantaneously : for this reason it is evidently indifferent whether we measure the time from the commencement or termination of the action of the impulsive force: and the term *initial velocity*, though there is no velocity, rigorously speaking, at the commencement of the motion, is perfectly allowable.

In short, when a system of material particles is projected into space and submitted to the action of surrounding bodies, two entirely different systems of forces act upon the particles. The first is a system of impulsive forces, of the nature described in Arts. 199, 200: these produce their effect in an indefinitely short time, after which they cease to act. The second system consists of forces of the nature described in the same Articles: these require a length of time of sensible duration to produce their effect. This latter system differs from the former merely in *intensity*.

We shall in this case have given *two* equations involving xyx, from which we are to obtain the *three* quantities $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$ or X, Y, Z: this shews that the problem is indeterminate.

The following is the way to proceed.

The two equations involving x, y and z must be differentiated twice with respect to t: by this means we have two equations involving the four quantities X, Y, Z, and velocity (v).

$$\text{But } v^2 = \frac{ds^2}{dt^2} = \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2};$$

$$\therefore \tfrac{1}{2}\frac{d \, . \, v^2}{dt} = X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt},$$

$$\tfrac{1}{2}\frac{d \, . \, v^2}{dx} = X + Y\frac{dy}{dx} + Z\frac{dz}{dx}.$$

This is a third equation involving X, Y, Z, v. By assuming a value of any one of these four quantities the other three may be determined, in terms of xyz.

RECTILINEAR MOTION.

231. PROP. *A body is acted on by a uniform force (that of gravity for instance) the motion being in the line of action of the force: required to determine the motion.*

Let x be the distance of the body at the time t from a fixed point in its course, measured in the direction of the force: and let g be the force.

Then the equation of motion is

$$\frac{d^2x}{dt^2} = g.$$

By integration we have

$$\frac{dx}{dt} = gt + C, \; C \text{ being an arbitrary constant.}$$

To determine C we must refer to the *initial* circumstances of the motion.

Suppose the body is projected with a velocity u in the direction in which the force acts.

$$\text{Then when } t = 0, \quad \frac{dx}{dt} = u; \quad \therefore \; u = C;$$

$$\therefore \; \frac{dx}{dt} = gt + u.$$

Integrating again

$$x = \tfrac{1}{2} g t^2 + ut + C'.$$

Let a be the distance of the body from the origin of x at the commencement of the motion: then the initial circumstances are that when $t = 0$, $x = a$; $\therefore$ $C' = a$;

$$\therefore \; x = a + ut + \tfrac{1}{2} g t^2,$$

or the space described in the time t is $ut + \frac{1}{2}gt^2$.

This is a necessary consequence of the second law of motion.

If the body be not projected then $u = 0$ and $x = a + \frac{1}{2}gt^2$.

If the body be projected with a velocity u in a direction *opposite* to that in which x is measured, then when $t = 0$, $-\dfrac{dx}{dt} = u$ since x is diminished as t increases:* and

$$x = a - ut + \tfrac{1}{2} g t^2.$$

* In Art. 191, it was shewn that if s be the space described in the time t by a body, and v its velocity at the end of that time, then $\dfrac{ds}{dt} = v$.

But if the space be measured in a direction opposite to that in which the *motion* takes place, then, b and s' being the distances of the point from which the space is measured at the commencement of the motion and at the end of the time t, then $s = b - s'$ and $-\dfrac{ds'}{dt} = v$.

Also in Art. 205, it was shewn that if f be the magnitude of the force at the end of the time t, then $\dfrac{d^2 s}{dt^2} = f$. If, as before, the space be measured in the direction opposite to that of the action of the *force*, then $-\dfrac{d^2 s'}{dt^2} = f$.

PROP. *A body falls towards a centre of force the intensity of which varies directly as the distance of the body from the centre: required to determine the motion.*

232. Let μ be the magnitude of the force at a distance unity from the centre of force: this is called the *absolute* force of the centre: a the distance of the body from the centre at the commencement of the motion, x the distance at the time t.

Then μx is the magnitude of the force at the distance x: and the equation of motion is

$$-\frac{d^2x}{dt^2} = \mu x,$$

the negative sign being taken because the tendency of the force is to diminish x;

$$\therefore\ 2\frac{dx}{dt}\frac{d^2x}{dt^2} = -2\mu x\frac{dx}{dt}$$

$$\text{integrating,}\quad \frac{dx^2}{dt^2} = C - \mu x^2,$$

C being an arbitrary constant to be determined by the initial circumstances of the motion: these are that when $t=0$, $x=a$, and the velocity, or $\frac{dx}{dt}$, $=0$; $\therefore\ C = \mu a^2$;

$$\therefore\ \frac{dx^2}{dt^2} = \mu(a^2 - x^2);$$

$$\therefore\ -\frac{dt}{dx} = \frac{1}{\sqrt{\mu}}\frac{1}{\sqrt{a^2-x^2}},$$

the negative sign being taken in extracting the square root because x diminishes as t increases.

$$\text{Integrating,}\quad t = \frac{1}{\sqrt{\mu}}\cos^{-1}\frac{x}{a} + C'$$

$$\text{when } t=0,\ x=a,\quad \therefore\ C'=0;$$

$$\therefore\ t = \frac{1}{\sqrt{\mu}} \cos^{-1}\frac{x}{a}$$

when $x = a$, the body arrives at the centre;

$$\therefore \text{ time of falling into the centre } = \frac{\pi}{2\sqrt{\mu}}.$$

The velocity is zero when $\frac{dx}{dt} = 0$, or when $x = a$ and $-a$: hence the body passes through the centre and stops at a distance on the other side equal to the original distance. From this point it will return to its original position and continually oscillate over the same space: the time of oscillation from rest to rest is $\frac{\pi}{\sqrt{\mu}}$. It is remarkable that this is independent of the initial distance of the body from the centre of force.

The expression for the time shews, that the body will oscillate backwards and forwards: for suppose α is the least positive value of $\cos^{-1}\frac{x}{a}$ for any given value of x, then

$$t = \frac{\alpha}{\sqrt{\mu}} \text{ or } \frac{2\pi - \alpha}{\sqrt{\mu}} \text{ or } \frac{2\pi + \alpha}{\sqrt{\mu}} \ldots\ldots$$

or, generally, $\frac{2n\pi \mp \alpha}{\sqrt{\mu}}$, n being any integer.

This proves that the body will periodically arrive at any given point of its path: the intervals of time between the successive arrivals being $\frac{2\pi - 2\alpha}{\sqrt{\mu}}$ and $\frac{2\alpha}{\sqrt{\mu}}$ alternately.

Prop. *Suppose the body in the last Proposition is projected with a velocity* u *in the line in which the force acts.*

233. As before we have

$$\frac{dx^2}{dt^2} = C - \mu x^2$$

when $x = a$, $\frac{dx}{dt} = u$ or $-u$ according as the direction of projection is from or towards the centre: in both cases

$$u^2 = C - \mu a^2$$

$$\frac{dx^2}{dt^2} = u^2 + \mu(a^2 - x^2).$$

Considering the motion *towards* the centre

$$-\frac{dt}{dx} = \frac{1}{\sqrt{\mu}} \frac{1}{\sqrt{a^2 + \frac{u^2}{\mu} - x^2}}$$

$$t = \frac{1}{\sqrt{\mu}} \cos^{-1} \frac{x}{\sqrt{a^2 + \frac{u^2}{\mu}}} + C$$

when $t = 0$, $x = a$;

$$\therefore t = \frac{1}{\sqrt{\mu}} \left\{ \cos^{-1} \frac{x}{\sqrt{a^2 + \frac{u^2}{\mu}}} - \cos^{-1} \frac{a}{\sqrt{a^2 + \frac{u^2}{\mu}}} \right\}.$$

The greatest distance to which the body goes from the centre is $\sqrt{a^2 + \frac{u^2}{\mu}}$, and the time of a complete oscillation from rest to rest is as before $\frac{\pi}{\sqrt{\mu}}$.

PROP. *A body falls towards a centre of force the intensity of which varies inversely as the square of the distance of the body: required to determine the motion.*

234. Let μ be the *absolute* force of the centre as before: then the force at distance x is equal to $\frac{\mu}{x^2}$: and the equation of motion is

$$-\frac{d^2x}{dt^2} = \frac{\mu}{x^2};$$

$$\therefore\ 2\frac{dx}{dt}\frac{d^2x}{dt^2} = -\frac{2\mu}{x^2}\frac{dx}{dt};$$

$$\text{integrating, } \frac{dx^2}{dt^2} = \frac{2\mu}{x} + C$$

$$\text{when } x = a,\ \frac{dx}{dt} = 0;\ \therefore\ 0 = \frac{2\mu}{x} + C$$

$$\frac{dx^2}{dt^2} = 2\mu\left(\frac{1}{x} - \frac{1}{a}\right)$$

$$-\frac{dt}{dx} = \sqrt{\frac{a}{2\mu}}\sqrt{\frac{x}{a-x}} = \sqrt{\frac{a}{2\mu}}\frac{x}{\sqrt{ax-x^2}};$$

$$\therefore\ \frac{dt}{dx} = \sqrt{\frac{a}{2\mu}}\frac{\frac{a}{2} - x - \frac{a}{2}}{\sqrt{ax-x^2}}$$

$$t = \sqrt{\frac{a}{2\mu}}\left\{\sqrt{ax-x^2} - \frac{a}{2}\text{vers}^{-1}\frac{2x}{a}\right\} + C$$

$$\text{when } t = 0,\ x = a;\ \therefore\ 0 = -\sqrt{\frac{a}{2\mu}}\frac{a\pi}{2} + C$$

$$t = \sqrt{\frac{a}{2\mu}}\left\{\frac{\pi}{2}a - \frac{a}{2}\text{vers}^{-1}\frac{2x}{a} + \sqrt{ax-x^2}\right\}$$

when the body arrives at the centre $x = 0$, therefore time of falling to the centre $= \frac{\pi}{\sqrt{\mu}}\left(\frac{a}{2}\right)^{\frac{3}{2}}$.

235. In a subsequent part of this work we shall see, that the attraction of the Earth on external bodies varies inversely as the square of the distance from its centre, supposing the Earth a sphere. And that the attraction on any bodies within the Earth varies directly as the distance from the centre.

It is for this reason that in the foregoing Propositions we have selected these particular laws of force. No other laws are known to exist in the universe.

PROP. *A body acted on by the constant force of gravity moves down an inclined plane: required to calculate the motion.*

236. Let the plane of the paper be the vertical plane in which the motion takes place: AB (fig. 78.) the intersection of this with the inclined plane: P the position of the body at the time t, A being its place when $t = 0$: α the angle the plane makes with the horizon. Now the forces which are acting upon the body at the time t are the force of gravity g, which acts vertically and the pressure of the plane on the body. If we resolve the forces in the direction of the motion we shall not introduce the pressure.

$$\text{Let } AP = x.$$

Now the part of g resolved along the line AP is $g \sin \alpha$, hence the equation of motion is

$$\frac{d^2x}{dt^2} = g \sin \alpha,$$

and the results will be precisely the same as those in Art. 231, if we there substitute $g \sin \alpha$ instead of g.

If we wish to know the pressure P upon the plane, by resolving the forces perpendicularly to the line of motion we have, since no space is described by the body in that direction,

$$0 = mg \cos \alpha - P \quad \text{(Art. 225.)}$$

Hence $P = mg \cos \alpha$, and is constant and is in proportion to the weight of the body in the ratio $\cos \alpha : 1$.

CURVILINEAR MOTION OF A PARTICLE.

PROP. *A body is acted on by the constant force of gravity, which acts in parallel lines: required to determine the motion of the body when it is projected in a direction not vertical.*

237. Let the axis of y be vertical and reckoned positive upwards and drawn through the point of projection. The motion will evidently take place wholly in a vertical plane.

Let the axis of x be drawn in this plane the origin being the point of projection A, (fig. 79.) Let also g be the accelerating force of gravity.

Then the equations of motion are

$$\frac{d^2x}{dt^2} = 0, \quad -\frac{d^2y}{dt^2} = g.$$

By integration

$$\frac{dx}{dt} = c, \quad \frac{dy}{dt} = c' - gt,$$

c and c' being constants to be determined by the circumstances of projection.

Let u be the velocity of projection, α the angle its direction makes with the axis of x.

$$\text{Then when } t = 0, \ \frac{dx}{dt} = u\cos\alpha, \ \frac{dy}{dt} = u\sin\alpha;$$

$$\therefore\ u\cos\alpha = c, \quad u\sin\alpha = c';$$

$$\therefore\ \frac{dx}{dt} = u\cos\alpha, \quad \frac{dy}{dt} = u\sin\alpha - gt.$$

Integrating again

$$x = ut\cos\alpha, \quad y = ut\sin\alpha - \tfrac{1}{2}gt^2 \ldots\ldots (1),$$

no constants are added after integration because when $t = 0$, $x = 0$ and $y = 0$ by the circumstances of the problem.

These two equations determine the position of the body at any time.

238. To find the curve described we eliminate t from equations (1); Art. 228;

$$\therefore\ y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}.$$

This is the equation to a parabola.

For it may be written

$$\left(x - \frac{u^2}{g}\cos\alpha\sin\alpha\right)^2 = -\frac{2u^2}{g}\cos^2\alpha\left(y - \frac{u^2}{2g}\sin^2\alpha\right).$$

And by transferring the origin to a point of which the co-ordinates are

$$\frac{u^2}{g}\cos\alpha\sin\alpha \text{ and } \frac{u^2}{2g}\sin^2\alpha$$

the equation becomes

$$x^2 = -\frac{2u^2}{g}\cos^2\alpha y,$$

which is the equation to a parabola with its axis vertical and measured downwards,

$$\text{and latus rectum} = \frac{2u^2}{g}\cos^2\alpha.$$

The *range* is the distance between the point of projection and the point where the body strikes the ground. The curve described is called the *projectile*.

PROP. *To find the range of the projectile, the time of flight, and the greatest height the body reaches.*

239. When $y = 0$, $x\tan\alpha - \dfrac{gx^2}{2u^2\cos^2\alpha} = 0$;

$$\therefore\ x = 0 \text{ and } x = \frac{2u^2}{g}\cos^2\alpha\tan\alpha = \frac{u^2}{g}\sin 2\alpha,$$

this latter value of x is the *range on a horizontal plane.*

If the body be projected from an inclined plane perpendicular to the plane of the projectile, then, if i be the angle of inclination of the plane to the horizon, $y = x\tan i$ is the equation to the intersection of this plane and the plane of motion: and the value of x when the body strikes the plane is found from

$$x\tan i = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha};$$

$$\therefore\ x=0, \text{ and } x = \frac{2u^2}{g}\cos^2\alpha(\tan\alpha - \tan i) = \frac{2u^2}{g}\frac{\cos\alpha\sin(\alpha-i)}{\cos i},$$

this latter value of x is the *range on the inclined plane.*

By (1) $x = ut \cos \alpha$;

therefore time of flight on the inclined plane

$$= \frac{x}{u \cos \alpha} = \frac{2u}{g} \frac{\sin(\alpha - i)}{\cos i}; \quad = \frac{2u}{g} \sin \alpha, \text{ if } i = 0.$$

When the body reaches its greatest height

$$\frac{dy}{dx} = 0; \therefore x = \frac{u^2}{g} \tan \alpha \cos^2 \alpha = \frac{u^2}{g} \sin \alpha \cos \alpha;$$

$$\therefore \text{ greatest height } = \frac{u^2}{g} \left\{ \sin^2 \alpha - \tfrac{1}{2} \sin^2 \alpha \right\} = \frac{u^2}{2g} \sin^2 \alpha.$$

CENTRAL FORCES.

240. Forces which continually tend towards a given point, and the intensity of which depends upon the distance from that point, whether fixed or in motion, are called *Central Forces.* All the forces with which we are acquainted in nature are of this description, as will appear in the sequel. For this reason we shall devote a large portion of these pages to the consideration of their action.

We shall, in the first place, investigate the most important general properties of orbits described by bodies moving under the influence of central forces, and in the next place determine the nature of the orbits when the law and intensity of the forces are given, and, conversely, determine the forces requisite to cause a body to describe given orbits.

Prop. *When a body is acted on by one central force the motion is wholly in one plane.*

241. Suppose xyz are the co-ordinates at the time t to a material particle moving about a centre of force, the origin of co-ordinates being at this centre: r the distance of the particle from the centre: and let P, some function of r, represent the intensity of the force at the distance r.

The resolved parts of this force parallel to the three axes of co-ordinates are

$$P\frac{x}{r},\quad P\frac{y}{r},\quad \text{and } P\frac{z}{r},$$

and since these tend to diminish the co-ordinates the equations of motion are

$$-\frac{d^2x}{dt^2} = P\frac{x}{r},\quad -\frac{d^2y}{dt^2} = P\frac{y}{r},\quad -\frac{d^2z}{dt^2} = P\frac{z}{r} \text{ .... (1).}$$

Multiplying the first by y and the second by x and subtracting the equations we have

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0;$$

$$\therefore\ x\frac{dy}{dt} - y\frac{dx}{dt} = h,$$

h being an arbitrary constant.

In like manner $z\frac{dx}{dt} - x\frac{dz}{dt} = h_1,$

$$y\frac{dz}{dt} - z\frac{dy}{dt} = h_2,$$

h_1 and h_2 being arbitrary constants, which, as well as h, are to be determined by the circumstances of the motion at any given time.

Now multiply these last three equations by z, y, x respectively and add them together;

$$\therefore\ 0 = hz + h_1y + h_2x.$$

This is the equation to an invariable plane passing through the origin of co-ordinates, its position depending on the values of $h\ h_1\ h_2$.

Hence the motion takes place wholly in a plane passing through the centre of force, the position depending upon the initial (or any other *given*) circumstances of the motion.

Prop. *The areas described by the body about the centre of force are proportional to the time.*

242. In consequence of the property proved in the last Proposition we shall refer the body's motion to two co-ordinates instead of three. Let the plane of motion be the plane xy.

Then the equations of motion are

$$\frac{d^2x}{dt^2} = -P\frac{x}{r} \;\ldots\ldots\; (1), \qquad \frac{d^2y}{dt^2} = -P\frac{y}{r} \;\ldots\ldots\; (2),$$

and, as before, we obtain

$$x\frac{dy}{dt} - y\frac{dx}{dt} = h,$$

and let A be the sectorial area swept out during the time t by the radius vector;

$$\therefore \frac{dA}{dt} = \tfrac{1}{2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) \text{ by Diff. Calc.} = \frac{h}{2};$$

$$\therefore A = \frac{ht}{2},$$

if t and A be both measured from the commencement of the motion. This proves that the area swept out by the radius-vector is proportional to the time of describing it.

When polar co-ordinates are used let θ be the angle that the radius-vector r makes with the axis of x; then $x = r\cos\theta$ and $y = r\sin\theta$: and by substitution

$$\frac{dA}{dt} = \tfrac{1}{2}r^2\frac{d\theta}{dt}; \qquad \therefore r^2\frac{d\theta}{dt} = h.$$

The following is an immediate consequence of this property.

Prop. *To prove that the velocity of the body at different parts of its path is inversely proportional to the perpendicular on the tangent.*

243. Velocity $= v = \dfrac{ds}{dt} = \dfrac{ds}{d\theta}\dfrac{d\theta}{dt}$

$$= \frac{r^2}{p}\frac{d\theta}{dt}, \text{ by the Differential Calculus,}$$

p is the perpendicular on the tangent at the distance r,

$$= \frac{r^2}{p} \frac{h}{r^2} \text{ by last Art.} = \frac{h}{p}.$$

PROP. *To prove that the velocity is independent of the path described.*

244. Multiply equations (1) (2) of Art. 242. by $2\frac{dx}{dt}$, $2\frac{dy}{dt}$ respectively and add them, then

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} + 2\frac{dy}{dt}\frac{d^2y}{dt^2} = -\frac{2P}{r}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

$$= -2P\frac{dr}{dt}, \quad \because x^2 + y^2 = r^2;$$

$$\text{but } v^2 = \frac{ds^2}{dt^2} = \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2};$$

$$\therefore \frac{d.v^2}{dt} = -2P\frac{dr}{dt};$$

$$\therefore v^2 = V^2 - 2\int_R^r P\,dr, \quad r = R \text{ when } v = V;$$

and since this, when integrated from one position of the body to another, will be a function only of the corresponding distances, it follows, that the velocity is independent of the orbit described, and at any given distance depends solely on the magnitude and law of the force and the velocity and distance of projection.

COR. This is true also when the body is acted on by any number of central forces tending to fixed centres.

There is one more property of central orbits which we shall demonstrate owing to its utility in determining the velocity whenever the force and orbit are known.

PROP. *To prove that the velocity at any point of a central orbit is that due to a body falling through one fourth of the chord of curvature at that point through the centre of force under the action of the force at that point supposed to remain constant.*

245. By last Article $v\dfrac{dv}{dr} = -P$.

Also by Art. 243. $v = \dfrac{h}{p}$:

differentiate the logarithm of each side of this equation;

$$\therefore \frac{1}{v}\frac{dv}{dr} = -\frac{1}{p}\frac{dp}{dr},$$

divide the first equation by this;

$\therefore v^2 = Pp\dfrac{dr}{dp} = 2P\,\tfrac{1}{4}$ chord of curvature through the centre of force at dist. r.

Hence the Proposition is true.

Having demonstrated these Properties of Central Orbits we shall proceed to the determination of the nature of the orbits themselves.

Prop. *A body being acted on by a central force: required to find the polar equation to its path.*

246. The equations of motion are

$$\frac{d^2x}{dt^2} = -P\frac{x}{r} \ldots (1), \quad \frac{d^2y}{dt^2} = -P\frac{y}{r} \ldots (2);$$

$$\therefore x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0,$$

$$x\frac{dy}{dt} - y\frac{dx}{dt} = \text{constant} = h,$$

putting $x = r\cos\theta$ and $y = r\sin\theta$; we have

$$r^2\frac{d\theta}{dt} = h.$$

Again, multiplying (1) and (2) by $2\dfrac{dx}{dt}$ and $2\dfrac{dy}{dt}$ and adding,

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} + 2\frac{dy}{dt}\frac{d^2y}{dt^2} = -\frac{2P}{r}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right);$$

$$\therefore \frac{d}{dt}\left\{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right\} = -2P\frac{dr}{dt}, \quad \because x^2 + y^2 = r^2,$$

and introducing polar co-ordinates

$$\frac{d}{dt}\left\{\left(\frac{dr^2}{d\theta^2} + r^2\right)\frac{d\theta^2}{dt^2}\right\} = -2P\frac{dr}{dt}.$$

$$\text{But } \frac{d\theta}{dt} = \frac{h}{r^2};$$

$$\therefore \frac{d}{d\theta}\left\{\frac{1}{r^4}\frac{dr^2}{d\theta^2} + \frac{1}{r^2}\right\} = -\frac{2P}{h^2}\frac{dr}{d\theta}.$$

$$\text{Put } \frac{1}{r} = u: \quad \text{and } \therefore -\frac{1}{r^2}\frac{dr}{d\theta} = \frac{du}{d\theta};$$

$$\therefore \frac{d}{d\theta}\left\{\frac{du^2}{d\theta^2} + u^2\right\} = \frac{2P}{h^2u^2}\frac{du}{d\theta},$$

and then performing the differentiation on the left-hand side and dividing by $2\dfrac{du}{d\theta}$,

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2}.$$

This is the differential equation to the orbit described. The force P being given in terms of r, we must integrate this equation: and the solution will be the equation to the orbit described.

The integral will contain three arbitrary constants, two introduced in the process of integration and the other, h, existing in the differential equation. These are determined by the initial (or any other given) circumstances of the motion: viz. the velocity, distance, and direction of projection.

The general integral determines only the *nature* of the orbit described: but the circumstances of the motion at any given time determine the *species* and *dimensions* of the orbit.

247. The differential equation $P = h^2u^2 \left\{\frac{d^2u}{d\theta^2} + u\right\}$ may be used to ascertain the law of force which must act upon a body to cause it to describe a given curve. To effect this we must determine the relation between u and θ from the equation to the orbit: we must then differentiate u twice with respect to θ and substitute the result in the expression for P, eliminating θ, if it occur, by means of the relation between u and θ. In this way we shall obtain P in terms of u alone, and therefore of r alone.

248. When we know the relation between r and θ, we make use of the equation $r^2 \frac{d\theta}{dt} = h$ to determine the time of describing a given portion of the orbit: or, conversely, to find the position of the body in its orbit at any time.

We proceed now to exemplify these principles by various applications.

Prop. *A body moves about a centre of force varying directly as the distance: required to determine the motion.*

249. Let μ be the absolute force: then $P = \mu r = \frac{\mu}{u}$.

In order to simplify the calculation we shall first suppose the body projected perpendicular to the radius vector.

Let V, R be the velocity and distance of projection;

$\therefore\ h = 2$ area described in $1'' = VR$ by Art. 243.

$$\therefore\ \frac{d^2u}{d\theta^2} + u = \frac{\mu}{V^2R^2u^3};$$

multiplying by $2\frac{du}{d\theta}$ and integrating

$$\frac{du^2}{d\theta^2} + u^2 = C - \frac{\mu}{V^2R^2u^2},$$

$$\text{when } \frac{1}{u} = R,\quad \frac{dr}{d\theta} = 0 \text{ and } \therefore\ \frac{du}{d\theta} = 0;$$

$$\therefore\ C = \frac{1}{R^2} + \frac{\mu}{V^2},$$

$$\therefore \frac{du^2}{d\theta^2} = \frac{V^2 + R^2\mu}{R^2V^2} - \frac{\mu}{V^2R^2u^2} - u^2;$$

$$\tfrac{1}{4}\left(\frac{d \,.\, u^2}{d\theta}\right)^2 = \left(\frac{V^2 - R^2\mu}{2\,R^2V^2}\right)^2 - \left(u^2 - \frac{V^2 + R^2\mu}{2\,R^2V^2}\right)^2;$$

extracting the square root, inverting, and integrating

$$2\,\theta + C = \sin^{-1}\frac{2\,R^2V^2u^2 - (V^2 + R^2\mu)}{V^2 - R^2\mu},$$

$$\text{when } \theta = 0, \quad u = \frac{1}{R}, \quad \therefore\ C = \sin^{-1} 1 = \frac{\pi}{2};$$

$$\therefore \frac{1}{r^2} = u^2 = \frac{(V^2 + R^2\mu) + (V^2 - R^2\mu)\cos 2\theta}{2\,R^2V^2}$$

$$= \frac{V^2\cos^2\theta + R^2\mu\sin^2\theta}{R^2V^2}$$

$$1 = \left(\frac{r\cos\theta}{R}\right)^2 + \left(\frac{\sqrt{\mu}\,r\sin\theta}{V}\right)^2.$$

Hence the orbit is an ellipse, the force being in the centre. The semiaxes are R and $\dfrac{V}{\sqrt{\mu}}$.

250. The periodic time may be found by integrating the equation $\dfrac{dt}{d\theta} = \dfrac{r^2}{h}$, (Art. 242). But the following method is more simple.

$$\text{Periodic time} = \frac{2 \text{ area of ellipse}}{h}, \text{ (see Art. 242.)}$$

$$= \frac{2\,\pi R\,\dfrac{V}{\sqrt{\mu}}}{VR} = \frac{2\,\pi}{\sqrt{\mu}}.$$

This result is remarkable; for it shews that the period is independent of the dimensions of the ellipse and depends solely on the *intensity* of the force.

251. Cor. 1. If the angle of projection be β instead of $\frac{\pi}{2}$ it will be found, that the orbit is still an ellipse, the force being in the centre; and if a, b be the semi-axes*,

* This may be demonstrated with greater facility by using the equations of motion, which are, in this case,

$$\frac{d^2x}{dt^2} = -\mu x, \quad \frac{d^2y}{dt^2} = -\mu y.$$

Multiplying them respectively by $2\frac{dx}{dt}$, $2\frac{dy}{dt}$ and integrating we have

$$\frac{dx^2}{dt^2} = \mu(h^2 - x^2), \quad \frac{dy^2}{dt^2} = \mu(k^2 - y^2) \quad \text{.................. (1)},$$

h and k being arbitrary constants, introduced in the above form for the sake of symmetry;

$$\therefore \frac{dy^2}{dx^2} = \frac{k^2 - y^2}{h^2 - x^2}, \quad \frac{1}{\sqrt{k^2 - y^2}} \frac{dy}{dx} = \frac{1}{\sqrt{h^2 - x^2}};$$

$$\therefore \sin^{-1}\frac{y}{k} = \sin^{-1}\frac{x}{h} + \sin^{-1} c \quad \text{.............................. (2)},$$

c being an arbitrary constant;

$$\therefore \frac{y}{k} = \frac{x}{h}\sqrt{1 - c^2} + c\sqrt{1 - \frac{x^2}{h^2}},$$

by transposing and squaring and transposing again

$$\frac{y^2}{k^2} + \frac{x^2}{h^2} - \frac{2\sqrt{1 - c^2}\,xy}{hk} = c^2.$$

This is the equation to an ellipse from the centre: since $B^2 - 4AC = \frac{4(1 - c^2)}{h^2k^2} - \frac{4}{h^2k^2}$ $= -\frac{4c^2}{h^2k^2}$ is essentially negative; A, B, C being the coefficients of y^2, xy, x^2 respectively.

In order to determine the constants h, k, c, let V be the velocity of projection, α the angle which the direction of projection makes with the axis of x, a, b, the co-ordinates to the point of projection: then equations (1) give

$$V^2\cos^2\alpha = \mu(h^2 - a^2), \quad V^2\sin^2\alpha = \mu(k^2 - b^2),$$

$$\text{and (2) gives } \sin^{-1}\frac{b}{k} - \sin^{-1}\frac{a}{h} = \sin^{-1} c,$$

by which h, k, c are known.

If, as in Art. 249, we suppose the body projected from the axis of x at right angles to that line, then $b = 0$, $\alpha = 90^0$;

$$\therefore h^2 = a^2, \quad \mu k^2 = V^2,$$

$$\sin^{-1} c = \sin^{-1}\frac{b}{k} - \sin^{-1}\frac{a}{h} = -\sin^{-1}, \text{ therefore } c^2 = 1,$$

and the equation to the orbit becomes

$$\frac{\mu}{V^2}y^2 + \frac{1}{a^2}x^2 = 1.$$

The equation to an ellipse of which the semi-axes are a and $\frac{V}{\sqrt{\mu}}$.

$$\frac{1}{a^2} \text{ and } \frac{1}{b^2} = \frac{V^2 + \mu R^2 \pm \sqrt{(V^2+\mu R^2)^2 - 4\mu V^2 R^2 \sin^2\beta}}{2V^2R^2\sin^2\beta} \text{ respectively.}$$

Hence periodic time $= \dfrac{2\pi ab}{(h=)\ VR\sin\beta} = \dfrac{2\pi}{\sqrt{\mu}}$,

the same result as before.

Cor. 2. The result of this Proposition is of great importance in Physical Optics. For the forces which act upon the disturbed molecules of the vibrating medium of light all vary as the distance so long as the displacements are not very great. Now the *colour* of the light is assumed to depend upon the *time of vibration* of the molecules: and the *intensity* of the light upon the extent and magnitude of the vibrations, that is, upon the quantity of motion. The preceding Proposition shews, then, that light may alter in intensity without changing in colour, since the time of vibration is independent of the magnitude of the motion, when the law of force is that of the direct distance.

Prop. *A body is acted on by a central force varying inversely as the square of the distance: required to determine the orbit described.* *

* Many Propositions of this description may be solved in the following manner.

By Arts. 243, 244, $v^2 = V^2 - 2\int_R^r P\,dr$, $v = \dfrac{h}{p}$; $\therefore \dfrac{h^2}{p^2} = V^2 - 2\int_R^r P\,dr$.

Ex. 1. Let $P = \mu r$; $\therefore \dfrac{h^2}{p^2} = V^2 + \mu R^2 - \mu r^2$,

which is the equation to an ellipse about the centre, the axes being given by the equations

$$a^2 + b^2 = \frac{V^2}{\mu} + R^2, \quad a^2 b^2 = \frac{h^2}{\mu}.$$

Ex. 2. Let $P = \dfrac{\mu}{r^2}$, $\therefore \dfrac{h^2}{p^2} = \dfrac{2\mu}{r} - \dfrac{2\mu}{R} + V^2$.

This is the equation to a conic section about the focus.

The equation to the ellipse is $\dfrac{1}{p^2} = \dfrac{2a}{b^2 r} - \dfrac{1}{b^2}$

.... hyperbola is $\dfrac{1}{p^2} = \dfrac{2a}{b^2 r} + \dfrac{1}{b^2}$

.... parabola is $\dfrac{1}{p^2} = \dfrac{1}{mr}$.

 In

252. Let μ be the absolute force: then $P = \frac{\mu}{r^2} = \mu u^2$, V, R, β the velocity, distance, and angle of projection: then $h = VR \sin\beta$ (Art. 243.)

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2},$$

multiplying by $2\frac{du}{d\theta}$ and integrating

$$\frac{du^2}{d\theta^2} + u^2 = \frac{2\mu}{h^2}u + C,$$

when $\frac{1}{u} = R$, $r\frac{d\theta}{dr}$, or the tangent of the angle between the radius vector and the tangent line, $= \tan\beta$:

$$\therefore\ u\frac{d\theta}{du} = -\tan\beta;$$

$$\therefore\ C = \frac{1}{R^2\tan^2\beta} + \frac{1}{R^2} - \frac{2\mu}{h^2R} = \frac{1}{R^2\sin^2\beta} - \frac{2\mu}{h^2R} = \frac{V^2R - 2\mu}{h^2R};$$

$$\therefore\ \frac{du^2}{d\theta^2} = \frac{V^2R - 2\mu}{h^2R} + \frac{\mu^2}{h^4} - \left(\frac{\mu}{h^2} - u\right)^2;$$

extracting the square root, inverting, and integrating

$$\theta + C' = \cos^{-1}\frac{\frac{\mu}{h^2} - u}{\sqrt{\frac{V^2R - 2\mu}{h^2R} + \frac{\mu^2}{h^4}}},$$

C' is found by the condition, that when $u = \frac{1}{R}$, $\theta = 0$.

In the case of the ellipse

$$\frac{a}{b^2} = \frac{\mu}{h^2}, \quad \frac{1}{b^2} = \frac{2\mu}{Rh^2} - \frac{V^2}{h^2};$$

$$\therefore\ a = \frac{R\mu}{2\mu - V^2R}, \quad b = \sqrt{\frac{Rh^2}{2\mu - V^2R}}.$$

The path is an ellipse, hyperbola, or parabola according as V^2 is less than, greater than, or equal to $\frac{2\mu}{R}$.

$$\text{Then } \frac{1}{r} = \frac{\mu}{h^2} + \sqrt{\frac{V^2R - 2\mu}{h^2R} + \frac{\mu^2}{h^4}} \cos(\theta + C')$$

is the equation to the path: it is the equation to a conic section from the focus, and may be written

$$\frac{1}{r} = \frac{1 + e\cos(\theta + C')}{a(1 \sim e^2)};$$

the angle $\theta + C'$ being measured from the shorter length of the axis major, and $2a$ and $2a\sqrt{1 - e^2}$ being the axes:

$$\text{Then } e^2 = \frac{V^2R - 2\mu}{R\mu^2} h^2 + 1, \text{ subs. for } h$$

$$= \frac{V^2R - 2\mu}{\mu^2} RV^2 \sin^2\beta + 1 \ldots (1);$$

$$\text{and } a(1 \sim e^2) = \frac{h^2}{\mu} = \frac{V^2R^2\sin^2\beta}{\mu} \ldots\ldots\ldots\ldots (2).$$

Now the conic section is an ellipse, parabola, or hyperbola according as e is less than, equal to, or greater than unity.

Hence, from equation (1), the orbit described is an ellipse, parabola, or hyperbola about the focus according as V^2 is less than, equal to, or greater than $\frac{2\mu}{R}$. This proves the remarkable property, that the species of the conic section described is independent of the direction of projection.

In the case of the ellipse and hyperbola the axis major $= 2a = \frac{2\mu R}{V^2R \sim 2\mu}$ and *this* is also independent of the direction of projection.

In the case of the parabola, the distance of the vertex from the focus, or $D = a(1 - e)\ (e = 1) = \frac{V^2R^2\sin^2\beta}{2\mu}$.

The position of the axis major with respect to the radius vector R, is determined by C', which is the angle between these two lines.

Put $\theta = 0$ and $r = R$ in the value of $\frac{1}{r}$:

$$\therefore \cos C' = \frac{a(1 \sim e^2)}{Re} - \frac{1}{e} = \frac{V^2 R \sin^2 \beta - \mu}{\mu e}.$$

By referring to Art. 234, we see that the velocity of a body falling from an infinite distance to a distance R from a centre of force $\frac{\mu}{r^2}$ is equal to $\sqrt{\frac{2\mu}{R}}$. Hence the orbit described about this centre of force will be an ellipse, parabola, or hyperbola according as the velocity is less than, equal to, or greater than that from infinity.

253. We might make use of the equation

$$P = h^2 u^2 \left(\frac{d^2 u}{d\theta^2} + u\right)$$

to discover the law of force when the orbit is given.

Thus if the orbit be a conic section with the force in one of the foci, and m be the distance of the pole from the nearest vertex, then the equation to the orbit is

$$u = \frac{1 + e\cos\theta}{m(1+e)}; \quad \therefore P = \frac{h^2 u^2}{m(1+e)} = \frac{h^2}{m(1+e)} \frac{1}{r^2};$$

or the only law of force is that of the inverse square of the distance.

If the orbit be the ellipse the centre being the centre of force, then, a and b being the semi-axes,

$$u^2 = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}; \quad u\frac{du}{d\theta} = \left(\frac{1}{b^2} - \frac{1}{a^2}\right)\cos\theta\sin\theta,$$

$$u\frac{d^2u}{d\theta^2} + \frac{du^2}{d\theta^2} = \left(\frac{1}{b^2} - \frac{1}{a^2}\right)(\cos^2\theta - \sin^2\theta);$$

$$\therefore P = \frac{h^2}{u}\left\{u^4 + u^3\frac{d^2u}{d\theta^2}\right\} = \frac{h^2}{u}\left\{u^4 - u^2\frac{du^2}{d\theta^2} + u^2\left(u\frac{d^2u}{d\theta^2} + \frac{du^2}{d\theta^2}\right)\right\}$$

$$= \frac{h^2}{u}\left\{\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)^2 - \left(\frac{1}{b^2} - \frac{1}{a^2}\right)^2\cos^2\theta\sin^2\theta\right.$$

$$\left. + \left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)\left(\frac{1}{b^2} - \frac{1}{a^2}\right)(\cos^2\theta - \sin^2\theta)\right\}$$

$$= \frac{h^2}{u}\left\{\frac{\cos^4\theta + 2\cos^2\theta\sin^2\theta + \sin^4\theta}{a^2b^2}\right\} = \frac{h^2}{a^2b^2}r,$$

and therefore the only law is that of the direct distance.

254. If the orbit be a circle the centre of force being in the centre of the circle: then a being the radius $r = a$ is its equation: and

$$P = h^2u^3 = \frac{h^2}{a^3}.$$

Also by Art. 242, $\frac{dt}{d\theta} = \frac{r^2}{h} = \frac{a^2}{h}$ in this case, and therefore the velocity is constant;

$$\therefore\ ht = a^2\theta + \text{const.}$$

when $t = 0$, suppose $\theta = 0$; and when $t = T$, the time of revolution, $\theta = 2\pi$;

$$\therefore\ hT = 2\pi a^2.$$

Let V be the velocity, then $h = Va$ Art. 243;

$$\therefore\ P = \frac{V^2}{a} \text{ and } T = \frac{2\pi a}{V}.$$

Since the velocity is uniform it follows that the force produces no effect upon the velocity: in short, the only effect of the force is to deflect the body from the rectilinear path which it would describe with the uniform velocity V if no force acted. Consequently the central force is a measure of the tendency that the body has at every instant to preserve a rectilinear course. This tendency is sometimes called the *Centrifugal Force;* and the central force is then called in reference to this the *Centripetal Force.*

When a particle describes a curve in space the force which acts upon it is employed partly in changing the velocity and partly in deflecting the course of the body. A force equal and opposite to the part of the force which deflects the course of the body is called the centrifugal force in this general case as well as in that specified above.

PROP. *To prove that the centrifugal force of a particle moving in space at any point of its course equals the square of the velocity divided by the radius of absolute curvature at that point, and acts in the osculating plane.*

255. If X, Y, Z be the accelerating forces acting on the particle, the equations of motion are

$$\frac{d^2x}{dt^2} = X, \quad \frac{d^2y}{dt^2} = Y, \quad \frac{d^2z}{dt^2} = Z.$$

Now if s, the arc described, be the independent variable the absolute radius of curvature (ρ) is given by the equations

$$\frac{1}{\rho} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2 \quad \ldots\ldots\ldots (1).$$

Hence if we change the independent variable in the equations of motion from t to s, we have

$$X = \frac{\frac{dt}{ds}\frac{d^2x}{ds^2} - \frac{dx}{ds}\frac{d^2t}{ds^2}}{\frac{dt^3}{ds^3}} = v^2\frac{d^2x}{ds^2} + \frac{dx}{ds}\frac{d^2s}{dt^2}, \quad v = \frac{ds}{dt},$$

in like manner

$$Y = v^2\frac{d^2y}{ds^2} + \frac{dy}{ds}\frac{d^2s}{dt^2}, \quad Z = v^2\frac{d^2z}{ds^2} + \frac{dz}{ds}\frac{d^2s}{dt^2}.$$

If, then, P be the resultant of X, Y, Z we have

$$P^2 = X^2 + Y^2 + Z^2 = v^4\left\{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2\right\}$$

$$+ 2v^2\frac{d^2s}{dt^2}\left\{\frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{dy}{ds}\frac{d^2y}{ds^2} + \frac{dz}{ds}\frac{d^2z}{ds^2}\right\}$$

$$+ \left\{\frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} + \frac{dz^2}{ds^2}\right\}\left(\frac{d^2s}{dt^2}\right)^2.$$

But $\frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} + \frac{dz^2}{ds^2} = 1$, and therefore by equation (1)

$$P^2 = \left(\frac{v^2}{\rho}\right)^2 + \left(\frac{d^2 s}{dt^2}\right)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Now $\frac{d^2 s}{dt^2}$ is the part of the force P that produces the change in velocity (Art. 206): and the other part $\frac{v^2}{\rho}$ acts at right angles to the former, as the form of equation (2) shews, and consequently is what we have termed the centrifugal force: this expression proves the first part of the Proposition.

The force P acts through the point (xyz), let

$$x_1 - x = A(z_1 - z), \quad y_1 - y = B(z_1 - z)$$

be the equations to its direction: then the cosines of the angles which this line makes with the axes are

$$\frac{A}{\sqrt{1 + A^2 + B^2}}, \quad \frac{B}{\sqrt{1 + A^2 + B^2}}, \quad \frac{1}{\sqrt{1 + A^2 + B^2}},$$

but these cosines are also

$$\frac{X}{\sqrt{X^2 + Y^2 + Z^2}}, \quad \frac{Y}{\sqrt{X^2 + Y^2 + Z^2}}, \quad \frac{Z}{\sqrt{X^2 + Y^2 + Z^2}}.$$

$$\text{Hence } A = \frac{X}{Z}, \quad B = \frac{Y}{Z}:$$

and the equations to the direction of the resultant are

$$Z(x_1 - x) = X(z_1 - z), \quad Z(y_1 - y) = Y(z_1 - z),$$

the equations to the tangent line, or the line in which the force $\frac{d^2 s}{dt^2}$ acts, are

$$(x_1 - x)\frac{dz}{ds} = (z_1 - z)\frac{dx}{ds}, \quad (y_1 - y)\frac{dz}{ds} = (z_1 - z)\frac{dy}{ds}:$$

Hence the equation to the plane passing through these lines, or the plane in which the centrifugal force acts, is

$$(z_1 - z)\left(X\frac{dy}{ds} - Y\frac{dx}{ds}\right) + (y_1 - y)\left(Z\frac{dx}{ds} - X\frac{dz}{ds}\right)$$
$$+ (x_1 - x)\left(Y\frac{dz}{ds} - Z\frac{dy}{ds}\right) = 0,$$

and substituting in this equation the values of X, Y, Z it becomes

$$(z_1 - z)\left(\frac{dy}{ds}\frac{d^2x}{ds^2} - \frac{dx}{ds}\frac{d^2y}{ds^2}\right) + (y_1 - y)\left(\frac{dx}{ds}\frac{d^2z}{ds^2} - \frac{dz}{ds}\frac{d^2x}{ds^2}\right)$$

$$+ (x_1 - x)\left(\frac{dz}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2z}{ds^2}\right) = 0,$$

which is the equation to the osculating plane at the point (xyz), the arc s being the independent variable.

Hence the second part of the Proposition is true.

After this digression respecting centrifugal force we shall return to the subject of central forces.

256. Kepler discovered by calculation depending on observations, that the planet Mars moves in an ellipse having the Sun in the focus. He also discovered that the areas described by the planet when near its perihelion and aphelion distances (that is, the nearest and farthest distances from the Sun) were proportional to the times of describing them. These two empiric laws have since been proved to hold for the other planets and also for every part of their course. Kepler likewise discovered that the squares of the periodic times of the planets about the Sun were in the same proportion as the cubes of their mean distances. These three laws are known by the name of *Kepler's Laws* and may be thus enunciated.

I. *The planets move in ellipses, each having one of its foci in the Sun's centre.*

II. *The areas swept out by each planet about the Sun are, in the same orbit, proportional to the time of describing them.*

III. *The squares of the periodic times of the planets about the Sun are proportional to the cubes of the mean distances.*

We shall shew how we are led by these empiric laws to conjecture respecting the nature of the force which acts upon the planetary system.

PROP. *To determine the nature of the force which acts upon the planetary system.*

257. Let XY be the forces which act on a planet parallel to two co-ordinate axes drawn through the Sun in the plane of motion of the planet: then the equations of motion are

$$\frac{d^2x}{dt^2} = X, \quad \frac{d^2y}{dt^2} = Y;$$

$$\therefore\ x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = xY - yX.$$

But by Kepler's Second Law the area is proportional to the time: therefore area $= c \,.\, t$, c being the area described in a unit of time:

$$\therefore\ \frac{d\,.\,\text{area}}{dt} \text{ or } \tfrac{1}{2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = c;$$

$$\therefore\ x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0;$$

$$\therefore\ xY - yX = 0; \quad \therefore\ \frac{X}{Y} = \frac{x}{y}.$$

This shews that the resolved parts of the force acting upon the planet are proportional to the co-ordinates from the Sun's centre: and therefore, by the composition of forces, the force itself must pass through the Sun's centre.

Hence the forces acting on the planets all pass through the Sun's centre.

Let $\frac{1}{r}$ or $u = \dfrac{1 + e\cos(\theta - a)}{a(1 - e^2)}$ be the equation to the elliptic orbit: Kepler's First Law.

Then the force P, since we have shewn it to be central,

$$= h^2u^2\left\{\frac{d^2u}{d\theta^2} + u\right\} \text{ Art. 246.}$$

$$= h^2u^2\left\{\frac{-e\cos(\theta - a)}{a(1 - e^2)} + \frac{1 + e\cos(\theta - a)}{a(1 - e^2)}\right\}$$

$$= \frac{h^2}{a(1 - e^2)}\,\frac{1}{r^2}.$$

Hence the *law* of the force acting upon the planets is that of the inverse square of the distance.

Let T be the periodic time of a planet and a the semi-axis major of its orbit, and therefore the *mean* distance from the Sun's centre.

$$\text{Then } T = \frac{2 \text{ area of ellipse}}{h} = \frac{2\pi a^2 \sqrt{1-e^2}}{h};$$

$$\therefore \frac{h^2}{a(1-e^2)} = \frac{4\pi^2 a^3}{T^2},$$

$$\text{and } P = \frac{4\pi^2 a^3}{T^2} \frac{1}{r^2}.$$

But by Kepler's Third Law $\frac{a^3}{T^2}$ is the same for all the planets. Hence not only is the *law* of force the same for all the planets; but the *absolute* force is the same: and consequently the same cause seems to act on all the planets.

From this calculation, then, we conclude that the Sun attracts the planets, and that with a force which varies as the inverse square of their distances from his centre.

258. The elliptical orbits of the planets are nearly *circular:* since, then, in a circle there is no *variation* of distance, it may at first sight be a matter of doubt whether the calculations which prove Kepler's Laws are sufficiently accurate to allow us to believe the *law of variation* of the Sun's attraction to be correctly determined.

This doubt is, however, easily removed. For in the case here contemplated the Third of Kepler's Laws determines both the *law* and *intensity* of the Sun's attraction.

In this case u is constant and $= \frac{1}{a}$;

$$\therefore P = \frac{h^2}{a^3} = \frac{4\pi^2 a}{T^2} = \frac{4\pi^2 a^3}{T^2} \frac{1}{a^2}.$$

But T^2 varies as a^3, for different planets;

$$\therefore P \text{ varies as } \frac{1}{a^2} \text{ for different planets,}$$

and therefore the *law* of attraction is that of the inverse square as before: and the *magnitude* is the same.

259. Now the greatest diameters of the planets are proved by observation to be exceedingly small when compared with their distances from the Sun. But in Art. 165, Cor. we have shewn that the constituent particles of bodies of this description, if they attract, will attract according to the same law as that according to which the bodies themselves attract. And we have just shewn (Art. 257.) that the Sun attracts the planets with a force varying inversely as the square of the distance from his centre. It is therefore highly probable that the particles of the Sun attract the particles of the planets, and vice versâ, with a force varying directly as the mass of the attracting particle and inversely as the square of the distance.

260. These consequences to which we have been led by Kepler's Laws are equally satisfied whether we suppose the centre alone of each body to have an inherent property of attraction, or each particle of the system to attract. But this ambiguity is removed by Dr. Maskelyne's observations on the stars from stations near the mountain Shehallien in Scotland. By these it was proved that the mountain produced a sensible effect in drawing the plumb line out of the vertical: see the *Philosophical Transactions*, 1775. Also some beautiful experiments by Cavendish on the attraction of leaden balls, recorded in the *Philosophical Transactions*, 1798, shew the same thing; that the property of attraction does not reside only in the centres of the heavenly bodies but in every portion of their mass. We are therefore led to conjecture that matter is endowed with a general gravitating principle by which every particle attracts every other particle according to the law before mentioned.

261. Were, however, this principle universally true, not only would the Sun attract the planets, but the planets would attract the Sun (which we have imagined *immoveable**) and likewise one another: and our calculations are erroneous, but these depend on Kepler's Laws. Wherefore it follows, that

* See Arts. 240......246. In these the centre of force is *fixed*.

either Kepler's Laws are not true, or that Universal Gravitation is not a Principle of Nature.

Now in point of fact observations of greater nicety than those made by Kepler prove that his laws are not accurately true, though they differ but slightly from the reality.

Here then is an additional argument (as far as it goes) in favour of Universal Gravitation. For since the magnitudes of the planets are very small in comparison with that of the Sun, we should anticipate that the perturbations of their elliptic motion about the Sun and of the position of the Sun in space by the action of the heavenly bodies would be small; and, consequently, that the deviation from Kepler's Laws would not be considerable.

262. Our investigations thus far are only a *first approximation* to the truth: it yet remains to be determined whether the perturbations actually experienced agree, both in their *nature* and *magnitude*, with those which are calculated on this hypothesis of Universal Gravitation. These are the real tests of the existence of such a principle. Probably many imaginary laws would explain the ordinary phenomena of the motion of the heavenly bodies; but that alone is the law of nature which will stand the test of the more refined calculations of the perturbations.

It is by the complete harmony which is found to subsist between the numerical results deduced from theory and observation, that we become convinced of the truth of the Law of Universal Gravitation. To prove this complete accordance is the object of Physical Astronomy.

263. Having stated the main arguments which lead us to conjecture that the motions of the heavenly bodies are regulated by a universal principle of attraction with which all matter is endued, we proceed to a more strict investigation of the consequences of this principle, and shall now enter upon the consideration of the motion of a given number of material particles attracting each other with forces varying directly as the mass of the attracting body and inversely as the square of the distance. This Problem is one of insuperable difficulty when considered in a general point of view, and has baffled the combined exertions of mathematicians from the days of Newton to

the present time. In our Solar System, however, the masses of the planets are so small in comparison with that of the Sun, and the inclinations of the planes of their orbits to one another is also so small that the Problem is rendered capable of solution by methods of approximation. But it must not be imagined, that the results are for this reason not to be relied upon. For by the process of *successive approximation* in which we begin by obtaining a *first* approximation, thence proceeding to a *second*, and so on, we can by extending our calculations approach as near the truth as we please: and although the number of calculations must, strictly speaking, be infinite in order to arrive, by this method, at an *exact* result, yet the error in stopping at the third or fourth approximation is so slight as in fact to be inappreciable to our senses. Suppose, for example, that the longitude of the Sun's centre is *calculated* to be $134^0.\ 0'.\ 1''$ at some given time and that the *real* longitude is 134^0: what difference does this make in a practical point of view? But even if we were able to obtain an exact solution of the Problem, yet in calculating numerical results we are obliged to reduce the whole to decimals; and though the labour in this case would be perhaps diminished, yet the result would still be only approximate.

We shall first calculate the motions of two bodies, considered as particles, attracting each other, and then proceed to the more general question.

CHAPTER III.

MOTION OF TWO MATERIAL PARTICLES ATTRACTING EACH OTHER.

PROP. *Two material particles attract each other with forces varying inversely as the square of their distance and directly as the mass of the attracting body: required to determine the motion of their centre of gravity.*

264. Let M and m be the masses of the two particles: r their distance at the time t: then, if the unit of attraction be the attraction of a unit of mass at a unit of distance, the accelerating force produced in M by the attraction of $m = \frac{m}{r^2}$;

and that produced in m by M's attraction $= \frac{M}{r^2}$.

Let xyz be co-ordinates to M at time t,
$x'y'z'$ .... m ....

Then resolving the attractions parallel to the axes, and attending to the directions in which the resolved parts act, the equations of motion of M are

$$\frac{d^2x}{dt^2} = -\frac{m(x-x')}{r^3}, \quad \frac{d^2y}{dt^2} = -\frac{m(y-y')}{r^3}, \quad \frac{d^2z}{dt^2} = -\frac{m(z-z')}{r^3}$$

and those of m are

$$\frac{d^2x'}{dt^2} = \frac{M(x-x')}{r^3}, \quad \frac{d^2y'}{dt^2} = \frac{M(y-y')}{r^3}, \quad \frac{d^2z'}{dt^2} = \frac{M(z-z')}{r^3}.$$

Multiply the first three equations by M and the last three by m, and add the first, second, and third of the first set to the first, second, and third of the second set respectively;

$$\left.\begin{aligned}\therefore\ M\frac{d^2x}{dt^2}+m\frac{d^2x'}{dt^2}=0,\quad M\frac{d^2y}{dt^2}+m\frac{d^2y'}{dt^2}=0,\\ M\frac{d^2z}{dt^2}+m\frac{d^2z'}{dt^2}=0.\end{aligned}\right\}\ldots\ldots(1).$$

Let $\bar{x}\,\bar{y}\,\bar{z}$ be the co-ordinates to the centre of gravity of the two bodies at the time t: then

$$(M+m)\,\bar{x}=Mx+mx',\quad (M+m)\,\bar{y}=My+my',$$

$$(M+m)\,\bar{z}=Mz+mz'.$$

Differentiating these twice with respect to t and making use of equations (1), we obtain

$$\frac{d^2\bar{x}}{dt^2}=0,\quad \frac{d^2\bar{y}}{dt^2}=0,\quad \frac{d^2\bar{z}}{dt^2}=0\ldots\ldots\ldots(2);$$

$$\therefore\ \frac{d\bar{x}}{dt}=a,\quad \frac{d\bar{y}}{dt}=b,\quad \frac{d\bar{z}}{dt}=c,$$

a, b, c being constants to be determined by the initial circumstances of the motion of the bodies.

Hence the velocity of the centre of gravity $=\sqrt{a^2+b^2+c^2}$, (Art. 210. Cor.) and is therefore uniform.

$$\text{Also }\frac{d\bar{x}}{d\bar{z}}=\frac{a}{c},\quad \frac{d\bar{y}}{d\bar{z}}=\frac{b}{c};$$

$$\therefore\ \bar{x}=\frac{a}{c}\bar{z}+a',\quad \bar{y}=\frac{b}{c}\bar{z}+b',$$

a', b' being constants to be determined as before.

These are the equations to the path of the centre of gravity; and, since they are the equations to a straight line in space, they prove that that point will move in a straight line.

If a, b, c each $=0$, then the expression for the velocity of the centre of gravity vanishes: and the general conclusion is, That *the centre of gravity of the two bodies will either remain at rest during the motion of the bodies, or move* *uniformly in a straight line.* Which of these will be the case is determined by the initial circumstances of the motion of the bodies.

PROP. *To determine the orbits the bodies describe about each other, and about their centre of gravity.*

265. Let us subtract the equations of motion for m from those of M respectively, and we obtain

$$\frac{d^2(x-x')}{dt^2} = -\frac{(M+m)(x-x')}{r^3}, \quad \frac{d^2(y-y')}{dt^2} = -\frac{(M+m)(y-y')}{r^3},$$

$$\frac{d^2(z-z')}{dt^2} = -\frac{(M+m)(z-z')}{r^3}.$$

These are the equations we should obtain by supposing either of the bodies at rest, and the force acting on the other to be the *sum* of the masses divided by the square of the distance.

Hence (Art. 252) each will describe relatively to the other a conic section, the nature of the path being determined by the circumstances of projection of the bodies.

266. To determine their paths about their centre of gravity, let $r_{,}$ and r' be the distances of M and m from that point at the time t: then

$$r_{,} = \frac{m}{M+m} r, \quad r' = \frac{M}{M+m} r.$$

Also, if P and Q be the two particles (fig. 80), G their centre of gravity,

$$\frac{PN}{PQ} = \frac{PN'}{PG}, \quad \therefore \frac{x-x'}{r} = \frac{x-\bar{x}}{r_{,}}:$$

and in the same way $\dfrac{y-y'}{r} = \dfrac{y-\bar{y}}{r_{,}}$ and $\dfrac{z-z'}{r} = \dfrac{z-\bar{z}}{r_{,}}$:

Now subtract equations (2) of Art. 264. from the equations of motion of m in that Article respectively:

$$\therefore \frac{d^2(x-\bar{x})}{dt^2} = -\frac{m(x-x')}{r^3} = -\frac{m^3}{(M+m)^2}\frac{x-\bar{x}}{r_{,}^3}$$

$$\frac{d^2(y-\bar{y})}{dt^2} = -\frac{m^3}{(M+m)^2}\frac{y-\bar{y}}{r_{,}^3} \text{ and } \frac{d^2(z-\bar{z})}{dt^2} = -\frac{m^3}{(M+m)^2}\frac{z-\bar{z}}{r_{,}^3}.$$

These are the equations of motion of M relatively to the centre of gravity of M and m, which as we have seen is at rest, or is moving uniformly in a straight line. They prove that the path about the centre of gravity is such as would be described about a force $\frac{m^3}{(M+m)^2}\cdot\frac{1}{r_{,}^2}$ residing in that point.

Hence the orbits of M and m relatively to the centre of gravity are conic sections, their nature and magnitude being determined by the circumstances of projection *relatively* to the centre of gravity of M and m.

PROP. *To compare the relative orbits of* M *and* m *about their centre of gravity.*

267. Let v, v' be the absolute velocities of projection of M and m: $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ the angles the directions of these velocities make with the axes.

V and V' the relative vels. of project. about centre of gravity,

R and R' the initial distances from the centre of gravity,

δ and δ' the relative angles of projection,

a and a' the semi-axes major of the orbits,

e and e' the eccentricities of the orbits,

μ and μ' the absolute forces.

Then by equations (1) (2) of Art. 252,

$$\frac{1-e^2}{1-e'^2} = \frac{2\mu - V^2R}{2\mu' - V'^2R'}\,\frac{RV^2\sin^2\delta}{R'V'^2\sin^2\delta'}\,\frac{\mu'^2}{\mu^2},$$

$$\text{and } \frac{a(1\sim e^2)}{a'(1\sim e'^2)} = \frac{V^2R^2\sin^2\delta}{V'^2R'^2\sin^2\delta'}\,\frac{\mu'}{\mu}.$$

$$\text{Also } \frac{R}{R'} = \frac{m}{M}, \text{ and } \frac{\mu'}{\mu} = \frac{M^3}{m^3} \text{ by Art. 266.}$$

To find V, V', δ, δ' we proceed as follows:

The velocities of the centre of gravity parallel to the axes are at first and therefore during the motion respectively (Art. 264.)

$$\frac{Mv\cos\alpha+mv'\cos\alpha'}{M+m},\ \frac{Mv\cos\beta+mv'\cos\beta'}{M+m},\ \frac{Mv\cos\gamma+mv'\cos\gamma'}{M+m}.$$

Also the *absolute* velocities of projection of M parallel to the axes are $v\cos\alpha$, $v\cos\beta$, $v\cos\gamma$: and therefore the *relative* velocities of projection of M about the centre of gravity parallel to the axes are

$$\frac{m(v\cos\alpha-v'\cos\alpha')}{M+m},\ \frac{m(v\cos\beta-v'\cos\beta')}{M+m},\ \frac{m(v\cos\gamma-v'\cos\gamma')}{M+m}.$$

Adding the squares of these, (Art. 210. Cor.) the square of the relative velocity of M about the centre of gravity (V^2) =

$$\frac{m^2}{(M+m)^2}\{(v\cos\alpha-v'\cos\alpha')^2+(v\cos\beta-v'\cos\beta')^2+(v\cos\gamma-v'\cos\gamma')^2\}$$

$$=\frac{m^2}{(M+m)^2}(v^2+v'^2-2vv'\cos A),$$

where A is the angle between the directions of projection of M and m: and therefore determined by the equation

$$\cos A=\cos\alpha\cos\alpha'+\cos\beta\cos\beta'+\cos\gamma\cos\gamma'.$$

Similarly $V'^2=\dfrac{M^2}{(M+m)^2}(v^2+v'^2-2vv'\cos A)$.

Let the line joining M and m at the commencement of the motion be the axis of x: then the cosine of the angle which the direction of V makes with the distance of projection (which coincides with the axis of x), or $\cos\delta$, equals the relative velocity parallel to the axis of x divided by the whole relative velocity (V) =

$$\frac{m}{M+m}\cdot\frac{v\cos\alpha-v'\cos\alpha'}{V}.$$

Similarly $\cos \delta' = \dfrac{M}{M+m} \cdot \dfrac{v' \cos \alpha' - v \cos \alpha}{V'}$.

Substituting these in the expression given above for $\dfrac{1-e^2}{1-e'^2}$ we find

$$\frac{1-e^2}{1-e'^2} = 1\,; \quad \therefore\; e = e',$$

or the orbits are similar to each other.

$$\text{Also } \frac{a}{a'} = \frac{m^4}{M^4} \cdot \frac{M^3}{m^3} = \frac{m}{M},$$

or the linear dimensions of the orbits of M and m are in the ratio of m to M.

268. Cor. 1. It follows from this that the perturbation of the Sun by any planet is very small, because his mass is so much the greater of the two masses.

In the same way it will be shewn that the combined effect of the heavenly bodies in moving the Sun is very slight; and therefore the error in Kepler's Laws, anticipated in Art. 261, owing to the supposed immobility of the Sun, is not very great. Thus far, then, we are confirmed in our hypothesis of Universal Gravitation.

269. Cor. 2. We have seen (Art. 257.) that if μ be the absolute force of a centre of which the law is that of the inverse square, and a the semi-axis major of the orbit described, the periodic time

$$(T) = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}} = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{M+m}}. \quad \text{(Art. 265.)}$$

M and m being the masses of the Sun and a planet.

Let m' be the mass of another planet: and a' the semi-axis major of its orbit, T' its period;

$$\therefore\; T' = \frac{2\pi a'^{\frac{3}{2}}}{\sqrt{M+m'}}, \quad \text{and } \therefore\; \frac{T^2}{T'^2} = \frac{a^3}{a'^3}\,\frac{M+m'}{M+m}.$$

This shews that Kepler's Third Law would not be true even if we suppose that the planets do not attract each other, unless their masses were equal to each other. The deviation, however, from the truth is extremely small.

270. The investigations in Arts. 252, 265, shew us that if our law of gravitation be true, the only orbits which a heavenly body will describe, supposed to be acted on only by the Sun, are an ellipse, a parabola, or a hyperbola with the Sun's centre in the focus.

The manner in which the magnitude and position of the orbit of a heavenly body is determined by actual observation will be found in Works on Plane Astronomy. We shall here briefly explain the process. There are six quantities which determine the position and magnitude of an elliptic or hyperbolic orbit, and the place of the body in its orbit: these are called the *elements* of the body's orbit, and are (1) the inclination of the orbit to the ecliptic, and (2) the longitude of the ascending node, these determine the *position of the plane of the orbit* in space: next (3) the longitude of the perihelion, (or point of the orbit nearest the Sun), which determines the *position of the orbit itself:* then (4) the mean distance, and (5) eccentricity, which determine the *magnitude* of the orbit, and lastly (6) the epoch, or the time of the planet's being in the perihelion, this determines the *position of the body itself* in its orbit.

The elements of a parabolic orbit are five in number, being the same as the above, if we replace the mean distance and eccentricity by the perihelion distance.

The elements of a circular orbit are only four in number, the eccentricity and longitude of the perihelion not being required.

In order to determine the numerical values of the elements of any heavenly body (supposed to move in a conic section with the Sun in the focus) two Trigonometrical equations* are deduced connecting the elements with the right ascension and

* For a parabolic and circular orbit see Maddy's Plane Astronomy, Chap. XIV. Woodhouse's Plane Astronomy, Chap. XXIV.

But for other orbits the reader may consult the Work of Lalande; Gauss's Theoria Motus Corporum Cœlestium; the Mécanique Céleste, Vol. I.; Lagrange's Mec. Analytique; Pontécoulant's Théorie Anal. du Système du Monde, and Mr Lubbock's Mathematical Tracts and various Papers in the Transactions of the Philosophical and Astronomical Societies.

declination of the body and the distance of the Earth from the Sun.

Since there are five or six quantities to be determined three independent observations must be made on the declination and right ascension of the body: when these are substituted successively in the two equations mentioned above we shall have six equations involving the elements: by means of which we shall be able to calculate the magnitude and position of the orbit.

271. By methods of this nature Kepler discovered his three planetary Laws.

Also Astronomers have in this way proved, that comets move in orbits most of which are parabolic, some elliptic, and others probably hyperbolic. In consequence of the vast distances to which comets penetrate into space, they are invisible except when near the Sun. During their appearance numerous observations are made, in order that the elements may be determined with the greatest possible accuracy. The calculations for parabolic motion are less laborious than for elliptic or hyperbolic motion. The elements are therefore first calculated on the supposition that the orbit is a parabola. If the elements thus calculated shew that the comet has passed so near any of the planets as to have experienced a sensible perturbation the elements must be corrected in a manner to be explained hereafter.

If a parabola will not coincide with the orbit calculations must be made for an ellipse or hyperbola. It is thus found that "three or four comets describe very long ellipses: and nearly all the others that have been observed are found to move in curves which cannot be distinguished from parabolas. There is reason to think that two or three comets move in hyperbolas." (Airy's *Gravitation*, page 15.)

272. Our calculations have been hitherto respecting the nature of the orbits described. We now proceed to deduce formulæ for determining the time that the body occupies in moving through a given angle; and conversely the angle described in a given time: by the former we know the time of the body being at a given place, and by the latter we know the place of the body at a given time.

PROP. *To find the time of motion of a planet or comet through any portion of an elliptic orbit, the Sun's centre being in the focus.*

273. Let θ and ϖ be the longitudes of the body and the perihelion, that is, the point of the orbit nearest the Sun: a the semi-axis major of the orbit: e the eccentricity: μ the sum of the masses of the Sun and the body (Art. 265): then the equation to the orbit is

$$\frac{1}{r} = \frac{1 + e\cos(\theta - \varpi)}{a(1 - e^2)}. \quad \text{Also } \frac{dt}{d\theta} = \frac{r^2}{h}, \text{ Art. 242.}$$

Now h must be determined in terms of the quantities above given, since the orbit to be described is known and not the original circumstances of projection. The following method, which we here apply to the ellipse, will answer our purpose in every case. By Art. 243, $h = vp$ at every point of a central orbit; v being the velocity and p the perpendicular from the centre of force on the tangent at that point: also by Art. 245, the velocity is that due to one-fourth the chord of curvature through the centre of force;

$$\therefore\ v^2 = \frac{2\mu}{r^2}\cdot\tfrac{1}{2}p\frac{dr}{dp}: \text{ but } p^2 = \frac{b^2 r}{2a - r} \text{ from the focus;}$$

$$\therefore\ h = vp = \sqrt{\frac{\mu}{r^2}p^3\frac{dr}{dp}} = \sqrt{\frac{\mu b^2}{a}} = \sqrt{\mu a(1 - e^2)}.$$

Then the time of moving from the perihelion through the angle $\theta - \varpi =$

$$t = \int_\varpi^\theta \frac{r^2 d\theta}{h} = \frac{a^{\frac{3}{2}}(1 - e^2)^{\frac{3}{2}}}{\sqrt{\mu}} \int_\varpi^\theta \frac{d\theta}{\{1 + e\cos(\theta - \varpi)\}^2}$$

$$= \frac{a^{\frac{3}{2}}(1 - e^2)^{\frac{3}{2}}}{\sqrt{\mu}} \int_\varpi^\theta \frac{d\theta}{\{(1 + e)\cos^2\frac{1}{2}(\theta - \varpi) + (1 - e)\sin^2\frac{1}{2}(\theta - \varpi)\}^2}$$

$$= \frac{2a^{\frac{3}{2}}(1 - e^2)^{\frac{3}{2}}}{\sqrt{\mu}} \int_\varpi^\theta \frac{\sec^2\frac{1}{2}(\theta - \varpi)\dfrac{d\tan\frac{1}{2}(\theta - \varpi)}{d\theta}d\theta}{\{(1 + e) + (1 - e)\tan^2\frac{1}{2}(\theta - \varpi)\}^2}.$$

To simplify this let

$$\tan \tfrac{1}{2}(\theta - \varpi) = \sqrt{\frac{1+e}{1-e}} \tan \frac{u}{2} \dots\dots\dots (1);$$

$$\therefore\ t = \frac{2a^{\frac{3}{2}}(1-e^2)^{\frac{3}{2}}}{\sqrt{\mu}} \int_0^u \frac{\left(1 + \frac{1+e}{1-e}\tan^2\frac{u}{2}\right)}{(1+e)^2 \sec^4 \frac{u}{2}} \frac{d}{du}\left(\sqrt{\frac{1+e}{1-e}} \tan\frac{u}{2}\right) du$$

$$= \frac{a^{\frac{3}{2}}}{\sqrt{\mu}} \int_0^u \left\{(1-e)\cos^2\frac{u}{2} + (1+e)\sin^2\frac{u}{2}\right\} du$$

$$= \frac{a^{\frac{3}{2}}}{\sqrt{\mu}} \int_0^u (1 - e\cos u)\, du$$

$$= \frac{a^{\frac{3}{2}}}{\sqrt{\mu}} (u - e\sin u), \text{ let } \frac{a^{\frac{3}{2}}}{\sqrt{\mu}} = \frac{1}{n};$$

$$\therefore\ nt = u - e\sin u \dots\dots\dots (2).$$

When θ is given we calculate u by (1), and substituting in (2) we know t.

The angle $\theta - \varpi$, or the excess of the longitude of the body over the longitude of the perihelion, is called the *true anomaly:* and nt is called the *mean anomaly*, since it varies uniformly with the time and coincides with the true anomaly at the end of each revolution, as the formulæ (1) (2) shew. Also the angle u is called the *eccentric anomaly*, since it equals the angle QCA (fig. 81), as may easily be proved: P is the body, APa the ellipse, S the focus, AQa a circle on Aa.

Cor. 1. If t be not measured from the epoch of passing the perihelion, but from the time when $u = u_{\prime}$, then

$$t = \frac{a^{\frac{3}{2}}}{\sqrt{\mu}} \{(u - u_{\prime}) - e(\sin u - \sin u_{\prime})\}.$$

Cor. 2. Whenever u increases by 2π, θ increases by 2π, and t by $\dfrac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}}$. This, then, is the periodic time of the planet: it is remarkable that it is independent of the eccentricity of the orbit.

To solve the converse of this Proposition, that is, to find the position of a heavenly body in its elliptic orbit at any time in terms of the time and the elements of the orbit, we must effect several expansions.

PROP. *To expand the true anomaly in terms of the eccentric anomaly.*

274. By last Article $\tan \dfrac{\theta - \varpi}{2} = \sqrt{\dfrac{1+e}{1-e}} \tan \dfrac{u}{2}$.

Substituting the exponential expressions for the tangents,

$$\frac{\varepsilon^{(\theta-\varpi)\sqrt{-1}} - 1}{\varepsilon^{(\theta-\varpi)\sqrt{-1}} + 1} = m \frac{\varepsilon^{u\sqrt{-1}} - 1}{\varepsilon^{u\sqrt{-1}} + 1}, \quad \sqrt{\frac{1+e}{1-e}} = m,$$

in which ε is the base of Naperian logarithms.

$$\therefore \varepsilon^{(\theta-\varpi)\sqrt{-1}} = \frac{(m+1)\,\varepsilon^{u\sqrt{-1}} - (m-1)}{(m+1) - (m-1)\,\varepsilon^{u\sqrt{-1}}} = \varepsilon^{u\sqrt{-1}} \frac{1 - \lambda\varepsilon^{-u\sqrt{-1}}}{1 - \lambda\varepsilon^{u\sqrt{-1}}}, \quad \lambda = \frac{m-1}{m+1};$$

$$\therefore (\theta - \varpi)\sqrt{-1} = u\sqrt{-1} + \log_\varepsilon (1 - \lambda\varepsilon^{-u\sqrt{-1}}) - \log_\varepsilon (1 - \lambda\varepsilon^{u\sqrt{-1}})$$

$$= u\sqrt{-1} + \lambda\,(\varepsilon^{u\sqrt{-1}} - \varepsilon^{-u\sqrt{-1}}) + \frac{\lambda^2}{2}\,(\varepsilon^{2u\sqrt{-1}} - \varepsilon^{-2u\sqrt{-1}}) + \dots$$

$$\therefore \theta - \varpi = u + 2\lambda \sin u + \frac{2\lambda^2}{2} \sin 2u + \frac{2\lambda^3}{3} \sin 3u + \dots$$

$$\text{in which } \lambda = \frac{\sqrt{1+e} - \sqrt{1-e}}{\sqrt{1+e} + \sqrt{1-e}} = \frac{1 - \sqrt{1-e^2}}{e}.$$

PROP. *To expand the eccentric anomaly in terms of the mean anomaly.*

275. By Art. 273, $u = nt + e \sin u$.

Hence by Lagrange's Theorem, putting $nt = z$,

$$u = z + e\sin z + \frac{e^2}{1.2}\frac{d\sin^2 z}{dz} + \frac{e^3}{1.2.3}\frac{d^2\sin^3 z}{dz^2} + \dots$$

$$= z + e\sin z + \tfrac{1}{2}e^2\sin 2z + \tfrac{1}{2}e^3(2\sin z - 3\sin^3 z) + \dots$$

$$= nt + e\sin nt + \tfrac{1}{2}e^2\sin 2nt + \tfrac{1}{8}e^3(3\sin 3nt - \sin nt) + \dots$$

PROP. *To expand sin* u, *sin* 2u,...*in terms of the mean anomaly.*

276. By Lagrange's Theorem,

$$\sin u = \sin z + e\sin z\frac{d\sin z}{dz} + \frac{e^2}{1.2}\frac{d}{dz}\left\{\sin^2 z\frac{d\sin z}{dz}\right\} + \dots$$

$$= \sin z + e\sin z\cos z + \tfrac{1}{2}e^2(2\cos^2 z\sin z - \sin^3 z) + \dots$$

$$= \sin z + \tfrac{1}{2}e\sin 2z + \tfrac{1}{8}e^2(2\sin 3z - \sin z) + \dots$$

$$= \sin nt + \tfrac{1}{2}e\sin 2nt + \tfrac{1}{8}e^2(3\sin 3nt - \sin nt) + \dots$$

Again, $\sin 2u = \sin 2z + e\sin z\dfrac{d\sin 2z}{dz} + \dots$

$$= \sin 2nt + 2e\sin nt\cos 2nt + \dots$$

$$= \sin 2nt + e(\sin 3nt - \sin nt) + \dots$$

$$\sin 3u = \sin 3nt + \dots$$

and so on.

PROP. *To expand the true anomaly in terms of the mean anomaly.*

277. By Art. 274 we have

$$\theta - \varpi = u + 2\lambda\sin u + \frac{2\lambda^2}{2}\sin 2u + \frac{2\lambda^3}{3}\sin 3u + \dots$$

$$\text{where } \lambda = \frac{1-\sqrt{1-e^2}}{e} = \frac{e}{2} + \frac{e^3}{8} + \dots$$

Then substituting for u, $\sin u$, $\sin 2u$ ... the values obtained in the last two Articles, and retaining powers of e as far as the cube,

$$\theta - \varpi = nt + e \sin nt + \tfrac{1}{2}e^2 \sin 2nt + \tfrac{1}{8}e^3 (3 \sin 3nt - \sin nt) + \dots$$
$$+ 2\lambda \left\{\sin nt + \tfrac{1}{2}e \sin 2nt + \tfrac{1}{8}e^2 (3 \sin 3nt - \sin nt) + \dots\right\}$$
$$+ \lambda^2 \left\{\sin 2nt + e (\sin 3nt - \sin nt) + \dots\dots\dots\dots\right\}$$
$$+ \tfrac{2}{3}\lambda^3 \sin 3nt + \dots\dots\dots\dots\dots\dots$$
$$= nt + \left(2e + \frac{e^3}{4}\right) \sin nt + \frac{5e^2}{4} \sin 2nt + \frac{13e^3}{12} \sin 3nt + \dots$$

which is true as far as terms involving e^3.

278. Cor. If the time t be not measured from the time of perihelion passage, suppose ϵ is the mean longitude of the body when $t = 0$; then the mean longitude at the time t is $nt + \epsilon$; and the mean anomaly is $nt + \epsilon - \varpi$: in this case, then,

$$\theta - \varpi = nt + \epsilon - \varpi + \left(2e + \frac{e^3}{4}\right) \sin (nt + \epsilon - \varpi)$$
$$+ \frac{5e^2}{4} \sin 2 (nt + \epsilon - \varpi) + \dots$$

ϵ is called the *epoch*.

Prop. *To expand the radius vector* r *in terms of the mean anomaly.*

279. The radius vector

$$r = \frac{a(1-e^2)}{1+e\cos(\theta-\varpi)} = \frac{a(1-e^2)}{(1+e)\cos^2\frac{1}{2}(\theta-\varpi) + (1-e)\sin^2\frac{1}{2}(\theta-\varpi)}$$

$$= \frac{a(1-e^2)\sec^2\frac{1}{2}(\theta-\varpi)}{1+e+(1-e)\tan^2\frac{1}{2}(\theta-\varpi)} = \frac{a(1-e^2)\left\{1 + \dfrac{1+e}{1-e}\tan^2\dfrac{u}{2}\right\}}{(1+e)\sec^2\dfrac{u}{2}}$$

$$= a\left\{(1-e)\cos^2\frac{u}{2} + (1+e)\sin^2\frac{u}{2}\right\} = a(1 - e\cos u).$$

But $u = nt + e \sin u$; putting $nt = z$,

$$\cos u = \cos z + e \sin z \frac{d \cos z}{dz} + \frac{e^2}{1.2} \frac{d}{dz}\left\{\sin^2 z \frac{d \cos z}{dz}\right\} + \dots$$
$$= \cos z - \tfrac{1}{2}e(1 - \cos 2z) - \tfrac{1}{8}e^2 (3\cos z - 3\cos 3z) + \dots$$
$$\therefore \frac{r}{a} = 1 + \frac{e^2}{2} - e\cos nt - \frac{e^2}{2}\cos 2nt - \frac{3e^3}{8}(\cos 3nt - \cos nt) + \dots$$

280. Cor. If t be measured as in Art. 278, then

$$r = a\left\{1 + \tfrac{1}{2}e^2 - e\cos(nt + \epsilon - \varpi) - \tfrac{1}{2}e^2\cos 2(nt + \epsilon - \varpi) - \ldots\right\}.$$

The time of describing a given portion of an elliptic hyperbolic or parabolic orbit may be found in terms of the radius vectors at the extremities of the arc and the chord of the arc. These expressions are useful in determining the elements of a heavenly body. They will be found in Maddy's *Plane Astronomy*, Chapter XIII. New Edition: and in the *Système du Monde* of M. Pontécoulant, Tom. I. Liv. II. Chap. V.

Prop. *To find the time of describing a given portion of a parabolic orbit about the Sun in the focus.*

281. We have $r^2\dfrac{d\theta}{dt} = h$: $h = \sqrt{2\mu D}$, and $r = \dfrac{D}{\cos^2\frac{1}{2}(\theta - \varpi)}$ is the equation to the parabola, θ and ϖ being the longitude of the comet and of its perihelion measured from the Sun, and D the perihelion distance;

$$\therefore\ t = \frac{D^{\frac{3}{2}}}{\sqrt{2\mu}}\int_{\varpi}^{\theta}\frac{d\theta}{\cos^4\dfrac{\theta - \varpi}{2}}$$

$$= \frac{2D^{\frac{3}{2}}}{\sqrt{2\mu}}\int_{\varpi}^{\theta}\frac{d\tan\frac{1}{2}(\theta - \varpi)}{d\theta}\left\{1 + \tan^2\tfrac{1}{2}(\theta - \varpi)\right\}d\theta$$

$$= \sqrt{\frac{2}{\mu}}\,D^{\frac{3}{2}}\left\{\tan\tfrac{1}{2}(\theta - \varpi) + \tfrac{1}{3}\tan^3\tfrac{1}{2}(\theta - \varpi)\right\},$$

t being measured from the time of the perihelion passage.

By this equation it is easy to calculate the time of describing a given angle.

Prop. *To find the position of the comet in a parabolic orbit at a given time.*

282. This would require the solution of the cubic equation in the last Article. This is, however, obviated in the following manner.

$$\text{Let } \sqrt{\frac{\mu}{2D^3}} = n;$$

$$\therefore\ nt = \tan\tfrac{1}{2}(\theta - \varpi) + \tfrac{1}{3}\tan^3\tfrac{1}{2}(\theta - \varpi).$$

A Table is formed consisting of two columns: one with values of t and the other with the corresponding values of $\theta - \varpi$ calculated from this formula for an orbit in which $n = 1$. Suppose, then, that we wish to find the position of a comet in a given parabolic orbit (the mean motion in which is n) at a given time t. We must multiply t by n and look for the value of $\theta - \varpi$ opposite the value of nt in the first column. This gives the position of the comet.

Prop. *To find the place of a comet at a given time in a very eccentric elliptic orbit.*

283. By Art. 273. $\dfrac{dt}{d\theta} = \dfrac{a^{\frac{3}{2}}(1-e^2)^{\frac{3}{2}}}{\sqrt{\mu}} \dfrac{1}{\{1 + e\cos(\theta - \varpi)\}^2}$.

Let D be the perihelion distance; $\therefore D = a(1-e)$;

$$\therefore \frac{dt}{d\theta} = \frac{D^{\frac{3}{2}}(1+e)^{\frac{3}{2}}}{\sqrt{\mu}} \frac{\sec^4 \frac{1}{2}(\theta - \varpi)}{\{(1+e) + (1-e)\tan^2 \frac{1}{2}(\theta - \varpi)\}^2}$$

$$= \frac{D^{\frac{3}{2}}}{\sqrt{\mu(1+e)}} \sec^4 \tfrac{1}{2}(\theta - \varpi) \left\{1 + \frac{1-e}{1+e}\tan^2 \tfrac{1}{2}(\theta - \varpi)\right\}^{-2}.$$

Expanding in powers of $1-e$, and neglecting powers of $1-e$ higher than the first, because $e = 1$ nearly;

$$\therefore nt = \tfrac{1}{2}\left(1 - \frac{1-e}{2}\right)^{-\frac{1}{2}} \int_{\varpi}^{\theta} \sec^4 \tfrac{1}{2}(\theta - \varpi)\{1 - (1-e)\tan^2 \tfrac{1}{2}(\theta - \varpi)\}\, d\theta$$

$$= \int_{\varpi}^{\theta} \frac{d\tan\frac{1}{2}(\theta - \varpi)}{d\theta} \{1 + \tan^2 \tfrac{1}{2}(\theta - \varpi)$$

$$+ (1-e)\left[\tfrac{1}{4} - \tfrac{3}{4}\tan^2 \tfrac{1}{2}(\theta - \varpi) - \tan^4 \tfrac{1}{2}(\theta - \varpi)\right]\}\, d\theta;$$

$$\therefore nt = \tan \tfrac{1}{2}(\theta - \varpi) + \tfrac{1}{3}\tan^3 \tfrac{1}{2}(\theta - \varpi)$$

$$+ (1-e)\left\{\tfrac{1}{4}\tan \tfrac{1}{2}(\theta - \varpi) - \tfrac{1}{4}\tan^3 \tfrac{1}{2}(\theta - \varpi) - \tfrac{1}{5}\tan^5 \tfrac{1}{2}(\theta - \varpi)\right\}.$$

The following is a convenient method for calculating the value of $\theta - \varpi$ for a given value of t.

Suppose $\theta' - \varpi$ is the true anomaly of a comet at the time t moving in a parabolic orbit of which D is the perihelion distance; then by Art. 282.

$$nt = \tan \tfrac{1}{2}(\theta' - \varpi) + \tfrac{1}{3}\tan^3 \tfrac{1}{2}(\theta' - \varpi).$$

Let $\theta - \varpi = \theta' - \varpi + x$: then putting this for $\theta - \varpi$ in the first expression for nt, and neglecting the squares and products of x and e, we have by Taylor's Theorem

$$nt = \tan \tfrac{1}{2}(\theta' - \varpi) + \tfrac{1}{3}\tan^3 \tfrac{1}{2}(\theta' - \varpi)$$

$$+ \frac{x}{2}\sec^4 \frac{\theta' - \varpi}{2} + \frac{1-e}{4}\tan \tfrac{1}{2}(\theta' - \varpi)\left\{1 - \tan^2 \tfrac{1}{2}(\theta' - \varpi)\right.$$

$$\left. - \tfrac{4}{5}\tan^4 \tfrac{1}{2}(\theta' - \varpi)\right\},$$

and eliminating nt from these last two equations

$$x = \tfrac{1}{10}(1-e)\tan \tfrac{1}{2}(\theta' - \varpi)\left\{4 - 3\cos^2 \tfrac{1}{2}(\theta' - \varpi) - 6\cos^4 \tfrac{1}{2}(\theta' - \varpi)\right\}.$$

A third column must now be added to the Table mentioned in Art. 282. consisting of values of $\dfrac{x}{1-e}$ for the corresponding values of t and $\theta - \varpi$. When this is constructed the manner of using it is as follows. Suppose $\sqrt{\dfrac{\mu}{2D^3}} = n$ in our orbit: then in the first column look for the time nt; and take the corresponding values of $\theta - \varpi$ and $\dfrac{x}{1-e}$: multiply the latter by $1 - e$, which will depend upon the form of the orbit, and then the true anomaly at the time t will be this quantity added to the value of $\theta - \varpi$ thus found.

CHAPTER IV.

EXPLANATION OF THE LUNAR PERTURBATIONS.

284. In the last Chapter we have calculated completely the motion of two bodies, considered as particles, attracting each other according to the assumed law of gravitation. When the various formulæ there obtained are applied to calculate the motion of the planets about the Sun, and for that purpose are reduced to tables, they manifest an agreement with observation so far complete as to leave no doubt of the correctness of the principles, which form the basis of the calculation; provided that observations be made at times separated by moderately long intervals. If, however, we proceed to a more rigorous nicety, and especially if we compare together observations which embrace a very long series of years, it is found that the agreement is not so perfect. Minute irregularities are detected, and the planets are found sometimes a little in advance, sometimes a little falling short, sometimes a little above or below, to the right or left, of their places, calculated on the theory of elliptic motion.

Now this is exactly what was to be anticipated. For if the principle of gravitation be universal the heavenly bodies disturb each other in their motion about the Sun, and so derange the elliptic form of the orbits and the equable description of areas by the radius vector.

285. It is our object in Chapters V and VI to deduce formulæ by which the mutual perturbations of the heavenly bodies may be calculated. The equations of motion for three or more particles attracting each other according to the law of gravitation have never yet been integrated. In fact, their integrals depend upon the integration of a function analogous

to the function V in Art. 169. See note to Art. 323. We must therefore have recourse to methods of approximation.

286. The peculiar configuration of the Solar System renders this approximation practicable, though under most other arrangements it would not be so; the bodies of our system are arranged either singly as the planets Mercury, Venus and Mars, or in groups as the Earth and the Moon, Jupiter, Saturn, and Herschel each with their Satellites, the central masses of the groups being much greater than that of the attending bodies: likewise the single bodies and the groups are always at considerable distances from each other, and describe orbits about the Sun very nearly circular and in planes nearly coinciding. There is, however, an exception in the case of the four asteroids Ceres, Vesta, Pallas, and Juno, the orbits of these being not very far different from each other in magnitude; but their masses are so small that in this way a compensation takes place. The mass of the Sun is of enormous magnitude in comparison with that of the other bodies; since, as we have remarked in Art. 284, the calculations made on the supposition that the Sun is the only attracting body nearly coincide with observation. Again the mass of the Earth is large when compared with that of the Moon; because the Earth moves about the Sun and the Moon about the Earth, nearly as if the Moon did not disturb the Earth's elliptic motion about the Sun, and as if the Sun did not disturb the Moon's elliptic motion about the Earth. In the same way we argue that the mass of Jupiter is much larger than that of his satellites by observing that Kepler's Laws are nearly verified, and so of the other bodies.

It is in consequence of this peculiar configuration of the Solar System that we are able to approximate to the solutions of our equations of motion by converging series.

287. In the present Chapter we intend to explain the nature of the perturbations of the Moon's motion about the Earth by the attraction of the Sun. We shall introduce a few calculations as interpretations of Newton's geometry into analytical language (*Principia* Book I. Prop. 66. and Book III); but shall reserve for the next Chapter the solution of the problem by systematic approximation.

We shall first explain a principle of great importance in calculating the combined effect of several small perturbing causes, which we shall find important throughout this and the two following Chapters.

Prop. *To explain the principle of the superposition of small motions.*

288. Let xyz be the co-ordinates of a body at the time t when undisturbed by any other body: a a very small numerical quantity which depends upon the disturbing force, of which the square and higher powers may therefore be always neglected.

$x + ax'$, $y + ay'$, $z + az'$ the co-ordinates of the body at time t when disturbed by the body (m') only.

$x + ax''$, $y + ay''$, $z + az''$ the co-ordinates of the body at time t when disturbed by the body (m'') only, and so on.

Now suppose the planets m', m'', ... all to disturb together.

In this case the alterations in x, y, z arising from the several planets will not be the same as before; but they will themselves suffer perturbations, since the action of each planet is now modified by that of all the others.

Thus the value of x will not become

$$x + a(x' + x'' + \ldots):$$

for each of the terms after the first will be modified, but since this modification arises from the disturbing forces, it follows, that the quantities to be added will be multiplied by a^2, a^3, ... and may therefore be neglected and, under this restriction, the co-ordinates of the body subjected to the combined perturbations of all the others will be

$$x + a(x' + x'' + \ldots\ldots\ldots),$$

$$y + a(y' + y'' + \ldots\ldots\ldots),$$

$$z + a(z' + z'' + \ldots\ldots\ldots), \text{ at the time } t.$$

Hence the perturbation in any quantities $\phi(x, y, z) = (u)$ which depends upon the co-ordinates of the planet will

$$= \frac{du}{dx} a (x' + x'' + \ldots\ldots) + \frac{du}{dy} a (y' + y'' + \ldots\ldots)$$

$$+ \frac{du}{dz} a (z' + z'' + \ldots\ldots).$$

And this $= \frac{du}{dx} ax' + \frac{du}{dy} ay' + \frac{du}{dz} az'$

$$+ \frac{du}{dx} ax'' + \frac{du}{dy} ay'' + \frac{du}{dz} az''$$

$$+ \ldots\ldots\ldots$$

But this latter form shews that the perturbation in $\phi (x, y, z)$ is equal to the sum of the separate perturbations of each planet supposing the others not to exist. Hence the Principle we enunciated is true. Its great use in our calculations is this, that it reduces the problem from one of several bodies to that of only three bodies. Hence the famous *Problem of Three Bodies.*

289. At an early period it was observed that the apparent motion of the Sun and Moon round the Earth was not uniform. This had been remarked by the Greek Astronomers. By observing the motion of the shadow of the gnomon they discovered a considerable difference in the intervals of time between the equinoxes and the solstices.

Hipparchus was the first who endeavoured to explain this: he supposed the orbits of the Sun and Moon described about the Earth to be eccentric circles, or circles of which the centres do not coincide with that of the Earth.

290. After a lapse of three centuries Ptolemy discovered that there was an error in the Moon's place in the heavens, which could not be accounted for on the hypothesis of Hipparchus: and he shewed that the magnitude of the error depended upon the position of the line of apsides, or axis major of the lunar orbit. This inequality was called the *Evection* of the Moon: we shall hereafter explain from what cause it arises

291. The next remarkable inequality of the Moon's motion was discovered by Tycho Brahe in the sixteenth century. This was proved to depend upon the angular distance of the Sun and Moon: and was greatest when the

Moon was about 45° or 135° from the Sun. In this respect this inequality, which was called the *Variation*, differed from the Evection, which Ptolemy found to be greatest when the Moon was ninety degrees from the Sun.

292. Tycho Brahe was the discoverer of one more inequality, which was called the *Annual Equation*, since it depends on the distance of the Sun from the Earth, and therefore goes through its changes in a year.

Not many years after these discoveries of Tycho Brahe, Kepler published to the world his Three Laws, which he had calculated with almost incredible labour and perseverance. Theory has led to the discovery of many other inequalities in the Moon's motion, but the above have been specified for their historical interest and because they are more sensible than the others.

293. All these, however, were merely bare facts, the results of continued and indefatigable observations and calculations. No common law appeared to connect them, no one cause was known of which they were necessary consequences. It was the glory of Newton, that he unravelled the mystery and demonstrated that these were all results of a universal principle with which matter is endowed by the Creator of the World.

Kepler and other Astronomers had conceived the notion of a universal gravitating principle: but it needed the master genius of Newton to demonstrate its existence.

We now proceed to explain the causes of these perturbations.

Prop. *To calculate the disturbing forces of the Sun on the Moon.*

294. The *disturbing forces* are the differences of the forces of the Sun on the Moon and Earth. Let E, M, S represent the masses of the Earth, Moon, and Sun. The law of force we assume to be that of the inverse square of the distance between the centres of the bodies. We shall consider the orbits of the Sun and Moon about the Earth nearly circular, since this is proved to be the case by observation*.

* The slight variations in the apparent magnitudes of the Sun and Moon convince us of this.

Let r be the distance between M and E (fig. 82),

r' .... S and E,

y .... S and M,

ω the angle SEM,

measured in the direction in which the hands of a watch move.

We must consider the motion of M about E as if E were fixed in order that we may discover the apparent perturbations: but the accelerating forces acting on E are $\frac{M}{r^2}$ and $\frac{S}{r'^2}$ in the directions EM and ES respectively: and in order that the relative motion may not be affected by supposing E fixed we must apply forces equal to these upon each body of the system in an opposite direction: the second law of motion shews the legitimacy of this process.

Hence the forces acting on M, E being considered fixed, are

$\frac{M+E}{r^2}$ in the direction ME,

$\frac{S}{y^2}$ .... MS,

$\frac{S}{r'^2}$ .... MK, MK being parallel to SE.

Then, resolving the second of these in the directions ME and ML (Art. 18), and combining the resolved parts of this with the other forces, the *disturbing* forces of S on M are

$\frac{Sr}{y^3}$ in the direction ME, and $\frac{Sr'}{y^3} - \frac{S}{r'^2}$ in the direction ML.

Again resolve these in the directions ME and perpendicular to this line: then the whole forces which act upon M about E at rest, are

$\frac{M+E}{r^2} + \frac{Sr}{y^3} - \frac{S}{r'^2}\left(\frac{r'^3}{y^3} - 1\right) \cos\omega$ in the direction ME,

$\frac{S}{r'^2}\left(\frac{r'^3}{y^3} - 1\right) \sin\omega$ in the direction perpendicular to ME,

and acting towards the nearest syzygy.

The Moon is said to be in *syzygy* when it is new or full; and in *quadrature* when ninety degrees from syzygy.

The first of the above forces deprived of its first term is called the *central disturbing force*: and the second is called the *tangential disturbing force*.

295. We proceed to obtain approximate expressions for these. Since the orbit of the Moon is nearly circular; and since $\frac{r}{r'}$ is a very small fraction, being about equal to $\frac{1}{400}$, we shall neglect its square and higher powers.

By the figure, $y^2 = r'^2 + r^2 - 2r'r\cos\omega$;

$$\therefore \frac{r'^3}{y^3} = \left(1 - \frac{2r}{r'}\cos\omega\right)^{-\frac{3}{2}} = 1 + \frac{3r}{r'}\cos\omega;$$

$$\therefore \text{central dist}^{\text{g}}\text{. force} = \frac{Sr}{r'^3} - \frac{3Sr}{r'^3}\cos^2\omega = -\frac{Sr}{2r'^3}(1+3\cos 2\omega)$$

$$\text{tangential dist}^{\text{g}}\text{. force} = \frac{3Sr}{r'^3}\sin\omega\cos\omega = \frac{3Sr}{2r'^3}\sin 2\omega.$$

In order still further to simplify these expressions; let f be the mean force of E upon M, E being supposed fixed: m the ratio of the periodic times of the Moon and Sun about the Earth ($m = \frac{1}{13}$ nearly): a and a' the mean distances of the Moon and Sun from the Earth: therefore by Art. 273. Cor. 2,

$$m^2 = \frac{4\pi^2 a^3}{M+E} \div \frac{4\pi^2 a'^3}{S+E} = \frac{S}{M+E}\,\frac{a^3}{a'^3} \text{ nearly};$$

$$\frac{M+E}{a^2} = f: \text{ and } \therefore \frac{Sr}{r'^3} = m^2 f\,\frac{a'^3}{r'^3}\,\frac{r}{a};$$

$$\therefore \text{central disturbing force} = -\frac{m^2}{2} f \frac{a'^3}{r'^3}\,\frac{r}{a}\,(1+3\cos 2\omega),$$

$$\text{tangential disturbing force} = \frac{3m^2}{2} f \frac{a'^3}{r'^3}\,\frac{r}{a}\sin 2\omega.$$

296. If we suppose the orbit of the Earth about the Sun and the undisturbed orbit of the Moon about the Earth to be circular, then $r' = a'$ and $r = a$; and

$$\text{central disturbing force} = -\tfrac{1}{2}m^2 f(1 + 3\cos 2\omega),$$

$$\text{tangential disturbing force} = \tfrac{3}{2}m^2 f \sin 2\omega.$$

297. The central disturbing force vanishes when

$$\omega = \tfrac{1}{2}\cos^{-1}(-\tfrac{1}{3}) = 55^0,\ 125^0,\ 235^0,\ 305^0 \text{ nearly}.$$

The points in the Moon's orbit determined by these angles are called *Octants*.

The central disturbing force is said to be *addititious* at points between those octants between which the quadratures lie, because at those points the above expression for the force is positive and consequently adds to the force of M to E. For a like reason the central disturbing force is said to be *ablatitious* between those octants between which the syzygies lie.

298. We proceed to examine the effects which the disturbing forces have upon the form and position of the Moon's orbit. We shall neglect quantities which depend upon the square and higher powers of the disturbing force. Whenever the undisturbed orbit is supposed to be nearly circular we may neglect all such terms as $\dfrac{dr^2}{d\theta^2}$. Thus

$$\text{radius of curvature} = \frac{\left\{r^2 + \dfrac{dr^2}{d\theta^2}\right\}^{\frac{3}{2}}}{r^2 + 2\dfrac{dr^2}{d\theta^2} - r\dfrac{d^2r}{d\theta^2}} = \frac{r^2}{r - \dfrac{d^2r}{d\theta^2}}.$$

$$\text{Also (velocity)}^2 = \left(\frac{ds}{dt}\right)^2 = r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2 = r^2\left(\frac{d\theta}{dt}\right)^2.$$

Again central force $= h^2u^2\left(u + \dfrac{d^2u}{d\theta^2}\right)$ by Art. 246.

$$= \frac{h^2}{r^2}\left\{\frac{1}{r} + \frac{2}{r^3}\frac{dr^2}{d\theta^2} - \frac{1}{r^2}\frac{d^2r}{d\theta^2}\right\} = \frac{h^2}{r^2}\left\{\frac{1}{r} - \frac{1}{r^2}\frac{d^2r}{d\theta^2}\right\}$$

$$= r^2 \left(\frac{d\theta}{dt}\right)^2 \left\{\frac{1}{r} - \frac{1}{r^2}\frac{d^2 r}{d\theta^2}\right\} \text{ by Art. 242.}$$

$$= \frac{(\text{vel.})^2}{\text{rad. of curvature}}.$$

We have introduced these expressions since we shall find them useful hereafter.

PROP. *To find the effect of the Sun's disturbing forces on the periodic time of the Moon.*

299. Since the tangential force passes through all its degrees of magnitude positive and negative during half a revolution of the Moon, it will compensate during one quarter of the revolution for any loss or gain that the angular velocity of the Moon may have experienced in the preceding quarter. In fact the *mean** tangential force equals zero. For this reason the tangential force has no effect on the periodic time.

For the same reason we neglect the periodic term of the central force. The *mean* central force $= f\left(1 - \frac{m^2}{2}\right)$. Since this is less than f, it follows that the mean distance is increased by the disturbing forces.

The absolute force $= fa^2\left(1 - \frac{m^2}{2}\right)$;

$\therefore$ the periodic time $= \dfrac{2\pi a^{\frac{3}{2}}}{\sqrt{\text{abs. force}}}$ nearly, (Art. 273. Cor. 2.)

$$= 2\pi\sqrt{\frac{a}{f}}\left(1 + \frac{m^2}{4}\right) \text{ nearly.}$$

This is greater than if there were no disturbing force (or $m = 0$); especially when we remember that a is greater than in the undisturbed orbit.

* By the *mean* value of a function of a variable angle θ we mean, the part of the function which is independent of periodical terms which pass through all their changes positive and negative as θ is increased by certain equal increments. Thus a is the mean value of $a + b\sin n\theta$.

Hence the periodic time is increased by the disturbing force. (*Principia*, Lib. I. Prop. LXVI. Cor. 6.)

300. Cor. 1. If we suppose the Earth's orbit about the Sun not circular, then by Art. 295, the mean central force

$$= f\left\{1 - \frac{m^2}{2}\frac{a'^3}{r'^3}\right\},$$

and the length of the month

$$= 2\pi\sqrt{\frac{a}{f}}\left\{1 + \frac{m^2}{4}\frac{a'^3}{r'^3}\right\}.$$

Hence the months are longest when the Earth is in perihelion, and shortest when in aphelion. This accounts for the Winter Months at this epoch being longer than the Summer Months.

301. Cor. 2. The mean velocity (V) of the Moon =

$$\frac{2\pi a}{\text{per. time}} = a\sqrt{\frac{f}{a}}\left(1 - \frac{m^2}{4}\right) = \sqrt{af}\left(1 - \frac{m^2}{4}\right).$$

Prop. *To find the effect of the Sun's disturbing force on the velocity of the Moon, supposing the undisturbed orbit of the Moon to be circular.*

302. Since the angle ω is referred to a line moveable in space we must adopt another means of measuring the position of the Moon. Let θ be the longitude of the Moon at the time t: then $m\theta$ is the longitude of the Sun, supposing that θ is measured from the time when the Sun and Moon were in Aries together; and supposing the orbits nearly circular: and therefore $\omega = (1 - m)\theta$:

$$\therefore \text{ tangential disturbing force} = \tfrac{3}{2}m^2 f \sin 2(1 - m)\theta,$$

and this is the only disturbing force which *directly* affects the velocity. We shall see in Art. 305, that the velocity is affected *indirectly* by the central disturbing force.

Now the space described by the Moon in the time t is $a\theta$; and the tangential force is the only force which acts in the line of the Moon's motion:

$$\therefore \; a\frac{d^2\theta}{dt^2} = -\frac{3m^2}{2} f \sin 2(1-m)\theta,$$

the *negative* sign being taken because the tangential force acts always towards the nearest syzygy, Art. 294, and consequently tends to diminish θ when the Moon is in the first and third quadrants, through which angles $\sin 2(1-m)\theta$ is positive, and to increase θ in the second and fourth quadrants, through which angles $\sin 2(1-m)\theta$ is negative.

Multiplying by $2a\frac{d\theta}{dt}$ and integrating

$$v^2 \text{ or } a^2\frac{d\theta^2}{dt^2} = \text{const.} + \frac{3m^2}{2(1-m)} fa\cos 2(1-m)\theta.$$

Let V be the mean velocity; then

$$v^2 = V^2 + \frac{3m^2}{2(1-m)} fa\cos 2(1-m)\theta$$

$$= V^2\left\{1 + \frac{3m^2}{2(1-m)}\cos 2(1-m)\theta\right\} \text{ neg}^g.\ m^4,$$

since $V^2 = fa\left(1 - \frac{1}{2}m^2\right)$ by Art. 301.

The effect, then, of the disturbing force is to increase the velocity above what it would be in the circular orbit, when the Moon is not more than 45° from syzygy; and in other positions to diminish it. This follows directly from the fact proved in Art. 294, that the tangential force always acts towards the nearest syzygy.

The velocity is greatest in syzygies and $= V\left\{1 + \frac{3m^2}{4(1-m)}\right\}$

.... least in quadratures and $= V\left\{1 - \frac{3m^2}{4(1-m)}\right\}$.

(*Principia*, Lib. I. Prop. LXVI. Cors. 2, 3.)

PROP. *To find the effect of the Sun's disturbing force on the form of the Moon's orbit; supposing the undisturbed orbit to be circular.*

303. The curvature at any point of the orbit is measured by the reciprocal of the radius of curvature: hence, by Art. 298, the curvature equals $\frac{\text{central force}}{(\text{velocity})^2}$.

Now the central force $= f - \frac{m^2}{2} f(1 + 3\cos 2\omega)$, Art. 296,

$$\text{also } (\text{velocity})^2 = V^2 \left\{1 + \frac{3m^2}{2(1-m)} \cos 2\omega\right\}, \text{ Art. 302.}$$

$$V = \text{mean vel.} = \sqrt{af}\left(1 - \frac{m^2}{4}\right) \text{ by Art. 301;}$$

$$\therefore \text{ curvature} = \frac{f}{V^2}\left\{1 - \frac{m^2}{2} - \frac{3m^2}{2}\left(1 + \frac{1}{1-m}\right)\cos 2\omega\right\}$$

$$= \frac{1}{a}\left\{1 - \frac{3m^2}{2}\left(1 + \frac{1}{1-m}\right)\cos 2\omega\right\}.$$

This is greatest in quadrature, when $\omega = 90^0$ and 270^0; and is least in syzygy, when $\omega = 0$ and 180^0.

This shews that the orbit will assume an oval form *with respect to the Sun*, having its minor axis in syzygy, (*Principia*, Lib. I. Prop. LXVI. Cor. 4). Its form in space will be an irregular curve, nearly circular, but not *re-entering*. Also the expression for the curvature shews that the equation to the orbit is $r = a(1 - x\cos 2\omega)$, the major and minor axes being $2a(1+x)$ and $2a(1-x)$.

Prop. *To find the ratio of the axes of the oval orbit.*

304. The equation to the orbit is $r = a(1 - x\cos 2\omega)$; but since ω is measured from a moveable line we must put $\omega = (1-m)\theta$ as in Art. 302;

$$\therefore r = a\left\{1 - x\cos 2(1-m)\theta\right\};$$

$$\therefore \text{ curvature} = \frac{1}{r} - \frac{1}{r^2}\frac{d^2 r}{d\theta^2} \text{ by Art. 298.}$$

$$= \frac{1}{a}\left\{1 + x\left[1 - 4(1-m)^2\right]\cos 2(1-m)\theta\right\}.$$

Equating this to the expression found in the last Article,

$$\frac{1}{a}\left\{1 + x\left[1 - 4(1-m)^2\right]\cos 2(1-m)\theta\right\}$$

$$= \frac{1}{a}\left\{1 - \frac{3m^2}{2}\left(1 + \frac{1}{1-m}\right)\cos 2(1-m)\theta\right\};$$

$$\therefore\ x = \frac{3m^2}{2}\,\frac{1 + \dfrac{1}{1-m}}{4(1-m)^2 - 1}.$$

If we put $m = \frac{1}{13.3}$, $x = \frac{1}{139}$ nearly;

$$\therefore\ \frac{1+x}{1-x} = \frac{70}{69},$$

the ratio in the *Principia* Lib. III. Prop. XXVIII.

This is the ratio of the axes of the oval orbit which moves round with the Sun while the Moon moves in it.

In Art. 302, we found the effect of the tangential disturbing force on the velocity of the Moon; and we have just shewn that the tangential and central disturbing forces draw the orbit of the Moon into an oval figure with respect to the Sun. We proceed, then, to calculate the velocity of the Moon when both disturbing forces are considered.

PROP. *To find the velocity of the Moon in the oval orbit.*

305. In Art. 302, we found the effect of the tangential force on the velocity: but the velocity will be affected by the change in form of the orbit: and thus we see the indirect effect of the central disturbing force upon the velocity.

Let v be the velocity of the Moon at any distance,

v_1 .... the mean distance,

then $v^2 = r^2 \frac{d\theta^2}{dt^2}$, neglecting $\frac{dr^2}{dt^2}$ which depends on the square of the disturbing force: Art. 298.

$$v^2 = \frac{1}{r^2}\left(r^2 \frac{d\theta}{dt}\right)^2 = \frac{h^2}{r^2}; \quad v_1{}^2 = \frac{h^2}{a^2};$$

$$\therefore \; v^2 = \frac{a^2}{r^2} v_1{}^2.$$

By substituting for $\frac{a^2}{r^2}$ we shall correct v^2 for the oval form of the orbit: and by substituting for $v_1{}^2$ we shall correct for the change in velocity in the circular orbit: and in this way the complete velocity in the oval orbit is found.

$$\frac{a^2}{r^2} = 1 + 2x \cos 2(1-m)\,\theta,$$

$$v_1{}^2 = V^2 \left\{1 + \frac{3m^2}{2(1-m)} \cos 2(1-m)\,\theta\right\}. \quad \text{See Art. 302;}$$

$$\therefore \; v^2 = V^2 \left\{1 + \left(2x + \frac{3m^2}{2(1-m)}\right) \cos 2(1-m)\,\theta\right\}.$$

306. Let $\delta\theta$ be the error in longitude owing to this change in the velocity: then

$$\frac{d\theta}{dt} = \frac{V}{a}, \quad \frac{d(\theta+\delta\theta)}{dt} = \frac{v}{r} \text{ nearly};$$

$$\therefore \; \frac{d.\delta\theta}{d\theta} = \frac{v}{V}\frac{a}{r} - 1$$

$$= \left\{1 + \left(x + \frac{3m^2}{4(1-m)}\right) \cos 2(1-m)\,\theta\right\}\left\{1 + x \cos 2(1-m)\,\theta\right\} - 1$$

$$= \left(2x + \frac{3m^2}{4(1-m)}\right) \cos 2(1-m)\,\theta;$$

$$\therefore \; \delta\theta = \left(\frac{x}{1-m} + \frac{3m^2}{8(1-m)^2}\right) \sin 2(1-m)\,\theta.$$

This error in longitude is greatest (disregarding its sign) when the Moon is 45° or 135° from the Sun on either side of syzygy: and therefore explains the cause of the error in the Moon's place discovered by Tycho Brahe, and called the *Variation* (Art. 291).

If we put $x = \dfrac{1}{139}$ and $m^2 = \dfrac{1}{179}$

$$\text{Variation} = m^2\left(1 + \frac{3}{8}\right)\sin 2(1-m)\theta \text{ nearly}$$

$$= \frac{11m^2}{8}\sin 2(1-m)\theta,$$

which accords with the rigorous approximation of the second order in the next Chapter, Art. 341.

We shall now suppose the undisturbed orbit of the Moon to be an ellipse of very small eccentricity; the Earth's centre being in the focus.

Our calculations will receive a remarkable degree of simplification by considering the perturbations of the Moon to affect, not the Moon itself directly, but the elements of its orbit, and so the Moon indirectly. The legitimacy of this hypothesis will appear from the following Proposition.

Prop. *To prove that the motion of the Moon may be represented by supposing it to move in an ellipse, the elements of which are continually changing.*

307. We have to shew that at every instant an ellipse can be drawn with one of its foci in the Earth's centre; its circumference passing through the Moon's centre; its tangent at the Moon in the direction of the Moon's motion at that instant; and the velocity in this ellipse calculated according to the principles of elliptic motion (see Art. 252.) equal to the velocity of the Moon at that time.

If an ellipse can be found which satisfies these conditions it is clear that we may suppose the Moon to be moving in its circumference at the proposed instant: and the perturbations of the Moon's motion will be found by calculating the change in the elements of this *instantaneous orbit*, as it is termed.

Two of the elements, viz. the inclination and longitude of the node, are fixed by the condition that the plane of the ellipse must pass through the direction of the Moon's motion and the centre of the Earth.

$$\text{Let } \frac{a}{r} = 1 + e\cos(\theta - \varpi)$$

be the equation to the required ellipse, neglecting powers of e higher than the first: also let $r_1\theta_1$ be the co-ordinates to the Moon in the plane of its orbit at the instant under consideration; v its velocity; ϕ the angle between the radius vector of the Moon and the direction of her motion; then the above conditions give

$$\frac{a}{r_1} = 1 + e\cos(\theta_1 - \varpi),$$

$$\tan\phi = \left(r\frac{d\theta}{dr} =\right)\frac{1}{e}\operatorname{cosec}(\theta_1 - \varpi),$$

$$v^2 = 2(E+M)\left\{\frac{1}{r_1} - \frac{1}{2a}\right\}.$$

From these three equations a, e, ϖ may be found. The sixth element, the epoch, determines the position of the Moon in the instantaneous ellipse.

Hence an ellipse can always be drawn as described in the enunciation.

308. We shall proceed to explain the nature of the alterations in the elements. Since these variations during a revolution of the Moon are small we shall, in accordance with the Principle proved in Art. 288, consider the variation of each element supposing all the others to remain invariable; and then add their effects together.

At any proposed instant the equation to the orbit is

$$\frac{a}{r} = 1 + e\cos(\theta - \varpi),$$

where a, e, ϖ have values depending upon the proposed instant. Hence also if v be the velocity

$$v^2 = 2\mu\left\{\frac{1}{r} - \frac{1}{2a}\right\}$$

$$= \frac{\mu}{a} \left\{1 + 2e \cos(\theta - \varpi)\right\}$$

$$= fa \left\{1 + 2e \cos(\theta - \varpi)\right\}, \text{ since } \frac{\mu}{a^2} = f.$$

We shall use the symbol δ to indicate differentiation, not with respect to the motion of the Moon, but with respect to the geometric deviations of the actual orbit of the Moon from the undisturbed orbit. Thus, if we wish to ascertain the change in the eccentricity in consequence of a change δv in the velocity, we have by differentiating the above equation

$$v\delta v = fa \cos(\theta - \varpi)\, \delta e,$$

which gives δe in terms of δv: and so of other perturbations. If we wish to ascertain the change in eccentricity arising from a change δf in the force f, we have the equation

$$0 = \delta f\, (1 + e \cos(\theta - \varpi) + f\delta e \cos(\theta - \varpi),$$

which gives δe in terms of δf.

We repeat the remark, that the direct tendency of the tangential disturbing force is to change the velocity: but the central disturbing force has not this direct effect. Therefore when we consider the effect of the central force on the elements a, e, ϖ, we must take the variation of the equation

$$v^2 = fa \left\{1 + 2e \cos(\theta - \varpi)\right\}$$

considering v constant: and when we consider the effect of the tangential force we must take the variation considering f constant.

We shall first find the effect of the disturbing forces on the position of the line of apsides.

An *apse* is a point in a polar orbit where the radius vector is perpendicular to the tangent.

Prop. *To find the effect of the mean central disturbing force upon the position of the line of apsides.*

309. The mean central force $= f\frac{a^2}{r^2} - \frac{m^2}{2} f \frac{r}{a}$, Art. 295.

We shall find it most convenient, in this case, to use the equation $\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2}$ proved in Art. 246, for central orbits.

In this case $h = aV = a\sqrt{af}\,(1 - \frac{1}{4}m^2)$;

$$\therefore\ \frac{d^2u}{d\theta^2} + u = \frac{1}{a}(1 + \tfrac{1}{2}m^2) - \frac{m^2}{2a^4}\frac{1}{u^3}.$$

Now $u - \frac{1}{a}$ varies as the eccentricity of the orbit; and therefore we neglect its square;

$$\therefore\ \frac{1}{u^3} = \left\{\frac{1}{a} + \left(u - \frac{1}{a}\right)\right\}^{-3} = a^3 - 3a^4\left(u - \frac{1}{a}\right);$$

$$\therefore\ \frac{d^2u}{d\theta^2} + u = \frac{1}{a}\left(1 + \frac{m^2}{2}\right) - \frac{m^2}{2a} + \frac{3m^2}{2}\left(u - \frac{1}{a}\right),$$

$$a\frac{d^2u}{d\theta^2} + \left(1 - \frac{3m^2}{2}\right)(au - 1) = 0,$$

the integral of this is

$$au \text{ or } \frac{a}{r} = 1 + e\cos\left(1 - \frac{3m^2}{4}\right)(\theta - \alpha),$$

e and α being arbitrary constants.

This is the equation to the path of the Moon, supposing the mean central disturbing force to be the only disturbing force. If the coefficient of $\theta - \alpha$ were unity it would be an ellipse. The following is the geometrical construction for the orbit.

Let $A'M'E$ be the ellipse supposing the coefficient of θ were unity (fig. 83). When $\theta - \alpha = 0$ the values of r in the real curve and the ellipse are the same. Let, then, $A'M$ be the real path of the Moon. Take the angle $A'EM = \theta - \alpha$: and the angle $A'EM' = (1 - \frac{3}{4}m^2)$ angle $A'EM$: and let EM' cut the ellipse in M'.

Then $\frac{a}{EM'} = 1 + e\cos A'EM'$ by the equation to the ellipse

$$= 1 + e\cos\left(1 - \frac{3m^2}{4}\right)(\theta - \alpha) \text{ by construction}$$

$$= \frac{a}{EM} \text{ by equation to Moon's path;}$$

$$\therefore EM' = EM.$$

Draw EA so that angle MEA = angle $M'EA'$: and make $EA = EA'$: we may then draw an ellipse about E as focus and through A and M exactly equal to $A'M'E$.

The motion of the Moon may therefore be geometrically represented by supposing it to move in an ellipse, the ellipse itself revolving in its own plane about the focus with an angular velocity $= \frac{dA'EA}{dt} = \frac{3m^2}{4}\frac{d\theta}{dt}$, or which is to the angular velocity of the Moon as $\frac{3}{4}m^2 : 1$ in the direction of the Moon's motion.

Hence the tendency of the mean central disturbing force is to make the line of apsides progress.

310. Observation shews that the above result is only half the amount of the progression. The reason will be explained in the next Chapter.

We cannot extend the approximation without introducing the tangential force, and therefore refer the reader to the Luner theory in the next Chapter, where the whole Problem is accurately solved to a second approximation. All that we shall here attempt is an explanation of the *tendency* of the disturbing forces in altering the motion of the Moon in her eccentric orbit.

Prop. *To explain the effect of the tangential disturbing force of the Sun on the line of apsides.*

311. Let the equation to the ellipse in which the Moon is moving at any instant be

$$\frac{a}{r} = 1 + e\cos(\theta - \varpi) \text{ neglecting } e^2 \ldots\ldots$$

Then by the principles of elliptic motion,

$$v^2 = 2\mu\left\{\frac{1}{r} - \frac{1}{2a}\right\} = \frac{\mu}{a}\left\{1 + 2e\cos(\theta - \varpi)\right\}.$$

But the tangential force is continually causing v to differ from what it would be in terms of θ on the supposition of elliptic motion. But, as we have shewn in Art. 307, we may still suppose the velocity to be the same as in the ellipse by altering ϖ, the longitude of the line of apsides, in such a way as to make a compensation. By taking the variation of the above expression for v^2 in terms of ϖ, we have

$$av\delta v = \mu e \sin(\theta - \varpi)\, \delta\varpi;$$

$$\therefore\ \delta\varpi = \frac{av}{\mu e} \frac{\delta v}{\sin(\theta - \varpi)}.$$

Now the effect of the tangential force is to diminish the velocity of the Moon as it is moving from syzygy to quadrature, and to increase the velocity as the Moon is moving from quadrature to syzygy; see Art. 302. Hence δv is negative as the Moon moves from syzygy to quadrature, and positive from quadrature to syzygy.

Also $\sin(\theta - \varpi)$ is positive or negative according as the Moon is moving from perigee to apogee, or from apogee to perigee.

Now the motion of the line of apsides during a revolution of the Moon is very small, since the disturbing forces are small. We may therefore suppose, in examining the effect that the particular position of the line of apsides with respect to syzygies and quadratures has upon their motion, that they remain stationary during a revolution of the Moon: otherwise we shall be introducing quantities which depend upon the square of the disturbing force, quantities which we have purposed to neglect. We shall consider the two cases when the line of apsides is in syzygies, and in quadratures: and then take the average effect.

I. *Suppose the line of apsides is in syzygies:* fig. 84.

Then as the Moon moves from quadrature to quadrature through perigee, (that is, through the arc qYQ), δv and $\sin(\theta - \varpi)$ are of different signs and therefore $\delta\varpi$ is negative, or the line of apsides is regressing: but as the Moon moves from quadrature to quadrature through apogee, (that is, through the arc Qyq) δv and $\sin(\theta - \varpi)$ have the same sign and therefore $\delta\varpi$ is positive, or the line of apsides is progressing. Now the time of moving from quadrature to quadrature through apogee is greater than that of moving through perigee, Art. 242.

Also the alteration in velocity by the tangential force is greater in the latter case than in the former.

Hence, when the line of apsides is in syzygies, it *progresses* on the whole during one revolution of the Moon.

II. *Suppose the line of apsides is in quadratures:* fig. 85.

Then as the Moon is moving from syzygy to syzygy through apogee, (that is, through the arc yqY) $\delta\varpi$ is negative, or the line of apsides is regressing, and as the Moon is moving from syzygy to syzygy through perigee $\delta\varpi$ is positive, or the line of apsides progressing. Now the time of moving through perigee is less than that of moving through apogee. And the alteration in velocity in the former case is less than in the latter.

Hence when the line of apsides is in quadratures, it on the whole *regresses* during a revolution of the Moon.

If we attentively examine the steps of the above investigation, and observe that the circumstances of regression in one position of the line of apsides closely resemble those of progression in the other, and *vice versâ*, it will easily be seen that in these two positions of the apsidal line the progression is very nearly the same as the regression.

When, however, intermediate positions of the apsidal line are considered the progression decidedly has the preponderance, and for this reason: regression takes place when the line of apsides is *distant* from the Sun; but progression takes place when the line of apsides is *near* the Sun. Hence regression, which, from its nature, causes the line of apsides to *meet* the Sun, moves it towards a progressing position: while progression, from its nature, causes the line of apsides to linger in a progressing situation. Hence, on the whole, the tangential force causes the line of apsides to progress. See Airy's *Gravitation*, p. 99.

Prop. *To explain the effect of the tangential force on the eccentricity of the Moon's orbit.*

312. We shall resume the equation $v^2 = fa\{1 + 2e\cos(\theta - \varpi)\}$ by which we can represent the real motion, if we suppose the elements to alter.

$$\text{Now } v\delta v = fa\cos(\theta - \varpi)\,\delta e;$$

$$\therefore\ \delta e = \frac{v}{fa}\,\frac{\delta v}{\cos(\theta - \varpi)}.$$

Now, in consequence of the tangential force, δv has the same sign and nearly the same magnitude for opposite positions of the Moon in her orbit, Art. 305: and $\cos(\theta - \varpi)$ has the same magnitude with different signs. Hence δe has different signs, but nearly the same magnitude for opposite positions of the Moon in her orbit: and therefore the tangential force has little effect on the eccentricity, since a partial compensation takes place in one part of the orbit for errors caused in the opposite part.

But we must examine this a little more accurately: for although compensation will take place pretty accurately when the Moon is about 90^0 from perigee and apogee, since the distance of the Moon from the Earth is then about the same, yet it will not be so complete when the Moon is near perigee and apogee, since at those points the Moon's distances differ more than in any other parts of her orbit. We shall therefore consider the case in which the Moon is near perigee and apogee.

1. When the line of apsides is *before* syzygies: fig. 86. Then δv is negative near perigee and apogee; and $\cos(\theta - \varpi)$ is positive near perigee and negative near apogee. Hence δe is negative when the Moon is near perigee and positive when near apogee. And therefore in this position of the line of apsides e is increasing when the greatest change takes place, and consequently on the whole the tendency of the eccentricity is to increase during each revolution of the Moon.

2. When the line of apsides is *behind* syzygies: fig. 87. Then it will easily be seen that δe is positive when the Moon is near perigee and negative when near apogee. Hence in this position of the line of apsides the tendency of the eccentricity is to decrease during each revolution of the Moon.

PROP. *To explain the effect of the central disturbing force on the eccentricity of the Moon's orbit.*

313. Now $fa = v^2 \{1 - 2e\cos(\theta - \varpi)\}$, and we have to find the variation of e corresponding to any variation in f: we have

$$\delta e = -\frac{a}{2v^2} \frac{\delta f}{\cos(\theta - \varpi)}.$$

Now δf has the same sign, and nearly the same magnitude in opposite parts of the Moon's orbit: Art. 295. Also $\cos(\theta-\varpi)$ has different signs in opposite parts of the orbit. Hence δe has different signs and nearly the same magnitude in opposite parts of the orbit, and a partial compensation will take place in the changes in eccentricity.

But, as in the last Article, we must examine this a little more accurately. The values of δf differ more at apogee and perigee than at any other parts of the orbit, since the distances of the Moon from the Earth differ most at these points. We shall therefore consider the changes in eccentricity when the Moon is near perigee and apogee.

1. Suppose the line of apsides is near *octants*: figs. 86, 87. When the Moon is near perigee and apogee, and therefore near octants, δf nearly vanishes and the eccentricity does not undergo any material change; and therefore the compensation of eccentricity is pretty accurate during a revolution of the Moon.

2. Suppose the line of apsides is near *syzygies*: fig. 84. When the Moon is near apogee and perigee, and therefore near syzygies, δf is negative (Art. 297); and $\cos(\theta-\varpi)$ is negative near apogee and positive near perigee. Hence δe is positive when the Moon is in perigee, and negative when in apogee. Wherefore e is decreasing when the change is greatest: and on the whole the eccentricity decreases when the line of apsides is near syzygies.

3. Suppose the line of apsides is near *quadratures*: fig. 85. Then δf is positive when the Moon is near perigee and apogee; and we shall find that the eccentricity is on the whole increasing during a revolution of the Moon.

314. The result of the last two Articles is as follows. The tangential force tends to increase or diminish the eccentricity according as the line of apsides is *before* or *behind* the Sun. And the central disturbing force tends to increase or diminish the eccentricity according as the line of apsides is nearly 90^0 from the Sun or near the Sun: in other parts a compensation takes place.

Hence, then, during the Sun's motion from the Moon's line of apsides through 90^0 the eccentricity of the Moon's orbit is decreasing: and during the Sun's motion towards the line of apsides the eccentricity is increasing. The eccentricity is greatest when the line of apsides is in syzygies and least when the line of apsides is in quadratures.

315. We shall now proceed to explain the effect of the disturbing forces on the inclination and the motion of the line of nodes.

Hitherto we have supposed the Moon to move in the plane of the ecliptic: let us now suppose that the planes of the ecliptic and the Moon's orbit are slightly inclined to each other, as is the case in nature. Let the disturbing force of the Sun on the Moon be resolved into two parts, one in the plane of the Moon's orbit, and the other perpendicular to this plane: this latter is the part which affects the inclination and the position of the line of nodes.

If we bear in mind that the Sun's disturbing force always acts *towards* the Sun when the Moon is nearer the Sun than the Earth, and *from* the Sun in the contrary case, we shall easily see, by referring to fig. 88, that the part of the disturbing force which is perpendicular to the plane of the Moon's orbit always acts *towards* the ecliptic, except when the Moon is between quadrature and the nearest node, in which case it acts *from* the ecliptic.

By the plane of the Moon's orbit we mean the plane drawn through the centres of the Moon and Earth, and through the direction of the Moon's motion at any instant. And since in consequence of the Sun's disturbing force the Moon is continually drawn out of the plane in which it is moving, the plane of the orbit is continually shifting its position by revolving about the Moon's radius vector as an instantaneous axis.

316. Before we begin to explain the effect on the inclination and line of nodes we shall enunciate the following Lemma, the truth of which is self-evident.

LEMMA. When a body is moving from or towards a plane and a force acts upon it in a direction from or towards the plane, then the inclination of the body's resulting motion will be increased or diminished according as the original motion and the force act in the same or opposite directions with respect to the plane.

Prop. *To explain the effect of the Sun's disturbing force upon the inclination of the Moon's orbit to the ecliptic and on the position of the line of nodes.*

317. I. *Suppose the line of nodes is in syzygies.*

It is clear that in this case the inclination and line of nodes will not be affected; since no part of the force acts perpendicularly to the plane of the orbit. The line of nodes would remain in this position were it not for the Sun's motion.

II. *Suppose the line of nodes is in advance of the Sun:* fig. 89.

Let Nn be the line of nodes: take $Nm = 90^0$ on the orbit: let Qq be the quadratures. Then as the Moon moves from N to Q she moves from the ecliptic, and the disturbing force acts from the ecliptic. Hence the inclination of the Moon's path (Art. 316), and therefore of the plane of her orbit, is increasing, and therefore in revolving about the radius vector EM the node N must move towards quadratures, or the line of nodes Nn must progress.

As the Moon moves from Q to m, her motion and the disturbing force act in opposite directions, and therefore the inclination is decreasing (Art. 316), and the line of nodes regressing.

As the Moon moves from m to n her motion and the disturbing force both tend towards the ecliptic and therefore the inclination of the plane of her orbit is increasing, and therefore in revolving about the radius vector EM' causes the point n to move back, or the line of nodes to regress.

In the other half of the orbit the effect will be exactly the same.

Hence, if ϕ be the angular distance of the line of nodes from syzygies (ϕ being less than 90^0), the inclination is increasing as the Moon is moving through an angle

$$= 2\,(NEQ + mEn) = 360^0 - 2\phi:$$

and is decreasing as the Moon is moving through the remaining angle 2ϕ. And the line of nodes regresses while the Moon is moving through an angle $180^0 + 2\phi$; and progresses while the Moon is moving through the remaining angle $180^0 - 2\phi$.

III. *Suppose the line of nodes is in quadratures.*

Then as the Moon moves from quadrature to syzygy the disturbing force and motion tend in different directions, and therefore the inclination is decreasing and the line of nodes is regressing. And as the Moon moves from syzygy to quadrature the inclination is increasing and therefore the line of nodes still regresses.

Wherefore the increase and decrease of inclination counteract each other; but the motion of the nodes is wholly *regressive.*

IV. *Suppose the line of nodes is behind the Sun:* fig. 90.

Then as the Moon moves from N to m the inclination is decreasing and the line of nodes regressing: and as she moves from m to q the inclination is increasing and the nodes regressing. As the Moon moves from q to n the inclination is decreasing and the nodes progressing.

Hence, if as before ϕ be the angular distance of the line of nodes from syzygy, the inclination is increasing as the Moon moves through an angle 2ϕ, and decreasing as she moves through the angle $360^0 - 2\phi$. But the nodes regress and progress respectively while the Moon is moving through the angles $180^0 + 2\phi$ and $180^0 - 2\phi$.

It appears, then, that on the whole the nodes *regress* pretty steadily: but the inclination is much more fluctuating and on the whole is not affected during a revolution of the line of nodes.

We introduce the two following Propositions as examples of the method used by Newton in the Third Book of the Principia: they will be found in Props. 30 and 31. Newton's geometry is translated into analysis.

PROP. *To calculate the motion of the nodes of the Moon's orbit considered circular.*

318. Let MM' be the arc described by the Moon in a unit of time, fig. 91: $M'L = 2$ space through which the dis-

turbing force would draw the Moon in the same time: Nn the line of nodes, Qq the line of quadratures, AB of syzygies: Mm is a tangent to the Moon's orbit at M meeting the ecliptic in m: join LM and produce it to meet the ecliptic in l: this gives the position of the tangent at M after the small time of describing MM' and therefore $\angle mEl$ represents the motion of the node.

Now LM' is parallel to the ecliptic and therefore can meet no line in the ecliptic: but it is in the same plane with lm, therefore LM' is parallel to lm.

$$\text{Hence } \frac{\text{motion of Node}}{\text{motion of Moon}} = \frac{\angle lEm}{\angle MEM'} = \frac{\sin mlE}{\angle MEM'} \frac{lm}{Em} \text{ nearly}$$

$$= \frac{\sin AEn}{\angle MEM'} \frac{LM'}{MM'} \frac{Mm}{Em} = \frac{\sin AEn}{\angle MEM'} \frac{LM'}{MM'} \sin MEm.$$

Now the disturbing force in the direction $M'L$

$$= \frac{Sr'}{y^3} - \frac{S}{r'^2} \text{ (see Art. 294.)} = \frac{S}{r'^2}\left(\frac{r'^3}{y^3} - 1\right)$$

$$= \frac{3Sr}{r'^3} \cos MEA, \quad \because y = r' - r\cos MEA;$$

$$\therefore LM' = \frac{3Sr}{r'^3} \cos MEA.$$

$$MM'. \angle MEM' = \frac{MM'^2}{r} = \frac{(\text{vel.})^2}{\text{rad.}} = \text{force of Moon to } E$$

$$= \frac{E+M}{r^2};$$

$$\therefore \frac{LM'}{MM'. \angle MEM'} = \frac{3S}{M+E} \frac{r^3}{r'^3} \cos MEA = 3m^2 \cos MEA,$$

Art. 295;

$\therefore$ motion of Node

$= 3m^2 \cos MEA \sin MEN \sin AEn$. motion of Moon.

Let N = longitude of the Node,

θ = ... Moon,

$m\theta$ = ... Sun,

supposing that the Sun, Moon, and Node were all in the first point of Aries when $\theta = 0$.

Hence the above equation gives

$$\frac{dN}{d\theta} = -3m^2 \cos(\theta - m\theta)\sin(\theta - N)\sin(m\theta - N),$$

the negative sign being taken because the mean motion of the node is regressive.

319. We shall now transform this by the ordinary trigonometrical formula

$$2\sin a\cos b = \sin(a+b) + \sin(a-b),$$

we have

$$\frac{dN}{d\theta} = -\frac{3m^2}{4}\left\{1 + \cos 2(\theta - m\theta) - \cos 2(\theta - N) - \cos 2(m\theta - N)\right\}.$$

For a first approximation we neglect the periodical terms and take the mean values:

$$\frac{dN}{d\theta} = -\frac{3m^2}{4} = -i \text{ suppose;}$$

$$\therefore N = -i\theta, \text{ constant} = 0.$$

For a second approximation we shall put this value of N in the periodical terms;

$$\therefore \frac{dN}{d\theta} = -i\left\{1 + \cos 2(1-m)\theta - \cos 2(1+i)\theta - \cos 2(m+i)\theta\right\};$$

$$\therefore N = -i\theta - \frac{i}{2(1-m)}\sin 2(1-m)\theta + \frac{i}{2(1+i)}\sin 2(1+i)\theta$$
$$+ \frac{i}{2(m+i)}\sin 2(m+i)\theta.$$

For a third approximation we shall put this value of N in $\frac{dN}{d\theta}$ after neglecting the terms divided by $1-m$ and $1+i$, because they are smaller than the term divided by $m+i$; then

$$N = -i\theta + \frac{i}{2(m+i)}\sin 2(m+i)\theta;$$

$$\therefore \frac{dN}{d\theta} = -i - i\cos 2(1-m)\theta + i\cos\left\{2(1+i)\theta - \frac{i}{m+i}\sin 2(m+i)\theta\right\}$$

$$+ i\cos\left\{2(m+i)\theta - \frac{i}{m+i}\sin 2(m+i)\theta\right\}.$$

If we expand these the last term gives

$$\frac{i^2}{m+i}\sin^2 2(m+i)\theta \text{ or } \frac{i^2}{2(m+i)} - \frac{i^2}{2(m+i)}\cos 4(m+i)\theta.$$

Hence we obtain

$$\frac{dN}{d\theta} = -i + \frac{i^2}{2(m+i)} + \text{periodic functions of } \theta,$$

and therefore the *mean* value of N is

$$N = -\left\{i - \frac{i^2}{2(m+i)}\right\}\theta = -\left\{1 - \frac{3m}{8\left(1+\frac{3m}{4}\right)}\right\}\frac{3m^2}{4}\theta.$$

If we expand in powers of m we have

$$N = -\left(\frac{3m^2}{4} - \frac{9m^3}{32} + \frac{27m^4}{128} + \dots\dots\right)\theta,$$

320. In this calculation we have supposed the Moon's angular velocity to be uniform. To correct for the oval orbit let N_1 be the corrected value of N.

Now the motion of the node varies as the magnitude of the disturbing force, which varies as the square of the time of the Moon's describing MM', and therefore as the square of the velocity at M inversely;

$$\therefore \frac{dN_1}{dN} = \frac{(\text{vel.})^2 \text{ in octants}}{(\text{vel.})^2 \text{ at } M} = 1 - \frac{3m^2}{2(1-m)}\cos 2(1-m)\theta;$$

$$\therefore \frac{dN_1}{d\theta} = -i\left\{1 - \frac{i}{2(m+i)} + \cos 2(1-m)\theta \dots\right\}$$

$$\times \left\{1 - \frac{2i}{1-m} \cos 2(1-m)\theta + \ldots\right\}$$

$$= -i\left\{1 - \frac{i}{2(m+i)} - \frac{2i}{1-m}\cos^2 2(1-m)\theta + \ldots\right\}$$

$$= -i\left\{1 - \frac{i}{2(m+i)} - \frac{i}{1-m}\right\} + \text{periodical terms,}$$

and the mean value of N_1 is

$$N_1 = -i\left(1 - \frac{i}{2(m+i)} - \frac{i}{1-m}\right)\theta$$

$$= -\frac{3m^2}{4}\left\{1 - \frac{3m}{8\left(1+\frac{3m}{4}\right)} - \frac{3m^2}{4(1-m)}\right\}\theta$$

$$= -\left(\frac{3m^2}{4} - \frac{9m^3}{32} - \frac{45}{128}m^4 + \ldots\right)\theta,$$

this correction does not affect the first and second terms.

PROP. *To calculate the inclination of the Moon's orbit to the ecliptic at any time.*

321. Let ENm be the line of nodes (fig. 92), El its position after a unit of time: Mp perpendicular to the plane of the ecliptic, pG perpendicular to the line of nodes EN; produce pG to cut El in g; join MG, Mg and draw Gr perpendicular to Mg: I the inclination of the plane of the Moon's orbit to the ecliptic.

Now $\delta I = \angle MGp - \angle Mgp = \angle GMg = \dfrac{Gr}{GM}$.

Also $\delta N = \angle GEg = \dfrac{Gg}{GE}$;

$$\therefore \delta I = \frac{Gr}{Gg}\frac{GE}{GM}\delta N = \sin I \,.\, \cot MEN \,.\, \delta N;$$

$$\therefore \frac{dI}{d\theta} = 3m^2 \sin I \cos MEA \cos MEN \sin AEn$$

$$= -3m^2 \sin I \cos(\theta - m\theta) \cos(\theta - N) \sin(m\theta - N)$$

$$= -\frac{3m^2}{4} \sin I \left\{\sin 2(\theta - N) - \sin 2(\theta - m\theta) + \sin 2(m\theta - N)\right\}.$$

This expression shews that I will always be small: and therefore $\sin I = I$ nearly; let γ be the mean value of I: also

$$N = -\frac{3m^2}{4}\theta = -i\theta, \text{ (see Art. 319.)}$$

$$\therefore \frac{dI}{d\theta} = -\frac{3m^2}{4}\gamma\left\{\sin 2(1+i)\theta - \sin 2(1-m)\theta + \sin 2(m+i)\theta\right\};$$

$$\therefore I = \frac{3m^2}{4}\gamma\left\{\frac{1}{2(1+i)}\cos 2(1+i)\theta - \frac{1}{2(1-m)}\cos 2(1-m)\theta + \frac{1}{2(m+i)}\cos 2(m+i)\theta\right\} + \text{const.}$$

The constant $= \gamma$, the mean value of I. Therefore, neglecting the first and second terms because they are of an order higher than the third,

$$I = \gamma\left\{1 + \frac{3m}{8\left(1 + \frac{3m}{4}\right)}\cos 2(m+i)\theta\right\}$$

$$= \gamma\left\{1 + \frac{3m}{8}\cos 2\,(\text{Sun's longitude—Node's longitude})\right\},$$

neglecting γm^2, &c.

This accords with Chapter V. Art. 342.

CHAPTER V.

LUNAR THEORY.

322. We now enter upon the calculation of the perturbations of the Moon by a process of systematic approximation; and shall proceed in the next Chapter to calculate those of the planets. In the Lunar and Planetary Theories we use different methods of calculation for this reason. The perturbations of the Moon are much greater than those of any planet, because the Sun, the mass of which is so enormous (Art. 286), is one of the disturbing bodies. Likewise the ratio of the distances of the disturbed and disturbing bodies from the central body about which they move is very different in the two theories; being about $\frac{1}{400}$ in that of the Moon, and sometimes so large as $\frac{3}{4}$ in that of the planets. The difference of the methods of approximation will be seen in the calculations of this and the following Chapter.

Before entering upon the immediate subject of the present Chapter we must investigate the following Proposition.

Prop. *A number of bodies considered as material particles attract each other with forces which vary inversely as the square of their distances, and directly as the mass of the attracting body: required the equations of motion of any one of the bodies relatively to a second.*

323. Let M, m, m', m'' .... be the masses of the bodies; M being that of the body about which the motion is to be calculated: and m the mass of that body of which the equations of motion are to be determined.

Let X, Y, Z be the co-ordinates of M,

$X+x$, $Y+y$, $Z+z$ . ... m,

$X+x'$, $Y+y'$, $Z+z'$ .... m'

....

Then the distance between m and m' is

$$\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{1}{2}},$$

and the attraction of m' on m is

$$\frac{m'}{(x'-x)^2+(y'-y)^2+(z'-z)^2}.$$

Let this be resolved into three parts parallel to the axes: that parallel to the axis of x is

$$\frac{m'(x'-x)}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}},$$

$$\text{or } \frac{1}{m}\frac{d}{dx}\left\{\frac{mm'}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{1}{2}}}\right\},$$

and so of the other bodies m'' ....

Now assume $\lambda = \Sigma \cdot \dfrac{mm'}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{1}{2}}}$,

which expression is the sum of the quantities found by dividing the product of every two of the masses m, m', m'' ...... by their respective distances.

Then the sum of the attractions of m', m'' ...... on m resolved parallel to the axis of x

$$=\frac{1}{m}\frac{d\lambda}{dx}.$$

In like manner $\dfrac{1}{m}\dfrac{d\lambda}{dy}$ and $\dfrac{1}{m}\dfrac{d\lambda}{dz}$ are the attractions of m', m''.... on m parallel to the axes of y and z.

Let $r\,r'\,r''$ .... be the distances of $m\,m'\,m''$ .... from M. Then the attraction of M on m parallel to x is $\dfrac{Mx}{r^3}$, and consequently the equation of motion of m in space parallel to x is

$$\frac{d^2(X+x)}{dt^2} = \frac{1}{m}\frac{d\lambda}{dx} - \frac{Mx}{r^3}.$$

But $\frac{mx}{r^3}$, $\frac{m'x'}{r'^3}$, ... are the attractions of mm' ... on M parallel to x: and therefore the equation of motion of M parallel to x is

$$\frac{d^2X}{dt^2} = \Sigma . \frac{mx}{r^3},$$

and by subtracting this from the equation above we have the equation of motion of m relatively to M

$$\frac{d^2x}{dt^2} + \frac{Mx}{r^3} + \Sigma . \frac{mx}{r^3} - \frac{1}{m}\frac{d\lambda}{dx} = 0.$$

And in like manner

$$\frac{d^2y}{dt^2} + \frac{My}{r^3} + \Sigma . \frac{my}{r^3} - \frac{1}{m}\frac{d\lambda}{dy} = 0,$$

$$\frac{d^2z}{dt^2} + \frac{Mz}{r^3} + \Sigma . \frac{mz}{r^3} - \frac{1}{m}\frac{d\lambda}{dz} = 0.$$

Now assume

$$R* = \frac{m'(xx'+yy'+zz')}{r'^3} + \frac{m''(xx''+yy''+zz'')}{r''^3} + \ldots . - \frac{\lambda}{m},$$

* The function R satisfies Laplace's Equation (Art. 168).

$$\text{For } \frac{dR}{dx} = \frac{m'x}{r'^3} + \ldots\ldots - \frac{1}{m}\frac{d\lambda}{dx}$$

$$= \frac{m'x'}{r'^3} + \ldots\ldots - \frac{1}{m}\Sigma . \frac{mm'(x'-x)}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}};$$

$$\therefore \frac{d^2R}{dx^2} = \frac{1}{m}\Sigma . \frac{mm'\{(y'-y)^2+(z'-z)^2-2(x'-x)^2\}}{\{(x'-x)^2+(y'-y)^2(z'-z)^2\}^{\frac{5}{2}}},$$

$$\text{and so } \frac{d^2R}{dy^2} = \frac{1}{m}\Sigma . \frac{mm'\{(x'-x)^2+(z'-z)^2-2(y'-y)^2\}}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{5}{2}}},$$

$$\frac{d^2R}{dz^2} = \frac{1}{m}\Sigma . \frac{mm'\{(x'-x)^2+(y'-y)^2+2(z'-z)^2\}}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{5}{2}}};$$

$$\therefore \frac{d^2R}{dx^2} + \frac{d^2R}{dy^2} + \frac{d^2R}{dz^2} = 0.$$

$$\text{then } \frac{dR}{dx} = \frac{m'x'}{r'^3} + \frac{m''x''}{r''^3} + \dots - \frac{1}{m}\frac{d\lambda}{dx}$$

$$= \Sigma . \frac{mx}{r^3} - \frac{mx}{r^3} - \frac{1}{m}\frac{d\lambda}{dx},$$

and the first equation of motion becomes

$$\left.\begin{aligned} &\frac{d^2x}{dt^2} + \frac{(M+m)\,x}{r^3} + \frac{dR}{dx} = 0 \\ &\text{and similarly} \\ &\frac{d^2y}{dt^2} + \frac{(M+m)\,y}{r^3} + \frac{dR}{dy} = 0 \\ &\frac{d^2z}{dt^2} + \frac{(M+m)\,z}{r^3} + \frac{dR}{dz} = 0 \end{aligned}\right\} \dots\dots (1).$$

These are the equations by which the motion of the Moon about the Earth, or of a planet about the Sun, is determined when under the action of all the other bodies of the Solar System. They have never yet been completely integrated. For this reason we must resort to approximation. To effect this R, which is called the *disturbing function*, must be developed in a converging series. The difference of the methods adopted in the Lunar and Planetary Theories depends upon the different modes of expanding the function R. In the Lunar Theory R is expanded in powers of the ratio of the distances of the Sun and Moon, a very small fraction nearly equal $\frac{1}{400}$; but in the Planetary Theory R is expanded in powers of the eccentricities and inclinations of the planetary orbits, all of which are very small, with the exception of those of Juno and Pallas; the eccentricities of these being about $\frac{1}{4}$ and the inclination of the orbit of Pallas to the ecliptic being about 35^0: but the masses of these planets are very small.

324. We intend throughout our calculations in this and the following Chapter to neglect quantities which depend upon the square and higher powers of the disturbing forces. In consequence of this we may calculate separately the perturbations caused by the Sun or a planet on the supposition that the

other heavenly bodies do not attract, and then add together the separate perturbations: this follows from the Principle explained in Art. 288.

PROP. *To obtain equations for calculating the radius vector of the Moon; and the inclination of the lunar orbit to the ecliptic.*

325. The equations of motion are by the last Article

$$\left.\begin{aligned}\frac{d^2x}{dt^2}+\frac{\mu x}{r^3}+\frac{dR}{dx}=0\\ \frac{d^2y}{dt^2}+\frac{\mu y}{r^3}+\frac{dR}{dy}=0\\ \frac{d^2z}{dt^2}+\frac{\mu z}{r^3}+\frac{dR}{dz}=0\end{aligned}\right\}\ \ldots\ldots\ (1),$$

where μ = mass of the Earth + mass of the Moon.

Let the plane of the ecliptic be the plane of xy: also let ρ be the projection of r on the ecliptic: s the tangent of inclination of ρ to the same plane: θ the longitude, the axis of x passing through Aries: then

$$x^2+y^2=\rho^2;\quad z^2+\rho^2=r^2;$$

$$x=\rho\cos\theta,\quad y=\rho\sin\theta,\quad z=\rho s.$$

Multiply the first equation by y, and the second by x, and subtract;

$$\therefore\ x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}=y\frac{dR}{dx}-x\frac{dR}{dy};$$

$$\therefore\ \frac{d}{dt}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)=\rho\left\{\sin\theta\frac{dR}{dx}-\cos\theta\frac{dR}{dy}\right\},$$

$$\text{or}\quad \frac{d}{dt}\left(\rho^2\frac{d\theta}{dt}\right)=\rho T,$$

$$\text{if we put }\sin\theta\frac{dR}{dx}-\cos\theta\frac{dR}{dy}=T.$$

Multiply each side by $\rho^2 \dfrac{d\theta}{dt}$;

$$\therefore \rho^2 \frac{d\theta}{dt} \frac{d}{dt}\left\{\rho^2 \frac{d\theta}{dt}\right\} = \rho^3 T \frac{d\theta}{dt};$$

$$\therefore \left(\rho^2 \frac{d\theta}{dt}\right)^2 = h^2 + 2\int \rho^3 T d\theta;$$

h^2 being a constant introduced by integration;

$$\therefore \frac{d\theta^2}{dt^2} = \frac{h^2}{\rho^4} + \frac{2}{\rho^4}\int \rho^3 T d\theta; \quad \rho = \frac{1}{u}$$

$$= h^2 u^4 + 2u^4 \int \frac{T d\theta}{u^3} \ldots\ldots\ldots (2).$$

Again, multiply the first and second equations (1) respectively by $2\dfrac{dx}{dt}$, $2\dfrac{dy}{dt}$ and add,

$$\therefore \frac{d}{dt}\left\{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right\} + \frac{2\mu}{r^3}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) + 2\frac{dx}{dt}\frac{dR}{dx} + 2\frac{dy}{dt}\frac{dR}{dy} = 0,$$

putting $x = \rho\cos\theta$, $y = \rho\sin\theta$, $x^2 + y^2 = \rho^2$;

$$\frac{d}{dt}\left\{\frac{d\rho^2}{dt^2} + \rho^2\frac{d\theta^2}{dt^2}\right\} + \frac{2\mu\rho}{r^3}\frac{d\rho}{dt} - 2\rho\frac{d\theta}{dt}\left(\sin\theta\frac{dR}{dx} - \cos\theta\frac{dR}{dy}\right)$$

$$+ 2\frac{d\rho}{dt}\left(\cos\theta\frac{dR}{dx} + \sin\theta\frac{dR}{dy}\right) = 0.$$

$$\text{Let } \frac{\mu\rho}{r^3} + \cos\theta\frac{dR}{dx} + \sin\theta\frac{dR}{dy} = P;$$

$$\therefore \frac{d}{dt}\left\{\frac{d\rho^2}{dt^2} + \rho^2\frac{d\theta^2}{dt^2}\right\} + 2P\frac{d\rho}{dt} - 2\rho\frac{d\theta}{dt}T = 0;$$

$$\therefore \frac{d}{d\theta}\left\{\frac{d\rho^2}{dt^2} + \rho^2\frac{d\theta^2}{dt^2}\right\} + 2P\frac{d\rho}{d\theta} - 2\rho T = 0, \quad \rho = \frac{1}{u};$$

$$\therefore \frac{d}{d\theta}\left\{\frac{1}{u^4}\frac{du^2}{dt^2}+\frac{1}{u^2}\frac{d\theta^2}{dt^2}\right\}-\frac{2P}{u^2}\frac{du}{d\theta}-\frac{2T}{u}=0.$$

Now $\frac{d\theta^2}{dt^2}=h^2u^4+2u^4\int\frac{Td\theta}{u^3}$ by (2),

by this equation we can eliminate t, and we have

$$\frac{d}{d\theta}\left\{\left(\frac{du^2}{d\theta^2}+u^2\right)\left(h^2+2\int\frac{Td\theta}{u^3}\right)\right\}-\frac{2P}{u^2}\frac{du}{d\theta}-\frac{2T}{u}=0,$$

performing the differentiation

$$2\frac{du}{d\theta}\left(\frac{d^2u}{d\theta^2}+u\right)\left(h^2+2\int\frac{Td\theta}{u^3}\right)$$

$$+\frac{2T}{u^3}\left(\frac{du^2}{d\theta^2}+u^2\right)-\frac{2P}{u^2}\frac{du}{d\theta}-\frac{2T}{u}=0;$$

$$\therefore \frac{d^2u}{d\theta^2}+u-\frac{\frac{P}{u^2}-\frac{T}{u^3}\frac{du}{d\theta}}{h^2+2\int\frac{Td\theta}{u^3}}=0 \;.........\; (3).$$

326. To obtain an equation for calculating the inclination of the radius vector to the ecliptic, we have by the last of equations (1),

$$\frac{d^2z}{dt^2}=-\frac{\mu z}{r^3}-\frac{dR}{dz}=-S \text{ suppose},$$

but $z=\frac{s}{u}$, $\therefore \frac{dz}{dt}=\frac{dz}{d\theta}\frac{d\theta}{dt}=\frac{1}{u^2}\left(u\frac{ds}{d\theta}-s\frac{du}{d\theta}\right)\frac{d\theta}{dt}$

$$=\left(u\frac{ds}{d\theta}-s\frac{du}{d\theta}\right)\sqrt{h^2+2\int\frac{Td\theta}{u^3}} \text{ by (2)};$$

$$\therefore \frac{d^2z}{dt^2}=\frac{d}{d\theta}\left\{\left(u\frac{ds}{d\theta}-s\frac{du}{d\theta}\right)\sqrt{h^2+2\int\frac{Td\theta}{u^3}}\right\}\frac{d\theta}{dt}$$

$$=\left\{\left(u\frac{d^2s}{d\theta^2}-s\frac{d^2u}{d\theta^2}\right)\sqrt{h^2+2\int\frac{Td\theta}{u^3}}\right.$$

$$\left.+\left(u\frac{ds}{d\theta}-s\frac{du}{d\theta}\right)\frac{T}{u^3\sqrt{h^2+2\int\frac{Td\theta}{u^3}}}\right\}\frac{d\theta}{dt}$$

$$=\left(u\frac{d^2s}{d\theta^2}-s\frac{d^2u}{d\theta^2}\right)u^2\left(h^2+2\int\frac{Td\theta}{u^3}\right)+\left(u\frac{ds}{d\theta}-s\frac{du}{d\theta}\right)\frac{T}{u}=-S;$$

$$\therefore\ \frac{d^2s}{d\theta^2}-\frac{s}{u}\frac{d^2u}{d\theta^2}+\frac{\frac{S}{u^3}+\frac{T}{u^3}\left(\frac{ds}{d\theta}-\frac{s}{u}\frac{du}{d\theta}\right)}{h^2+2\int\frac{Td\theta}{u^3}}=0;$$

but by the last Article

$$\frac{s}{u}\frac{d^2u}{d\theta^2}+s-\frac{\frac{Ps}{u^3}-\frac{Ts}{u^4}\frac{du}{d\theta}}{h^2+2\int\frac{Td\theta}{u^3}}=0,$$

adding these last equations we have

$$\frac{d^2s}{d\theta^2}+s+\frac{\frac{S-Ps}{u^3}+\frac{T}{u^3}\frac{ds}{d\theta}}{h^2+2\int\frac{Td\theta}{u^3}}=0\ \ldots\ldots\ldots\ (4).$$

327. It is necessary that we estimate the comparative magnitude of the various small quantities involved in our calculations.

Let e, e' be the eccentricities of the lunar and solar orbits; k the tangent of the mean inclination of the lunar orbit to the ecliptic; m the ratio of the Sun's mean motion to the Moon's mean motion, a and a' the mean distances of the Moon and Sun from the Earth: the values of these quantities are nearly

$$e=\tfrac{1}{20},\quad e'=\tfrac{1}{60},\quad k=\tfrac{1}{12},\quad m=\tfrac{1}{13},$$

these we shall reckon of the first order of small quantities.

But $\frac{a}{a'} = \frac{1}{400}$ nearly is a quantity of the second order of magnitude, since it $= e^2$ nearly. The Sun's disturbing force is greatest when the Moon is between the Sun and Earth: in which case it $= \frac{m'}{(a'-a)^2} - \frac{m'}{a'^2}$: the ratio which this bears to the action of the Earth on the Moon

$$= \left\{\frac{m'}{(a'-a)^2} - \frac{m'}{a'^2}\right\} \div \frac{\mu}{a^2} = \frac{m'}{\mu}\,\frac{2a^3}{a'^3} \text{ nearly} = 2m^2$$

by Kepler's third law. Hence the disturbing force is of the second order.

We proceed to expand the values of T, P, S.

Prop. *To expand the values of* T, P, S *neglecting small quantities of the fourth order.*

328. By Arts. 325, 326,

$$T = \sin\theta \frac{dR}{dx} - \cos\theta \frac{dR}{dy};$$

$$P = \frac{\mu\rho}{r^3} + \cos\theta \frac{dR}{dx} + \sin\theta \frac{dR}{dy};$$

$$S = \frac{\mu z}{r^3} + \frac{dR}{dz}, \text{ and by Arts. 323, 324,}$$

$$R = \frac{m'(xx' + yy' + zz')}{r'^3} - \frac{m'}{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}^{\frac{1}{2}}},$$

m' = mass of Sun; $x'y'z'$ co-ordinates of Sun: r' = dist. of Sun.

Let $x = \rho\cos\theta$, $y = \rho\sin\theta$, $z = \rho s$ as before;

$$x' = r'\cos\theta',\ y' = r'\sin\theta',\ z' = 0,$$

since the plane of xy is the plane of the ecliptic.

$$\therefore\ R = \frac{m'\rho}{r'^2}\cos(\theta - \theta') - \frac{m'}{\{\rho^2 + z^2 + r'^2 - 2\rho r'\cos(\theta-\theta')\}^{\frac{1}{2}}}$$

$$R = \frac{m'\rho}{r'^2}\cos(\theta - \theta') - \frac{m'}{r'}\left\{1 - \frac{2\rho}{r'}\cos(\theta - \theta') + (1 + s^2)\frac{\rho^2}{r'^2}\right\}^{-\frac{1}{2}}$$

$$= -\frac{m'}{r'} + \frac{m'}{2r'}\left\{1 + s^2 - 3\cos^2(\theta - \theta')\right\}\frac{\rho^2}{r'^2}$$

$$= -\frac{m'}{r'} - \frac{m'}{4r'}\left\{1 - 2s^2 + 3\cos 2(\theta - \theta')\right\}\frac{\rho^2}{r'^2}.$$

Also $\theta = \tan^{-1}\dfrac{y}{x}$, $\rho = \sqrt{x^2 + y^2}$, $s = \dfrac{z}{\sqrt{x^2 + y^2}}$.

Hence $\dfrac{dR}{dx} = \dfrac{dR}{d\theta}\dfrac{d\theta}{dx} + \dfrac{dR}{d\rho}\dfrac{d\rho}{dx} + \dfrac{dR}{ds}\dfrac{ds}{dx}$

$$= -\frac{3m'\rho}{2r'^3}\sin 2(\theta - \theta')\sin\theta - \frac{m'\rho}{2r'^3}\left\{1 + 3\cos 2(\theta - \theta')\right\}\cos\theta$$

$$= -\frac{m'\rho}{2r'^3}\cos\theta - \frac{3m'\rho}{2r'^3}\cos(\theta - 2\theta');$$

$$\frac{dR}{dy} = \frac{dR}{d\theta}\frac{d\theta}{dy} + \frac{dR}{d\rho}\frac{d\rho}{dy} + \frac{dR}{ds}\frac{ds}{dy}$$

$$= \frac{3m'\rho}{2r'^3}\sin 2(\theta - \theta')\cos\theta - \frac{m'\rho}{2r'^3}\left\{1 + 3\cos 2(\theta - \theta')\right\}\sin\theta$$

$$= -\frac{m'\rho}{2r'^3}\sin\theta + \frac{3m'\rho}{2r'^3}\sin(\theta - 2\theta');$$

$$\frac{dR}{dz} = \frac{dR}{d\theta}\frac{d\theta}{dz} + \frac{dR}{d\rho}\frac{d\rho}{dz} + \frac{dR}{ds}\frac{ds}{dz} = \frac{m's\rho}{r'^3}.$$

Hence $T = \sin\theta\dfrac{dR}{dx} - \cos\theta\dfrac{dR}{dy} = -\dfrac{3m'\rho}{2r'^3}\sin 2(\theta - \theta')$;

$$P = \frac{\mu\rho}{r^3} + \cos\theta\frac{dR}{dx} + \sin\theta\frac{dR}{dy}$$

$$= \frac{\mu\rho}{\{\rho^2 + \rho^2 s^2\}^{\frac{3}{2}}} - \frac{m'\rho}{2r'^3} - \frac{3m'\rho}{2r'^3} \cos 2(\theta - \theta')$$

$$= \frac{\mu}{\rho^2}\left(1 - \frac{3s^2}{2}\right) - \frac{m'\rho}{2r'^3} - \frac{3m'\rho}{2r'^3} \cos 2(\theta - \theta');$$

$$S = \frac{\mu z}{r^3} + \frac{dR}{dz} = \frac{\mu\rho s}{\{\rho^2 + \rho^2 s^2\}^{\frac{3}{2}}} + \frac{m' s\rho}{r'^3}$$

$$= \frac{\mu}{\rho^2}\left(s - \frac{3s^3}{2}\right) + \frac{m' s\rho}{r'^3}.$$

PROP. *To integrate the differential equations, first approximation.*

329. We here neglect all small quantities of an order higher than the first, and therefore the disturbing force (Art. 327): hence by last Article

$$T = 0, \quad P = \frac{\mu}{\rho^2} = \mu u^2, \quad S = \frac{\mu s}{\rho^2} = \mu s u^2,$$

and the differential equations (3) (4) of Arts. 325, 326 become

$$\frac{d^2 u}{d\theta^2} + u - \frac{\mu}{h^2} = 0;$$

$$\text{and } \frac{d^2 s}{d\theta^2} + s = 0.$$

The solutions of these equations are

$$u = \frac{\mu}{h^2}\{1 + e\cos(\theta - a)\} = b\{1 + e\cos(\theta - a)\}, \quad b = \frac{\mu}{h^2},$$

and $s = k\sin(\theta - \gamma)$: e, a, k, γ are constants.

The first of these proves that the orbit of the Moon is an ellipse: and the second proves that the tangent of the latitude bears a constant ratio to the sine of the longitude reckoned from the node, and therefore the Moon moves in a constant plane.

PROP. *To shew that to integrate the differential equations to a second approximation we must introduce all terms of the* third *order, in which the coefficient of θ is either nearly equal to unity, or is small.*

330. By approximating to the values of the small quantities we shall arrive at a differential equation in u of the form

$$\frac{d^2u}{d\theta^2} + u + a + a' \cos(n\theta + n') + \ldots\ldots = 0,$$

the integral of which is of the form

$$u = -a + A\cos(\theta + B) + C\cos(n\theta + n') + \ldots$$

A, B being arbitrary constants, and C...constants to be determined by putting this value of u in the differential equation. This gives

$$C(1 - n^2) = -a'; \quad \therefore C = \frac{-a'}{1 - n^2},$$

from which we learn that if n nearly $= 1$, then C will be large. Wherefore when the coefficient of the argument of a cosine or sine is nearly unity we must retain coefficients of the *third* order, since these terms rise into importance by the process of integration.

Again, the function R and therefore the differential equation in u contains terms depending on r': and the reciprocal of r'

$$= b'\{1 + e'\cos(\theta' - a')\} = b'\{1 + e'\cos(m\theta + \beta - a') + \ldots\}$$

the accented letters refer to the Sun: m = ratio of the Moon's period to the Sun's period: β = longitude of the Sun when the Moon is in Aries. Hence

$$\frac{dt}{d\theta} \text{ calculated from } \frac{dt}{d\theta} = \frac{1}{hu^2}\left\{1 + 2\int\frac{T\,d\theta}{h^2u^3}\right\}^{-\frac{1}{2}}$$

(Art. 325, equation (2)) will contain a term $C\cos(m\theta + \beta - a')$, and therefore t contains a term

$$\frac{C}{m}\sin(m\theta + \beta - a'):$$

hence C must be calculated to the *third* order. Wherefore all terms in which the coefficient of θ is small must be calculated to the third order; as well as those in which the coefficient of θ is nearly equal to unity.

PROP. *To calculate* $\sin 2(\theta-\theta')$ *and* $\cos 2(\theta-\theta')$ *to the first order.*

331. We need calculate these only to the first order because they occur only in terms multiplied by quantities of the second order.

$$\frac{dt}{d\theta} = \frac{1}{hu^2}\left\{1 + 2\int\frac{T\,d\theta}{h^2u^3}\right\}^{-\frac{1}{2}} = \frac{1}{hu^2}, \text{ first order:}$$

$$= \frac{1}{b^2h}\left\{1 - 2e\cos(\theta-a)\right\},\ b^2h = n = \text{Moon's mean motion;}$$

$$\therefore\ nt = \theta - 2e\sin(\theta - a),$$

$t=0$ when the Moon's mean longitude $=0$; also let the Sun's mean longitude then $=\beta$:

$$\therefore\ n't + \beta = \theta' - 2e'\sin(\theta'-a'),\ n' = \text{Sun's mean motion.}$$

Now $\dfrac{n'}{n} = m$: hence multiplying the first equation by m and subtracting, we have

$$\theta' - 2e'\sin(\theta'-a') = m\theta + \beta;\ \text{neg}^{g}.\ me, \text{ of second order;}$$

$$\therefore\ \theta' = m\theta + \beta + 2e'\sin(m\theta + \beta - a');$$

$$\therefore\ \sin 2(\theta-\theta') = \sin\left\{[(2-2m)\theta - 2\beta] - 4e'\sin(m\theta+\beta-a')\right\}$$

$$= \sin\{(2-2m)\theta - 2\beta\} - 4e'\cos\{(2-2m)\theta-2\beta\}\sin(m\theta+\beta-a')$$

$$= \sin\{(2-2m)\theta-2\beta\} - 2e'\sin\{(2-m)\theta-\beta-a'\}$$
$$+ 2e'\sin\{(2-3m)\theta - 3\beta + a'\}$$

$$\cos 2(\theta-\theta') = \cos\{(2-2m)\theta-2\beta\}$$
$$+ 4e'\sin\{(2-2m)\theta-2\beta\}\sin(m\theta+\beta-a')$$
$$= \cos\{(2-2m)\theta-2\beta\} + 2e'\cos\{(2-3m)\theta-3\beta+a'\}$$
$$- 2e'\cos\{(2-m)\theta-\beta-a'\}.$$

PROP. *To calculate* $\frac{\mathrm{T}}{\mathrm{h}^2\mathrm{u}^3}$, $\int\frac{\mathrm{T}\,d\theta}{\mathrm{h}^2\mathrm{u}^3}$, $\frac{\mathrm{T}}{\mathrm{h}^2\mathrm{u}^3}\frac{d\mathrm{u}}{d\theta}$ *to the third order.*

332. By Art. 328, $\frac{T}{h^2u^3} = -\frac{3m'}{2u^4h^2r'^3}\sin 2(\theta-\theta')$

$$= -\frac{3m'b'^3}{2h^2b^4}\{1+e\cos(\theta-\alpha)\}^{-4}\{1+e'\cos(\theta'-\alpha')\}^3\sin 2(\theta-\theta')$$

$$= -\tfrac{3}{2}m^2\{1-4e\cos(\theta-\alpha)+3e'\cos(m\theta+\beta-\alpha')\}$$
$$\times\{\sin[(2-2m)\theta-2\beta]-2e'\sin[(2-m)\theta-\beta-\alpha']$$
$$+2e'\sin[(2-3m)\theta-3\beta+\alpha']\}$$
$$= -\tfrac{3}{2}m^2\{\sin[(2-2m)\theta-2\beta]-2e\sin[(1-2m)\theta-2\beta+\alpha]\}.$$

Again $u=b\{1+e\cos(\theta-\alpha)\}$; $\therefore \frac{du}{d\theta} = -be\sin(\theta-\alpha)$;

$$\therefore \frac{T}{h^2u^3}\frac{du}{d\theta} = \tfrac{3}{4}bm^2e\cos\{(1-2m)\theta-2\beta+\alpha\}.$$

Again $\int\frac{T\,d\theta}{h^2u^3} = \tfrac{3}{2}m^2\left\{\frac{1}{2-2m}\cos[(2-2m)\theta-2\beta]\right.$
$$\left.-\frac{2e}{1-2m}\cos[(1-2m)\theta-2\beta+\alpha]\right\}$$
$$= \tfrac{3}{4}m^2\cos\{(2-2m)\theta-2\beta\}-3m^2e\cos\{(1-2m)\theta-2\beta+\alpha\}$$

PROP. *To calculate* $\frac{\mathrm{P}}{\mathrm{h}^2\mathrm{u}^2}$ *to the third order.*

333. By Art. 328.

$$\frac{P}{h^2u^2} = b(1-\tfrac{3}{2}s^2)-\frac{m'}{2u^3h^2r'^3}\{1+3\cos 2(\theta-\theta')\}.$$

First. $b(1-\tfrac{3}{2}s^2) = b\{1-\tfrac{3}{4}k^2+\tfrac{3}{4}k^2\cos 2(\theta-\gamma)\}$.

Secondly.

$$-\frac{m'}{2u^3h^2r'^3} = -\frac{m'b'^3}{2h^2b^3}\{1+e\cos(\theta-\alpha)\}^{-3}\{1+e'\cos(\theta'-\alpha')\}^3$$
$$= -\tfrac{1}{2}bm^2\{1-3e\cos(\theta-\alpha)+3e'\cos(m\theta+\beta-\alpha')\};$$

both terms must be retained, since in the first the coefficient of $\theta = 1$, and in the second it is small.

Thirdly. $-\dfrac{3m'}{2u^3h^2r'^3}\cos 2(\theta - \theta') =$

$$-\tfrac{3}{2}bm^2\{1 - 3e\cos(\theta - a) + 3e'\cos(m\theta + \beta - a')\}$$

$$\times \{\cos[(2 - 2m)\theta - 2\beta] + 2e'\cos[(2 - 3m)\theta - 3\beta + a']$$

$$- 2e'\cos[(2 - m)\theta - \beta - a']\}.$$

Multiplying these together by the formula $2\cos a\cos b = \cos(a - b) + \cos(a + b)$, neglecting quantities of the third order, except those in which the coefficient of θ is small or nearly unity, we have this third part of $\dfrac{P}{h^2u^2} =$

$$-\tfrac{3}{2}bm^2\{\cos[(2 - 2m)\theta - 2\beta] - \tfrac{3}{2}e\cos[(1 - 2m)\theta - 2\beta + a]\}.$$

Hence the value of $\dfrac{P}{h^2u^2}$ is

$$b\{1 - \tfrac{3}{4}k^2 + \tfrac{3}{4}k^2\cos 2(\theta - \gamma)\} - \tfrac{1}{2}bm^2\{1 - 3e\cos(\theta - a)$$

$$+ 3e'\cos(m\theta + \beta - a') + 3\cos[(2 - 2m)\theta - 2\beta]$$

$$- \tfrac{9}{2}e\cos[(1 - 2m)\theta - 2\beta + a]\}.$$

PROP. *To form the differential equation for* u.

334. By Art. 325, equation (3),

$$\frac{d^2u}{d\theta^2} + u - \frac{\dfrac{P}{u^2} - \dfrac{T}{u^3}\dfrac{du}{d\theta}}{h^2 + 2\displaystyle\int\frac{Td\theta}{u^3}} = 0.$$

By expanding the reciprocal of the denominator of the fractional part and neglecting the square of the disturbing force which is of the fourth order, and neglecting all other quantities of the fourth order, and observing that P contains a term μu^2, or by Art. 329 bh^2u^2, which is not small, we have

$$\frac{d^2u}{d\theta^2} + u - \frac{P}{h^2u^2} + \frac{T}{h^2u^3}\frac{du}{d\theta} + \frac{2b}{h^2}\int\frac{Td\theta}{u^3} = 0.$$

By the last two Articles, we have

$$\frac{d^2u}{d\theta^2} + u - b\left(1 - \tfrac{3}{4}k^2 - \tfrac{1}{2}m^2\right) - \tfrac{3}{2}bm^2e\cos(\theta - \alpha)$$

$$- \tfrac{3}{4}bk^2\cos 2(\theta - \gamma) + 3bm^2\cos\left\{(2 - 2m)\theta - 2\beta\right\}$$

$$- \tfrac{15}{2}bm^2e\cos\left\{(1-2m)\theta - 2\beta + \alpha\right\} + \tfrac{3}{2}bm^2e'\cos(m\theta + \beta - \alpha') = 0.$$

Now this equation cannot be integrated, as it stands, according to the method mentioned in Art. 330; because the term $\frac{3}{2}bm^2e\cos(\theta - \alpha)$ would introduce an infinite coefficient into the expression of u since the coefficient of θ = unity. But this may be remedied by putting for $be\cos(\theta - \alpha)$ in the term $\frac{3}{2}bm^2e\cos(\theta - \alpha)$, which is of the third order, its first approximate value $u - b$: then the equation becomes

$$\frac{d^2u}{d\theta^2} + \left(1 - \tfrac{3}{2}m^2\right)(u - b) + \tfrac{1}{4}b\left(3k^2 + 2m^2\right) - \tfrac{3}{4}bk^2\cos 2(\theta - \gamma)$$

$$+ 3bm^2\cos\left\{(2-2m)\theta - 2\beta\right\} - \tfrac{15}{2}bm^2e\cos\left\{(1-2m)\theta - 2\beta + \alpha\right\}$$

$$+ \tfrac{3}{2}bm^2e'\cos(m\theta + \beta - \alpha') = 0.$$

Let $1 - \frac{3}{2}m^2 = c^2$; then if we neglect all coefficients of the second order, we have

$$\frac{d^2u}{d\theta^2} + c^2(u - b) = 0, \quad \therefore\ u = b\left\{1 + e\cos(c\theta - \alpha)\right\}.$$

Now although c differs from unity only by a quantity of the second order, yet $\cos(c\theta - \alpha)$ will differ very sensibly from $\cos(\theta - \alpha)$ after several revolutions of the Moon. Wherefore the peculiarity of the differential equation in u (mentioned in the last page) when we proceed to a second approximation teaches us, that the value of u in Art. 329 will cease to be a first approximate value after several revolutions of the Moon; the true first approximate value being $b\left\{1 + e\cos(c\theta - \alpha)\right\}$. We must therefore carefully retrace our steps, and replace θ by $c\theta$ in every place where θ is introduced in consequence of its depending immediately on the first approximate value of u. This may very easily be accomplished by putting $\alpha + (1-c)\theta$ instead of α in every place where it occurs: for α enters the equations solely in consequence of its dependence on θ and u by the equation $u = b\{1 + \cos(\theta - \alpha)\}$.

The same will be the case with the value of s, as we shall shew in the next Proposition. We shall write down the corrected equations of u and s in Art. 336.

PROP. *To form the differential equation for* s.

335. By Art. 326, equation (4),

$$\frac{d^2 s}{d\theta^2} + s + \frac{\dfrac{S - Ps}{u^3} + \dfrac{T}{u^3}\dfrac{ds}{d\theta}}{h^2 + 2\displaystyle\int \frac{T\,d\theta}{u^3}} = 0.$$

Now $\dfrac{S - Ps}{h^2 u^3} = \dfrac{3m's}{2u^4 h^2 r'^3}\{1 + \cos 2(\theta - \theta')\}$

$$= \frac{3m'kb'^3}{2h^2 b^4}\sin(\theta - \gamma)\{1 + \cos[2(1-m)\theta - 2\beta]\}$$

$$= \tfrac{3}{2}m^2 k\{\sin(\theta - \gamma) - \tfrac{1}{2}\sin[(1-2m)\theta - 2\beta + \gamma]\},$$

retaining those terms of the third order which have the multiplier of θ nearly $= 1$.

Again, $\dfrac{ds}{d\theta} = k\cos(\theta - \gamma)$;

$$\therefore \frac{T}{h^2 u^3}\frac{ds}{d\theta} = -\frac{3m'k}{2u^4 h^2 r'^3}\cos(\theta - \gamma)\sin 2(\theta - \theta')$$

$$= -\tfrac{3}{2}m^2 k\cos(\theta - \gamma)\sin\{2(1-m)\theta - 2\beta\}$$

$$= -\tfrac{3}{4}m^2 k\sin\{(1-2m)\theta - 2\beta + \gamma\}.$$

Then the equation in s becomes

$$\frac{d^2 s}{d\theta^2} + s + \tfrac{3}{2}m^2 k\{\sin(\theta - \gamma) - \sin[(1-2m)\theta - 2\beta + \gamma]\} = 0.$$

This (as in the case of the equation in u) cannot be integrated by the method explained in Art. 330, because the term $\frac{3}{2}m^2k\sin(\theta-\gamma)$ would introduce an infinite coefficient into the value of s. But by putting for $k\sin(\theta-\gamma)$ its first approximate value s in the term $\frac{3}{2}m^2k\sin(\theta-\gamma)$, which is of the third order, this difficulty is overcome; the differential equation then becomes

$$\frac{d^2s}{d\theta^2}+(1+\tfrac{3}{2}m^2)\,s-\tfrac{3}{2}\,m^2k\sin\{(1-2m)\,\theta-2\beta+\gamma\}=0.$$

Let $1+\frac{3}{2}m^2=g^2$; then if we neglect the coefficient of the second order, we have

$$\frac{d^2s}{d\theta^2}+g^2s=0;\ \therefore\ s=k\sin(g\theta-\gamma).$$

Hence (as in the last Article) although g differs from unity only by a quantity of the second order, yet $\sin(g\theta-\gamma)$ will differ sensibly from $\sin(\theta-\gamma)$ after several revolutions of the Moon. Therefore $k\sin(\theta-\gamma)$ ceases to be a first approximation of s after several revolutions of the Moon: and we must retrace our steps and put $g\theta$ for θ in every place where θ enters in consequence of its immediately depending on s. This may be done by putting $\gamma+(1-g)\,\theta$ for γ in every place where γ occurs.

PROP. *To integrate the differential equations in* u *and* s.

336. After replacing θ by $c\theta$ and $g\theta$ in all such places as θ enters the equations in consequence of its immediate dependence on u and s respectively, the equations of Arts. 334, 335, become

$$\frac{d^2u}{d\theta^2}+c^2(u-b)+\tfrac{1}{4}b\,(3k^2+2m^2)-\tfrac{3}{4}\,bk^2\cos 2\,(g\theta-\gamma)$$
$$+\,3bm^2\cos\{(2-2m)\,\theta-2\beta\}-\tfrac{15}{2}\,bm^2e\cos\{(2-2m-c)\,\theta-2\beta+a\}$$
$$+\,\tfrac{3}{2}\,bm^2e'\cos(m\theta+\beta-a')=0;$$

and $$\frac{d^2s}{d\theta^2}+g^2s-\tfrac{3}{2}\,m^2k\sin\{(2-2m-g)\,\theta-2\beta+\gamma\}=0.$$

To integrate the first assume $u =$

$$b\{A + e\cos(c\theta - a) + B\cos 2(g\theta - \gamma) + C\cos[(2-2m)\theta - 2\beta] + D\cos[(2-2m-c)\theta - 2\beta + a] + E\cos(m\theta + \beta - a')\},$$

A, B, C, D, E being indeterminate coefficients: to find these substitute this value of u in the differential equation, and equate the coefficients to zero: then, remembering that $c^2 = 1 - \frac{3}{2}m^2$ and $g^2 = 1 + \frac{3}{2}m^2$, and neglecting small quantities of orders higher than the second, we have

$$c^2 A = c^2 - \tfrac{3}{4}k^2 - \tfrac{1}{2}m^2, \quad \therefore\ A = 1 - \tfrac{3}{4}k^2 - \tfrac{1}{2}m^2$$

$$(c^2 - 4g^2)B = \tfrac{3}{4}k^2, \quad \therefore\ B = -\tfrac{1}{4}k^2$$

$$\{c^2 - (2-2m)^2\}C = -3m^2, \quad \therefore\ C = m^2$$

$$\{c^2 - (2-2m-c)^2\}D = \tfrac{15}{2}m^2 e, \quad \therefore\ D = \tfrac{15}{8}me$$

$$(c^2 - m^2)E = -\tfrac{3}{2}m^2 e', \quad \therefore\ E = -\tfrac{3}{2}m^2 e'.$$

Hence $u = b\{1 - \frac{3}{4}k^2 - \frac{1}{2}m^2 + e\cos(c\theta - a) - \frac{1}{4}k^2\cos 2(g\theta - \gamma)$

$$+ m^2\cos[(2-2m)\theta - 2\beta] + \tfrac{15}{8}me\cos[(2-2m-c)\theta - 2\beta + a] - \tfrac{3}{2}m^2 e'\cos(m\theta + \beta - a)\}.$$

337. Again, to integrate the equation in s assume

$$s = k\sin(g\theta - \gamma) + F\sin\{(2-2m-g)\theta - 2\beta + \gamma\},$$

F being an indeterminate coefficient. Substitute this in the differential equation, and we have

$$\{g^2 - (2-2m-g)^2\}F = \tfrac{3}{2}m^2 k, \quad \therefore\ F = \tfrac{3}{8}mk;$$

$$\therefore\ s = k\sin(g\theta - \gamma) + \tfrac{3}{8}mk\sin\{(2-2m-g)\theta - 2\beta + \gamma\}.$$

We shall now make use of these values of u and s to calculate the distance and longitude of the Moon.

PROP. *To find the distance of the Moon from the Earth.*

338. Let r be this distance; $\therefore\ r = \rho\sqrt{1 + s^2}$;

$\therefore\ \frac{1}{r} = u(1 - \frac{1}{2}s^2)$, neglecting quantities of the fourth order,

$$= u \left\{1 - \tfrac{1}{4}k^2 + \tfrac{1}{4}k^2 \cos 2(g\theta - \gamma)\right\}, \text{ (Art. 337.)}$$

$$= b \left\{1 - k^2 - \tfrac{1}{2}m^2 + e\cos(c\theta - a) + m^2 \cos[(2-2m)\theta - 2\beta]\right.$$

$$+ \tfrac{15}{8} me \cos[(2 - 2m - c)\theta - 2\beta + a]$$

$$\left. - \tfrac{3}{2} m^2 e' \cos(m\theta + \beta - a')\right\}. \text{ (Art. 336.)}$$

$$= \mathrm{b} \left\{1 + e\cos(c\theta - a) + m^2 \cos[(2-2m)\theta - 2\beta]\right.$$

$$\left. + \tfrac{15}{8} me \cos[(2-2m-c)\theta - 2\beta + a] - \tfrac{3}{2} m^2 e' \cos(m\theta + \beta - a')\right\},$$

where $\mathrm{b} = b(1 - k^2 - \frac{1}{2}m^2)$.

Prop. *To find the longitude of the Moon in terms of the time.*

339. By Art. 325. equation (2) we have

$$\frac{dt}{d\theta} = \frac{1}{hu^2}\left\{1 + 2\int \frac{Td\theta}{h^2u^3}\right\}^{-\frac{1}{2}} = \frac{1}{hu^2}\left\{1 - \int \frac{Td\theta}{h^2u^3}\right\}.$$

Then substituting for $\int \frac{Td\theta}{h^2u^3}$ and u by Arts. 332, 336, and retaining those terms of the third order in which the coefficient of θ is small (Art. 330),

$$\frac{dt}{d\theta} = \frac{1}{hb^2}\left\{1 + \tfrac{3}{2}k^2 + m^2 + \tfrac{3}{2}e^2 - 2e\cos(c\theta - a)\right.$$

$$+ \tfrac{3}{2}e^2 \cos 2(c\theta - a) + \tfrac{1}{2}k^2 \cos 2(g\theta - \gamma)$$

$$- \tfrac{11}{4}m^2 \cos[(2-2m)\theta - 2\beta] - \tfrac{15}{4} me \cos[(2-2m-c)\theta - 2\beta + a]$$

$$\left. + 3m^2 e' \cos(m\theta + \beta - a')\right\}.$$

Then putting $hb^2(1 - \frac{3}{2}k^2 - m^2 - \frac{3}{2}e^2) = n$, multiplying by n and integrating,

$$nt = \theta - 2e\sin(c\theta - a) + \tfrac{3}{4}e^2 \sin 2(c\theta - a) + \tfrac{1}{4}k^2 \sin 2(g\theta - \gamma)$$

$$- \tfrac{11}{8}m^2 \sin\{(2-2m)\theta - 2\beta\} - \tfrac{15}{4} me \sin\{(2-2m-c)\theta - 2\beta + a\}$$

$$+ 3me' \sin(m\theta + \beta - a').$$

To obtain θ in terms of t we proceed as follows. Transpose all the terms but θ to one side of the equation. If all small quantities are neglected $\theta = nt$; then for a *first approximation* we neglect small quantities of the second order, and put $\theta = nt$ in the small terms;

$$\therefore \theta = nt + 2e \sin (cnt - \alpha).$$

For a *second approximation* we put this value of θ in small terms and neglect small quantities of the third order;

$$\therefore \theta = nt + 2e \sin (cnt - \alpha) + \tfrac{5}{4} e^2 \sin 2 (cnt - \alpha) - \tfrac{1}{4} k^2 \sin 2 (gnt - \gamma)$$

$$+ \tfrac{11}{8} m^2 \sin \{(2-2m) nt - 2\beta\} + \tfrac{15}{4} me \sin \{(2 - 2m - c) nt - 2\beta + \alpha\}$$

$$- 3me' \sin (mnt + \beta - \alpha').$$

340. These expressions for the radius vector and the longitude of the Moon shew, that her distance from the Earth preserves nearly a constant value, fluctuating between very small limits: and that her longitude varies nearly as the time of motion, departing from this law only by small quantities.

It will be an interesting enquiry to examine these formulæ for the radius vector and the longitude, and see whether they will enable us to explain the various inequalities that observations have pointed out in the motion of the Moon. The principle of the superposition of small motions (Art. 288.) allows us to examine the cause of each small term upon the supposition that all the other small terms do not exist.

PROP. *To interpret the physical meaning of the various terms in the analytical expressions for the radius vector and the longitude of the Moon.*

341. The first variable term of the reciprocal of the radius vector, or $be \cos (c\theta - \alpha)$, may be thus interpreted.

If $c = 1$ this term would be the ordinary variation from circular motion when a body moves in an ellipse: but c does not$=1$. Let E be the focus and aEA' the axis major (fig. 83.) of the ellipse of which the equation is $\frac{1}{r} = b \left\{1 + e \cos \left(\theta - \frac{\alpha}{c}\right)\right\}$:

then $\theta - \frac{\alpha}{c}$ is measured from the line EA': A' is a point in the

Moon's orbit; let $A'M$ be her orbit; and let $\angle A'EM = \theta - \frac{a}{c}$: also let $\angle A'EM' = c \angle A'EM$, and let EM' cut the ellipse above mentioned in M': then

$$\frac{1}{EM} = \text{b}\left\{1 + e\cos c \angle A'EM\right\} = \text{b}\left\{1 + e\cos A'EM'\right\} = \frac{1}{EM'};$$

$$\therefore\ EM = EM'.$$

Draw EA equal to EA' making an angle equal to $\angle MEM'$ with EA': then through the variable points A and M an ellipse can always be drawn having its focus in E and equal in dimensions to the ellipse on aA'. Hence this inequality shews, that, if we neglect all the other terms, the Moon's motion may be represented by supposing that it moves in an ellipse, the perigee of which revolves about the Earth with an angular velocity = $(1-c)\frac{d\theta}{dt} = \frac{3m^2}{4}\frac{d\theta}{dt}$ nearly. *Principia*, Lib. I. Prop. 66. Cor. 7.

The two terms of the longitude

$$2e\sin(cnt - a) + \tfrac{5}{4}e^2\sin 2(cnt - a)$$

correspond to the above term in the reciprocal of the radius vector; as may be seen by comparing the form of the terms with those in the expansion of $\theta - \varpi$ in Art. 277.

The second term in the reciprocal of the radius vector is

$$\text{b}m^2\cos\left\{(2 - 2m)\theta - 2\beta\right\}.$$

Now $(1-m)\theta - \beta = \theta - (m\theta + \beta) =$ long^e. of Moon − long^e. of Sun

= angular distance of Moon from the Sun.

Hence this inequality has its greatest positive value when the Moon is in syzygies, and its greatest negative value when the Moon is in quadratures. This agrees with the *Principia*, Prop. 66. Cor. 5. and also with Art. 303.

The term $\frac{11}{8}m^2\sin\left\{(2-2m)nt - 2\beta\right\}$ in the longitude corresponds to the above term: and is the inequality called the *Variation* discovered by Tycho Brahe (Art. 291, 305).

The third term in the reciprocal of the radius vector is

$$\tfrac{15}{8}\text{b}me\cos\left\{(2 - 2m - c)\theta - 2\beta + a\right\}.$$

Since $2 - 2m - c$ nearly equals unity it will be seen that this term is nearly analogous to the first term, though of much less importance because of the smallness of its coefficient. We shall take it in conjunction with that term (see *Airy's Tracts*, *Lunar Theory*), neglecting the motion of the perigee and other small quantities.

$$\text{Then } \frac{1}{r} = \text{b}\left\{1 + e\cos(\theta - \alpha) + \tfrac{15}{8}me\cos(\theta - 2\beta + \alpha)\right\},$$

neglecting the other terms

$$= \text{b}\left\{1 + e\cos(\theta - \alpha) + \tfrac{15}{8}me\cos[\theta - \alpha + 2(\alpha - \beta)]\right\}$$

$$= \text{b}\left\{1 + [e + \tfrac{15}{8}me\cos 2(\alpha - \beta)]\cos(\theta - \alpha)\right.$$

$$\left. - \tfrac{15}{8}me\sin 2(\alpha - \beta)\sin(\theta - \alpha)\right\}$$

$$= \text{b}\left\{1 + e[1 + \tfrac{15}{8}m\cos 2(\alpha - \beta)]\cos[\theta - \alpha + \tfrac{15}{8}m\sin 2(\alpha - \beta)]\right\},$$

as will easily be seen upon expanding this latter expression and neglecting small quantities of the third order.

Hence the effect of this third term in the reciprocal of the Moon's distance is to increase the eccentricity of the elliptic orbit by $\frac{15}{8}me\cos 2(\alpha - \beta)$; and to diminish the longitude of the perigee by $\frac{15}{8}m\sin 2(\alpha - \beta)$.

If we suppose the Sun to be stationary during one revolution of the Moon, β = longitude of the Sun: therefore

$$\text{eccentricity} = e\left\{1 + \tfrac{15}{8}m\cos 2(\text{long. perigee} - \text{long. Sun})\right\}$$

$$\text{long. perigee (corrected)} = \alpha - \tfrac{15}{8}m\sin 2(\text{long. perigee} - \text{long. Sun}).$$

The term $\frac{15}{4}me\sin\{(2 - 2m - c)nt - 2\beta + \alpha\}$ in the longitude exactly corresponds with the term above. It is called the *Evection*, and was discovered by Ptolemy (Art. 290). When the perigee is in syzygies, then $\alpha - \beta = 0$ or π, and the eccentricity is increased by $\frac{15}{8}me$: and when the perigee is in quadratures the eccentricity is diminished by that quantity: *Principia*, Lib. I. Prop. 66. Cor. 9. and Art. 314.

The last term in the reciprocal of the radius vector is

$$-\tfrac{3}{2}\text{b}\,m^2e'\cos(m\theta + \beta - \alpha').$$

This is of the third order: but the corresponding term in the longitude, *viz.* $-3me' \sin(mnt+\beta-\alpha')$, is of the second order. This inequality in the longitude depends upon the Sun's mean anomaly: when the Sun is in perigee and apogee then $(mnt+\beta)-\alpha'=0$ and π, and this term vanishes: when the Sun is moving from perigee to apogee the term is negative, and positive as the Sun moves from apogee to perigee: hence the Moon is behind or before her mean place (in consequence of this inequality) according as the Sun is moving from perigee to apogee or from apogee to perigee. This is the ***Annual Equation***: (Art. 292, 300). Also see *Principia*, Lib. I. Prop. 66. Cor. 6. and Lib. III. *Scholium to Lunar Theory*.

There is another term $-\frac{1}{4}k^2 \sin(2gnt-2\gamma)$ in the longitude. This depends upon the Moon's distance from the mean place of her node, and nearly equals the difference between her longitudes measured on the ecliptic and her orbit: hence it is called the ***Reduction***.

PROP. *To explain the physical meaning of the terms in the analytical expression for the inclination of the Moon's orbit to the ecliptic.*

342. The first term is $k\sin(g\theta-\gamma)$.

Let N be the ascending node when $\theta-\frac{\gamma}{g}=0$, fig. 93. Take $\angle NEM'=\theta-\frac{\gamma}{g}$: and $\angle M'En=g\,.\,\angle M'EN$: also let M be the Moon, $\tan MEM'=s$: then n is the node, moving backwards. For in the right-angled triangle $MM'n$, we have

$$\sin M'n=\tan MM'\cot MnM',\quad \tan MnM'=\tan AB=k;$$

$$\therefore\ s=\tan MM'=k\sin g\ \angle M'EN=k\sin(g\theta-\gamma).$$

Hence the meaning of this term is that the node regresses with an angular velocity $=(g-1)\frac{d\theta}{dt}=\frac{3m^2}{4}\frac{d\theta}{dt}$.

The second term $\frac{3}{8}mk\sin\{(2-2m-g)\theta-2\beta+\gamma\}$ is best considered in connexion with the first, as we did the Evection: (Airy's *Tracts*).

Neglecting the motion of the Node

$$s=k\left\{\sin(\theta-\gamma)+\tfrac{3}{8}m\sin(\theta-\gamma+2\gamma-2\beta)\right\}$$

$$=k\left\{1+\tfrac{3}{8}m\cos 2(\gamma-\beta)\right\}\sin(\theta-\gamma)+\tfrac{3}{8}mk\sin 2(\gamma-\beta)\cos(\theta-\gamma)$$

$$=k\left\{1+\tfrac{3}{8}m\cos 2(\gamma-\beta)\right\}\sin\left\{\theta-\gamma+\tfrac{3}{8}m\sin 2(\gamma-\beta)\right\}.$$

Hence the effect of the second term in s is to increase the tangent of inclination of the lunar orbit by $\frac{3}{8}mk\cos 2(\gamma-\beta)$ or $\frac{3}{8}mk\cos 2$ (long. node − long. Sun), and to diminish the longitude of the node, calculated on the supposition of its uniformly regressing, by the angle $\frac{3}{8}m\sin 2(\gamma-\beta)$ or $\frac{3}{8}m\sin 2$ (long. node − long. Sun). *Principia*, Lib. III. Props. 33. and 35.

The inclination of the orbit is greatest when the node is in syzygies, and least when in quadratures: see Art. 321, and *Principia*, Lib. I. Prop. 66. Cor. 10.

343. The angle described by the perigee during a revolution of the Moon, as calculated in Art. 341, equals $\frac{3}{4}m^2 . 2\pi = \frac{3}{2}m^2\pi = 1^0. 30'$ nearly: but its true value as proved by observation is about twice this. This apparent discrepancy between theory and observation shook Clairaut's belief in Newton's law of gravitation, and induced him to propose a new and more complicated law; pamphlets were already printed and about to be circulated by Clairaut, when he discovered that by extending the approximation the value of c is

$$1-\tfrac{3}{4}m^2-\tfrac{225}{32}m^3,$$

the third term of which, owing to the largeness of the coefficient, nearly equals the second term: and therefore reconciles the apparent difference.

344. The value of g is $1+\frac{3}{4}m^2-\frac{9}{32}m^3$, and therefore the ratio of the motion of the perigee to that of the node

$$=\left(\tfrac{3}{4}m^2+\tfrac{225}{32}m^3\right)\div\left(\tfrac{3}{4}m^2-\tfrac{9}{32}m^3\right)=\left(1+\tfrac{75}{8}m\right)\left(1+\tfrac{3}{8}m\right)=2 \text{ nearly}.$$

This ratio is much larger than for one of Jupiter's satellites, because for that system m is very small indeed. *Principia*, Lib. III. Prop. 23.

345. If m_1 be the ratio of the mean motion of Jupiter to that of one of his satellites; then the progression of the perijove and regression of the node during a revolution of the satellite each $= \frac{3}{2}\pi m_1^2$. Hence the regression of the node of this satellite *during a given time* equals the regression of the Moon's node $\times$ ($m_1^2 \div$ per. time of satellite) $\div$ ($m^2 \div$ per. time of Moon)

$$= \left(\frac{\text{mean motion of Jupiter}}{\text{mean motion of Earth}}\right)^2 \left(\frac{\text{mean motion of Moon}}{\text{mean motion of satellite}}\right) \text{regress}^{\text{n}}\text{. of Moon's node.}$$

The same formula is true for the satellites of Saturn.

The progression of the perijove $= \frac{3}{2}\pi m_1^2$, and that of the Moon nearly $= 3\pi m^2$ (Art. 343): hence the progression of the perijove *during a given time*

$$= \tfrac{1}{2}\left(\frac{\text{mean motion of Jupiter}}{\text{mean motion of Earth}}\right)^2 \left(\frac{\text{mean motion of Moon}}{\text{mean motion of satellite}}\right) \text{regress}^{\text{n}}\text{. of Moon's perigee.}$$

The same is true for the satellites of Saturn.

If the series for c were more converging (Art. 343), then the $\frac{1}{2}$ which multiplies this expression would be 1. *Principia*, Lib. III. Prop. 23. Newton omits the $\frac{1}{2}$ and says "diminui tamen debet motus augis sic inventus in ratione 9 ad 5 vel 2 ad 1 circiter, ob causam hîc exponere non vacat." So it seems that Newton had some way of accounting for this apparent anomaly.

The reader that wishes to enter more deeply into the calculation of the lunar inequalities must consult a memoir by Baron Damoiseau in the *Mémoires présentés par divers savans à l'Académie Royale des Sciences;* Tom. I: the *Lunar Theory* of Messrs. Plana and Carlini; and that of Mr Lubbock. In these works the approximation is carried so far as to enable us to deduce all the inequalities from theory alone.

346. In this Chapter we have given the inequalities of the first and second order: those of the third order are fifteen in number, these and some of the inequalities of the fourth and higher orders will be found in the *Méc. Cél.* Liv. VII. We shall mention some of the more interesting results.

Among the periodical inequalities of the Moon's motion in longitude, that which depends on the simple angular distance of the Sun and Moon is important on account of the great light it throws upon the Sun's parallax. The parallax is found to be 8.56 seconds, being the same as several astronomers have found from the last transit but one of Venus over the Sun: *Méc. Cél.* Liv. VIII. § 24.

An inequality, which is not less important, is that which depends upon the longitude of the Moon's node: as it did not appear to depend on the theory of gravity, it was neglected by most astronomers; till a more thorough examination led Laplace to discover that its cause is the oblateness of the Earth: it gives an oblateness $= \frac{1}{305.05}$: *Méc. Cél.* Liv. VII. § 24.

There is also an inequality in the Moon's latitude, which Laplace discovered by theory: he shewed that it arises from the oblateness of the Earth's figure: it gives the oblateness $= \frac{1}{304.6}$: *Méc. Cél.* Liv. VII. § 25.

These two inequalities prove that the Moon's gravity to the Earth arises from the attraction of all the particles of the Earth, and not of the centre alone. (Art. 260.)

By examining the records of ancient eclipses of the Moon it was found that the Moon's mean motion was continually accelerated. The cause of this was long sought for in vain; till Laplace discovered by theory that it depends upon the variation (the secular variation, see Art. 377.) of the eccentricity of the Earth's orbit. All the observations which have been made during the last century and a half, have put beyond a doubt this result of analysis. When the acceleration of the Moon's mean motion was known, but not accounted for, conjectures were started as to its depending on the resistance of a medium, or the transmission of gravity; but analysis shews that neither of these causes produces any sensible alteration. *Méc. Cél.* Liv. VII. § 23.

PROP. *To prove that the centre of gravity of the Earth and Moon very nearly describes an ellipse about the Sun.*

347. Let $x' y' z'$ be the co-or. of the Earth from the Sun,

$x_1 y_1 z_1$ .... Moon from the Sun,

$x\ y\ z$ .... Moon from the Earth,

$\bar{x}\ \bar{y}\ \bar{z}$ .... centre of grav. of Earth and Moon from the Sun.

m', E, M the masses of the Sun, Earth, and Moon.

$r' r_1$ the distances of the Earth and Moon from the Sun.

r the distance of the Moon from the Earth.

Then $x_1 - x' = x, \quad y_1 - y' = y, \quad z_1 - z' = z.$

The ratio of E to m' equals 1 : 354936 and may therefore be neglected.

The equations of motion of the Earth about the Sun, the Moon being the disturbing body, are

$$\left.\begin{aligned} \frac{d^2x'}{dt^2} + \frac{m'x'}{r'^3} + \frac{dR'}{dx'} = 0 \\ \frac{d^2y'}{dt^2} + \frac{m'y'}{r'^3} + \frac{dR'}{dy'} = 0 \\ \frac{d^2z'}{dt^2} + \frac{m'z'}{r'^3} + \frac{dR'}{dz'} = 0 \end{aligned}\right\} \dots\dots\dots\dots\dots (1);$$

$$R' = \frac{M(x'x_1 + y'y_1 + z'z_1)}{\{x_1^2 + y_1^2 + z_1^2\}^{\frac{3}{2}}} - \frac{M}{\sqrt{(x_1 - x')^2 + (y_1 - y')^2 + (z_1 - z')^2}}.$$

The equations of motion of the Moon about the Earth, the Sun being the disturbing body, are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} + \frac{(E+M)\,x}{r^3} + \frac{dR}{dx} = 0 \\ \frac{d^2y}{dt^2} + \frac{(E+M)\,y}{r^3} + \frac{dR}{dy} = 0 \\ \frac{d^2z}{dt^2} + \frac{(E+M)\,z}{r^3} + \frac{dR}{dz} = 0 \end{aligned}\right\} \dots\dots\dots (2);$$

$$R = -\frac{m'(x'x + y'y + z'z)}{\{x'^2 + y'^2 + z'^2\}^{\frac{3}{2}}} - \frac{m'}{\sqrt{(x' + x)^2 + (y' + y)^2 + (z' + z)^2}};$$

since $-x'$, $-y'$, $= z'$ are the co-ordinates of the Sun from the Earth.

Now $\bar{x} = \dfrac{Ex' + Mx_1}{E+M}, = x' + \dfrac{Mx}{E+M}$, also $= x_1 - \dfrac{Ex}{E+M}$;

$$\therefore \frac{d^2\bar{x}}{dt^2} = \frac{d^2x'}{dt^2} + \frac{M}{E+M}\frac{d^2x}{dt^2}, \text{ by equations (1) (2),}$$

$$= -\frac{m'x'}{r'^3} - \frac{Mx}{r^3} - \frac{dR'}{dx'} - \frac{M}{E+M}\frac{dR}{dx}$$

$$= -\frac{m'x'}{r'^3} - \frac{Mx}{r^3} - \frac{Mx_1}{r_1^{\,3}} + \frac{M(x_1 - x')}{r^3} + \frac{M}{E+M}\left\{\frac{m'x'}{r'^3} - \frac{m'(x+x')}{r_1^{\,3}}\right\}$$

$$= -\frac{E}{E+M}\frac{m'x'}{r'^3} - \frac{M}{E+M}\frac{m'x_1}{r_1^{\,3}}, \text{ neglecting } \frac{Mx_1}{r_1^{\,3}},$$

substituting for x' and x_1 in terms of x and $\bar{x}$,

$$= -\frac{m'}{r'^3}\frac{E}{E+M}\left(\bar{x} - \frac{Mx}{E+M}\right) - \frac{m'}{r_1^{\,3}}\frac{M}{E+M}\left(\bar{x} + \frac{Ex}{E+M}\right).$$

Now $\dfrac{1}{r'^3} = \left\{\left(\bar{x} - \dfrac{Mx}{E+M}\right)^2 + \left(\bar{y} - \dfrac{My}{E+M}\right)^2 + \left(\bar{z} - \dfrac{Mz}{E+M}\right)^2\right\}^{-\frac{3}{2}}$

$$= \frac{1}{\bar{r}^3}\left\{1 + \frac{3M}{E+M}\frac{x\bar{x} + y\bar{y} + z\bar{z}}{\bar{r}^2} + \dots\dots\right\}.$$

Also $\dfrac{1}{r_1^{\,3}} = \left\{\left(\bar{x} + \dfrac{Ex}{E+M}\right)^2 + \left(\bar{y} + \dfrac{Ey}{E+M}\right)^2 + \left(\bar{z} + \dfrac{Ez}{E+M}\right)^2\right\}^{-\frac{3}{2}}.$

$$= \frac{1}{\bar{r}^3}\left\{1 - \frac{3E}{E+M}\frac{x\bar{x} + y\bar{y} + z\bar{z}}{\bar{r}^2} + \dots\dots\right\};$$

$$\therefore \frac{d^2\bar{x}}{dt^2} = -\frac{m'\bar{x}}{\bar{r}^3}$$

$$-\frac{3EMm'}{(E+M)^2}\frac{x\bar{x} + y\bar{y} + z\bar{z}}{\bar{r}^5}\left\{\bar{x} - \frac{Mx}{E+M} - \bar{x} - \frac{Ex}{E+M}\right\} + \dots$$

$$= -\frac{m'\bar{x}}{\bar{r}^3} + \frac{3EMm'}{(E+M)^2}\frac{x^2\bar{x} + xy\bar{y} + xz\bar{z}}{\bar{r}^5} + \dots$$

$= -\frac{m'\bar{x}}{\bar{r}^3}$ + terms multiplied by the products and powers of $\frac{x}{r}$, $\frac{y}{r}$ and $\frac{z}{r}$ higher than the first.

Now $\frac{r}{\bar{r}} = \frac{1}{400}$ nearly, and x, y, z cannot be greater than r: hence if we neglect small quantities of the fourth order, we have

$$\left.\begin{aligned} \frac{d^2\bar{x}}{dt^2} + \frac{m'\bar{x}}{\bar{r}^3} = 0 \\ \text{and similarly} \quad \frac{d^2\bar{y}}{dt^2} + \frac{m'\bar{y}}{\bar{r}^3} = 0 \\ \frac{d^2\bar{z}}{dt^2} + \frac{m'\bar{z}}{\bar{r}^3} = 0 \end{aligned}\right\} .$$

These equations shew that the path of the centre of gravity is a conic section in one plane the Sun being in the focus, (Arts. 241, 246, 252); it evidently must be an ellipse.

348. Mr Airy, Astronomer Royal, has proposed a method for determining the mass of the Moon, which depends upon this Proposition. Since the centre of gravity of the Earth and Moon describes an ellipse about the Sun, it follows that the Earth does not describe an ellipse about the Sun: this deviation from elliptic motion depends upon the mass of the Moon, and can easily be calculated by theory: and thence can be determined the error in the Sun's right ascension and declination on the supposition of the orbit of the Earth being an ellipse. Now when Venus is near inferior conjunction she is only a third of the distance of the Earth from the Sun, and consequently the errors in her right ascension and declination will be much greater than in the Sun's. Some observations for this end will be found in the *Memoirs of the Royal Astronomical Society*, Vol. V. p. 223.

CHAPTER VI.

PLANETARY THEORY.

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349. We have already stated that the perturbations of the Moon are far larger than those of the planets, because the Sun, the mass of which is enormous and distance not proportionably great, is one of the disturbing bodies.

The perturbations of the planets, on the other hand, are very minute; and are not detected in short periods of time. These might, however, be calculated in the manner pursued in calculating the longitude, latitude, and radius vector of the Moon: but since the approximation is made by means of series which proceed by powers of the ratio of the distances of the disturbed and disturbing bodies from the central one, and since this fraction is much smaller in the Lunar Theory than in the Planetary Theory, it is necessary to retain many more terms in the calculation of the perturbations of the planets than in that of the perturbations of the Moon; and consequently the process is much slower in the former than in the latter calculation. For this reason R is expanded in powers of the eccentricities and inclinations of the orbits of the planets instead oi the ratio of the distances of the disturbed and disturbing bodies: and the calculation then conducted as in the last Chapter.

350. But we shall make use of an entirely different mode of calculation. It is to Lagrange that we are indebted for the method we are about to lay before our readers of calculating the Planetary Perturbations.

If at any instant the disturbing forces were to cease acting, the planet would move in an exact ellipse; and this ellipse and the actual orbit of the planet would manifestly have a common tangent, and the actual velocity of the planet and that calculated for the motion in this ellipse according to the elliptic theory would be the same. For this reason this ellipse is called the *ellipse of curvature* to the orbit at that instant: it is also denominated the *instantaneous ellipse* of the planet.*

* That this ellipse is an ellipse of curvature to the actual orbit, and that the contact is of the first order only may also be thus shewn.

The most general equations to an ellipse in space are the equations to a plane and to a surface of the second order. These contain *twelve* arbitrary constants. Now these constants are in our case, connected by the following relations.

1. The plane of the ellipse must pass through the Sun's centre. This gives *one* relation connecting the constants.

2. The focus of the ellipse must lie in the Sun's centre. This gives *three* more relations.

3. The co-ordinates of the planet at the instant under consideration must satisfy the equations of the ellipse.

This gives *two* more relations.

4. The velocities of the planet in the two orbits must be the same and their directions also at the instant under consideration: or, which comes to the same thing, the three velocities parallel to the axes of co-ordinates must be the same.

This gives *three* more relations.

5. The velocity in the ellipse must be equal to that which results from the theory of elliptic motion, viz: $2\mu\left(\frac{1}{r}-\frac{1}{2a}\right)$.

This gives *one* more relation.

These five considerations give *ten* relations among the constants of the equations to the ellipse. Hence these equations in our case involve only *two arbitrary constants*.

Now let xyz be the co-ordinates to the common point of the orbits: $x+\delta x$, $y+\delta y$, $z+\delta z$ the co-ordinates of a point near this in the path of the planet: and $x+\delta x'$, $y+\delta y'$, $z+\delta z'$ the co-ordinates of a point in the instantaneous orbit corresponding to the above: also let t become $t+\tau$:

$$\text{then } \delta x = \frac{dx}{dt}\tau + \frac{d^2x'}{dt^2}\frac{\tau^2}{2}+\ldots$$

$$\delta x' = \frac{dx'}{dt}\tau + \frac{d^2x'}{dt^2}\frac{\tau^2}{2}+\ldots$$

$$\therefore\ \delta x - \delta x' = \left(\frac{d^2x}{dt^2}-\frac{d^2x'}{dt^2}\right)\frac{\tau^2}{2}+\ldots\ldots \text{ by condition (4).}$$

$$\text{Similarly } \delta y - \delta y' = \left(\frac{d^2y}{dt^2}-\frac{d^2y'}{dt^2}\right)\frac{\tau^2}{2}+\ldots$$

$$\delta z - \delta z' = \left(\frac{d^2z}{dt^2}-\frac{d^2z'}{dt^2}\right)\frac{\tau^2}{2}+\ldots$$

Hence the distance between the new points in the curves

$$\sqrt{(\delta x-\delta x')^2+(\delta y-\delta y')^2+(\delta z-\delta z')^2}$$

$$=\left\{\left(\frac{d^2x'}{dt^2}-\frac{d^2x'}{dt^2}\right)^2+\left(\frac{d^2y}{dt^2}-\frac{d^2y'}{dt^2}\right)^2+\left(\frac{d^2z}{dt^2}-\frac{d^2z'}{dt^2}\right)^2\right\}^{\frac{1}{2}}\frac{\tau^2}{2}+\&c.\ \ldots$$

In

From what precedes it is evident that the motion of the planet may be represented by supposing it to move in an ellipse of which the elements are continually and slowly changing. If we know the elements of the instantaneous ellipse at any proposed instant, we have nothing to do but to calculate the position of the planet in this ellipse by the ordinary formulæ in Chap. III.

351. Since the perturbations of the planetary motions are very small, it follows from the Principle of the superposition of small motions, that the perturbations will be the sum of the perturbations produced by the several disturbing bodies considered separately: (Art. 288). We shall therefore in the following calculations consider only one disturbing body.

PROP. *To explain the process of integrating the equations of motion of the disturbed planet.*

352. The equations of motion of a disturbed planet are by Art. 323.

$$\frac{d^2x}{dt^2} + \frac{\mu x}{r^3} + \frac{dR}{dx} = 0,$$

$$\frac{d^2y}{dt^2} + \frac{\mu y}{r^3} + \frac{dR}{dy} = 0,$$

$$\frac{d^2z}{dt^2} + \frac{\mu z}{r^3} + \frac{dR}{dz} = 0,$$

where $R = \dfrac{m'(xx' + yy' + zz')}{\{x'^2 + y'^2 + z'^2\}^{\frac{3}{2}}} - \dfrac{m'}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$

μ = mass of Sun + mass of disturbed planet

m' = mass of the disturbing planet.

In order to make the ellipse have a contact nearer than that of the first order [which it has by condition (4)] we must make the first term of this expression $=0$: but this we cannot do since it requires *three* conditions to be satisfied and we have only *two disposeable constants*.

Hence the ellipse described in the text is an ellipse of contact to the real orbit, the contact being of the first order.

We shall first integrate these equations of motion omitting the disturbing forces: by this process we shall obtain six integrals of the first order, containing six arbitrary constants. These six constants must be determined in terms of the six elements of the planet's orbit (Art. 270.); the inclination, the longitude of the node, the mean distance, the eccentricity, the longitude of the perihelion, and the epoch. By eliminating the three quantities $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ from these six integrals we have the three final integrals of the equations of motion.

We shall then proceed to the integration of the equations of motion taking into consideration the disturbing forces.

The six integrals of the first order obtained on the supposition that there were no disturbing forces, contain the six arbitrary constants and also the quantities x, y, z, $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$. Now since the actual orbit and the instantaneous orbit touch each other in the point (xyz), and since the velocities in these two orbits at that point are the same (Art. 350), it follows that x, y, z, $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ are the same in the actual orbit and in the instantaneous orbit. Consequently we shall consider these six integrals to be still the integrals of the equations of motion when the disturbing forces are not neglected; with this difference, that now we must consider the six constants as variable quantities. To determine their values we must differentiate the six integrals with respect to t and eliminate $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$ by the equations of motion. In this way six equations will be obtained for calculating the six variable quantities.

We shall then use the equations which connect the elements of the instantaneous orbit with these six quantities (which are the six arbitrary constants when the disturbing forces are neglected), and in this manner obtain equations for calculating the variations which the elements of the instantaneous orbit undergo: so that if at any epoch these elements are known, they may be calculated for any other epoch. Then these elements being put in the series of Arts. 278, 280, we know the position of the planet at any given time.

We proceed now to the investigation of the several Propositions necessary for these results.

PROP. *To integrate the equations of motion of an undisturbed planet.*

353. These equations are

$$\left.\begin{aligned}\frac{d^2x}{dt^2}+\frac{\mu x}{r^3}=0\\ \frac{d^2y}{dt^2}+\frac{\mu y}{r^3}=0\\ \frac{d^2z}{dt^2}+\frac{\mu z}{r^3}=0\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots (1),$$

where $\mu = M + m$ = mass of Sun + mass of planet.

Multiply the first by y and the second by x and take their difference; then

$$x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}=0;$$

$$\left.\begin{aligned}\therefore\ x\frac{dy}{dt}-y\frac{dx}{dt}=\text{const.}=h\\ \text{in like manner}\quad z\frac{dx}{dt}-x\frac{dz}{dt}=h_1\\ y\frac{dz}{dt}-z\frac{dy}{dt}=h_2\end{aligned}\right\}\ldots\ldots\ldots (2).$$

These are three of the first integrals and they contain the three arbitrary constants h, h_1, h_2.

Again, multiply the equations of motion by $2\frac{dx}{dt}$, $2\frac{dy}{dt}$, $2\frac{dz}{dt}$ respectively, add them, and integrate: then

$$\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}+\frac{dz^2}{dt^2}-\frac{2\mu}{r}=\text{const.}=c\ \ldots\ldots\ldots\ldots (3).$$

This is a fourth integral of the first order and contains the arbitrary constant c.

Again, multiply the first and second of the equations of motion by the second and third of (2) respectively: then by subtraction we have

$$h_1 \frac{d^2x}{dt^2} - h_2 \frac{d^2y}{dt^2} = -\frac{\mu x}{r^3}\left(z\frac{dx}{dt} - x\frac{dz}{dt}\right) + \frac{\mu y}{r^3}\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right)$$

$$= \frac{\mu}{r^3}(x^2+y^2+z^2)\frac{dz}{dt} - \frac{\mu z}{r^3}\left(x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}\right)$$

$$= \frac{\mu}{r}\frac{dz}{dt} - \frac{\mu z}{r^2}\frac{dr}{dt} = \mu\frac{d\left(\frac{z}{r}\right)}{dt};$$

$$\left.\begin{aligned} \therefore\ h_1\frac{dx}{dt} - h_2\frac{dy}{dt} &= \frac{\mu z}{r} + f \\ \text{so also}\ h_2\frac{dz}{dt} - h\frac{dx}{dt} &= \frac{\mu y}{r} + f_1 \\ h\frac{dy}{dt} - h_1\frac{dz}{dt} &= \frac{\mu x}{r} + f_2 \end{aligned}\right\} \text{..............} (4).$$

Thus we have three more integrals of the first order, containing the arbitrary constants f, f_1, f_2.

It would appear, then, that we have seven, and not only six, integrals of the first order: but we can shew that any one of these seven is a consequence of the other six: and the constants h, h_1, h_2, f, f_1, f_2 are connected by an equation.

For by the last of equations (2) and (4) we have

$$h_2 f_2 = h h_2 \frac{dy}{dt} - h_1 h_2 \frac{dz}{dt} - \frac{\mu x}{r} h_2$$

$$= h\left(h_2\frac{dy}{dt} - h_1\frac{dx}{dt}\right) + h_1\left(h\frac{dx}{dt} - h_2\frac{dz}{dt}\right) - \frac{\mu x}{r} h_2$$

$$= -hf - h_1 f_1 - \frac{\mu}{r}(xh_2 + yh_1 + zh) \text{ by equations (4)}$$

$$= -hf - h_1 f_1 \text{ by equations (2)}$$

$$\therefore\ hf + h_1 f_1 + h_2 f_2 = 0 \ldots\ldots\ldots\ldots (5)$$

or the arbitrary constants have a necessary relation, and therefore the seven integrals found above are not independent integrals.

And moreover, since the seven integrals contained in (2) (3) (4) do not involve the time t *explicitly* it would appear that by eliminating $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ we should obtain three final integrals functions of xyz without t: but this evidently cannot be the case. It follows, then, that the seven integrals must be equivalent to only five independent integrals: and the constants h, h_1, h_2, c, f, f_1, f_2 are connected by another relation. This relation is found as follows. Add the squares of equations (4);

$$\therefore\ f^2 + f_1^2 + f_2^2 + \frac{2\mu}{r}(fz + f_1 y + f_2 x) + \mu^2$$

$$= (h^2 + h_1^2 + h_2^2)\left\{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2}\right\} - \left\{h\frac{dz}{dt} + h_1\frac{dy}{dt} + h_2\frac{dx}{dt}\right\}^2$$

$$= (h^2 + h_1^2 + h_2^2)\left(\frac{2\mu}{r} + c\right) \text{ by equations (3) and (2).}$$

But by equations (4) $fz + f_1 y + f_2 x + \mu r$

$$= h\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) + h_1\left(z\frac{dx}{dt} - x\frac{dz}{dt}\right) + h_2\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right)$$

$$= h^2 + h_1^2 + h^2;$$

$$\therefore\ f^2 + f_1^2 + f_2^2 = \mu^2 + (h^2 + h_1^2 + h_2^2)\,c \ldots\ldots (6),$$

this is the relation sought for.

We are unable to obtain a sixth integral of the first order by direct integration: and must therefore integrate the integrals already obtained to get a relation involving the time: this will be one of the final integrals.

To obtain the other two final integrals we must eliminate the differential coefficients $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ from the integrals of the first order: to effect this multiply the equations (2) respectively by z, y, x and add

$$\therefore\ hz + h_1 y + h_2 x = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7),$$

this proves that the undisturbed planet moves in a plane.

Again, multiply equations (4) by z, y, x respectively and add: then

$$fz + f_1 y + f_2 x + \mu\sqrt{x^2 + y^2 + z^2}$$

$$= h\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) + h_1\left(z\frac{dx}{dt} - x\frac{dz}{dt}\right) + h_2\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right)$$

$$= h^2 + h_1^2 + h_2^2 \text{ by (2)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

This is the equation to a surface of revolution of the second order, the origin of co-ordinates being in the focus: the equation to the plane generated by the directrix of the generating conic section being

$$fz' + f_1 y' + f_2 x' = h^2 + h_1^2 + h_2^2.$$

For the perpendicular from any point (xyz) of the surface on this plane

$$= \frac{fz + f_1 y + f_2 x - h^2 - h_1^2 - h_2^2}{\sqrt{f^2 + f_1^2 + f_2^2}} = -\frac{\mu r}{\sqrt{f^2 + f_1^2 + f_2^2}}.$$

Now r is the same for all points equally distant from the origin. Hence the surface must be one of revolution about an axis perpendicular to the plane of which the equation is

$$fz' + f_1 y' + f_2 x' = h^2 + h_1^2 + h_2^2.$$

Also the ratio of the perpendicular to the distance r is constant: and this is a property peculiar to the focus of conic sections. Hence the surface is a surface of the second order from the focus: and by combining this with the equation (7) we learn, that the planet moves in a conic section the Sun being in the focus: (Art. 252).

To obtain the third integral add the squares of equations (2): then

$$(x^2+y^2+z^2)\left\{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2}\right\} - \left\{x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}\right\}^2 = h^2 + h_1^2 + h_2^2,$$

$$\text{and } r^2 = x^2 + y^2 + z^2;$$

$$\therefore \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2} - \frac{dr^2}{dt^2} = \frac{h^2 + h_1^2 + h_2^2}{r^2} \ldots\ldots (9).$$

Let θ be the longitude of the planet at the time t measured on the plane of its orbit: then

$$\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2} = r^2\frac{d\theta^2}{dt^2} + \frac{dr^2}{dt^2};$$

$$\therefore \frac{dt}{d\theta} = \frac{r^2}{\sqrt{h^2 + h_1^2 + h_2^2}}.$$

In this we must substitute for r in terms of θ by means of the two other integrals: and in integrating we shall introduce the sixth independent arbitrary constant: this constant is called the *epoch*, since it depends upon the epoch of the planet's perihelion passage.

Having integrated the equations of motion for an undisturbed planet we proceed to the following Proposition.

PROP. *To calculate the elements of the orbit in terms of the arbitrary constants introduced by the integration.*

354. Let i be the inclination of the plane of the orbit to the plane of the ecliptic; the ecliptic we shall take to be the plane of xy,

Ω the longitude of the node, the axis of x being drawn through the first point of aries,

ϖ the longitude of the perihelion projected on the ecliptic,

$2a$ the axis-major of the orbit,

e the eccentricity,

ϵ the epoch.

The equation to the plane of the orbit is $zh + yh_1 + xh_2 = 0$:

$$\therefore \cos i = \frac{h}{\sqrt{h^2 + h_1^2 + h_2^2}}, \text{ and } \tan i = \sqrt{\frac{h_1^2 + h_2^2}{h^2}}.$$

By putting $z = 0$ in the equation to the plane of the orbit, we have $h_1 y + h_2 x = 0$, the equation to the line of nodes:

$$\therefore \tan \Omega = -\frac{h_2}{h_1}.$$

At the perihelion r is a minimum; $\therefore x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt} = 0$,

$$\text{also } \tan \varpi = \frac{y}{x} \text{ at that point:}$$

we must therefore find the value of this ratio at the perihelion: for this end we have

$$\frac{\mu y}{r} = h_2 \frac{dz}{dt} - h\frac{dx}{dt} - f_1 \text{ by (4) of last Article}$$

$$= y\left(\frac{dz^2}{dt^2} + \frac{dx^2}{dt^2}\right) - \frac{dy}{dt}\left(z\frac{dz}{dt} + x\frac{dx}{dt}\right) - f_1 \text{ by (2)}$$

$$= y\left(\frac{dz^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dx^2}{dt^2}\right) - f_1 \text{ at the perihelion;}$$

so also $\frac{\mu x}{r} = x\left(\frac{dz^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dx^2}{dt^2}\right) - f_2$ at the perihelion;

$$\therefore \tan \varpi = \frac{y}{x} \text{ at perihelion} = \frac{f_1}{f_2}.$$

At the extremities of the axis-major $\frac{dr}{dt} = 0$, and therefore equations (3) and (9) of last Article give

$$\frac{h^2 + h_1^2 + h_2^2}{r^2} = \frac{2\mu}{r} + c;$$

$$\therefore r = -\frac{\mu}{c} \pm \frac{\sqrt{\mu^2 + (h^2 + h_1^2 + h_2^2)c}}{c};$$

$$\therefore a = -\frac{\mu}{c},$$

$$\text{and } e = \frac{\sqrt{\mu^2 + (h^2 + h_1^2 + h_2^2)c}}{\mu} = \frac{\sqrt{f^2 + f_1^2 + f_2^2}}{\mu}$$

by equation (6) of last Article.

Lastly to find the *epoch* (ϵ) we must integrate the equation $\frac{dt}{d\theta} = \frac{r^2}{\sqrt{h^2 + h_1^2 + h_2^2}}$ after having substituted for r.

Having thus obtained the elements of the undisturbed orbit in terms of the constants we will proceed to shew the importance of these expressions in determining the perturbations of a disturbed planet.

PROP. *To integrate the equations of motion of a disturbed planet.*

355. The equations of motion are

$$\frac{d^2x}{dt^2} + \frac{\mu x}{r^3} + \frac{dR}{dx} = 0,$$

$$\frac{d^2y}{dt^2} + \frac{\mu y}{r^3} + \frac{dR}{dy} = 0,$$

$$\frac{d^2z}{dt^2} + \frac{\mu z}{r^3} + \frac{dR}{dz} = 0.$$

In conformity with the method of the variation of parameters invented by Lagrange, and explained in Art. 352, we shall assume that the following integrals (taken from Art. 353.) satisfy these equations, $h\ h_1\ h_2\ c\ f\ f_1\ f_2$ being *variables*,

$$h = x\frac{dy}{dt} - y\frac{dx}{dt},$$

$$h_1 = z\frac{dx}{dt} - x\frac{dz}{dt}.$$

$$h_2 = y\frac{dz}{dt} - z\frac{dy}{dt},$$

$$c + \frac{2\mu}{r} = \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2},$$

$$f + \frac{\mu z}{r} = h_1\frac{dx}{dt} - h_2\frac{dy}{dt},$$

$$f_1 + \frac{\mu y}{r} = h_2 \frac{dz}{dt} - h\, \frac{dx}{dt},$$

$$f_2 + \frac{\mu x}{r} = h\, \frac{dy}{dt} - h_1 \frac{dz}{dt},$$

and we now proceed to shew how to determine the values of the variables $h\,h_1\,h_2\,c\,f\,f_1\,f_2$ in order that this may be the case.

Differentiate all these equations with respect to t and eliminate $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$ by means of the equations of motion:

we have $\frac{dh}{dt} = y\,\frac{dR}{dx} - x\frac{dR}{dy}$,

$$\frac{dh_1}{dt} = x\,\frac{dR}{dz} - z\,\frac{dR}{dx},$$

$$\frac{dh_2}{dt} = z\,\frac{dR}{dy} - y\,\frac{dR}{dz},$$

$$\frac{dc}{dt} = -2\left\{\frac{dR}{dx}\frac{dx}{dt} + \frac{dR}{dy}\frac{dy}{dt} + \frac{dR}{dz}\frac{dz}{dt}\right\} = -2\frac{d(R)}{dt},$$

the brackets surrounding R implying that the *total* differential coefficient with respect to t is to be taken, *but this only in so far as* R *is a function of* x y z*.

$$\frac{df}{dt} = \frac{dh_1}{dt}\,\frac{dx}{dt} - \frac{dh_2}{dt}\,\frac{dy}{dt} - h_1\frac{dR}{dx} + h_2\frac{dR}{dy},$$

$$\frac{df_1}{dt} = \frac{dh_2}{dt}\,\frac{dz}{dt} - \frac{dh}{dt}\,\frac{dx}{dt} - h_2\frac{dR}{dz} + h\,\frac{dR}{dx},$$

$$\frac{df_2}{dt} = \frac{dh}{dt}\,\frac{dy}{dt} - \frac{dh_1}{dt}\,\frac{dz}{dt} - h\,\frac{dR}{dy} + h_1\frac{dR}{dz}.$$

* R is also a function of t in consequence of being a function of $x'y'z'$, but the bracket is meant to imply that R is to be differentiated only in so far as it is a function of $x\,y\,z$.

356. The inclinations of the planes of the planetary orbits to the ecliptic are very small; the asteroids (of which the masses, however, are very small) being excepted. This is the case also with the eccentricities. We shall consequently neglect powers of these quantities higher than the square.

By referring to the value of R (Art. 352.) it will be seen that $\frac{dR}{dx}$, $\frac{dR}{dy}$, $\frac{dR}{dz}$ all vary as m' the mass of the disturbing planet, which in our system is always extremely small in comparison with that of the Sun: we shall therefore neglect these quantities when they have small multipliers, and also their squares.

The difference between all angles and distances measured on the plane of the orbit and their projections on the ecliptic varies as the versine of the orbit's inclination, and therefore as the square of the angle of inclination nearly. This shews that in calculating the perturbations of the mean distance, the eccentricity, the longitude of the perihelion, and the epoch, we may neglect i and therefore $h_1^2+h_2^2$ and consequently h_1 and h_2, and also f, Art. 353, equation (5): hence the equations of last Article become

$$\frac{dh}{dt} = y\frac{dR}{dx} - x\frac{dR}{dy},$$

$$\frac{dc}{dt} = -2\frac{d(R)}{dt},$$

$$\frac{df_1}{dt} = \frac{dx}{dt}\left\{x\frac{dR}{dy} - y\frac{dR}{dx}\right\} + h\frac{dR}{dx},$$

$$\frac{df_2}{dt} = -\frac{dy}{dt}\left\{x\frac{dR}{dy} - y\frac{dR}{dx}\right\} - h\frac{dR}{dy}.$$

357. When we have expanded the function R then we must calculate the terms of these equations which involve the partial differential coefficients of R. After this we shall obtain the variations of the elements of the instantaneous orbit of the planet in terms of these variations of the arbitrary quantities $h\,h_1 h_2\,c\,f\,f_1 f_2$. Then by integrating these we shall know the

elements of the instantaneous orbit. Let $a_{,} e_{,} \varpi_{,} \epsilon_{,} i_{,} \Omega_{,}$ be these elements at the time t; the subscript accents being used to denote that *the elements are variable.* Then by substituting these in

$$\frac{r}{a_{,}} = 1 + \frac{e_{,}^2}{2} - e_{,} \cos(n_{,}t + \epsilon_{,} - \varpi_{,}) - \frac{e_{,}^2}{2} \cos 2(n_{,}t + \epsilon_{,} - \varpi_{,}) - \ldots.$$

$$\text{and } \theta = n_{,}t + \epsilon_{,} + \left(2e_{,} - \frac{e_{,}^3}{4}\right) \sin(n_{,}t + \epsilon_{,} - \varpi_{,})$$

$$+ \frac{5e_{,}^2}{4} \sin 2(n_{,}t + \epsilon_{,} - \varpi_{,}) + \ldots\ldots$$

we know the position of the planet in its orbit; the position of the orbit being known by $i_{,}$ and $\Omega_{,}$.

At present, however, we shall proceed to the transformation of R to polar co-ordinates.

PROP. *To determine* $\frac{dR}{dx}$, $\frac{dR}{dy}$ *in terms of* $\frac{dR}{d\theta}$, $\frac{dR}{dr}$.

358. In calculating these disturbing forces we may suppose r and θ the same as their projections on the plane of xy: for otherwise we should be retaining quantities varying as the product of the square of the inclination and disturbing force;

$$\therefore\ x = r\cos\theta,\quad y = r\sin\theta,\quad r^2 = x^2 + y^2,\quad \tan\theta = \frac{y}{x};$$

$$\therefore\ \frac{dR}{dx} = \frac{dR}{dr}\frac{dr}{dx} + \frac{dR}{d\theta}\frac{d\theta}{dx} = \frac{dR}{dr}\cos\theta - \frac{dR}{d\theta}\frac{\sin\theta}{r},$$

$$\frac{dR}{dy} = \frac{dR}{dr}\frac{dr}{dy} + \frac{dR}{d\theta}\frac{d\theta}{dy} = \frac{dR}{dr}\sin\theta + \frac{dR}{d\theta}\frac{\cos\theta}{r}.$$

359. But since R is to be expanded in terms of t and the elements we must still further transform these partial differential coefficients.

Upon examining the expansions of r and θ we see that $\epsilon_{,}$ and $\varpi_{,}$ are remarkably connected with $n_{,}t$: r is a function of $n_{,}t + \epsilon_{,} - \varpi_{,}$, and θ equals $n_{,}t + \epsilon_{,} +$ a function of $n_{,}t + \epsilon_{,} - \varpi_{,}$; and $\epsilon_{,}$ and $\varpi_{,}$ occur in no other way in r and θ

and consequently in R. Hence by an analytical artifice we may consider R as a function of $\epsilon_{,}$ and $\varpi_{,}$ in consequence of its being a function of r and θ, and may change the variables from r and θ to $\epsilon_{,}$ and $\varpi_{,}$: this will be better understood by reading the next Proposition.

PROP. *To obtain* $\frac{d(R)}{dt}$, $\frac{dR}{d\theta}$, $\frac{dR}{dr}$ *in terms of the partial differential coefficients of* R *with respect to the elements.*

360. In $\frac{d(R)}{dt}$ R is supposed to be differentiated only inasmuch as it depends on the co-ordinates of the disturbed planet; viz. r and θ. Now by examining the expansions of r and θ we see that wherever t occurs $\epsilon_{,}$ is connected with it in the expression $n_{,}t + \epsilon_{,}$, and $\epsilon_{,}$ occurs in no other place in r and θ: hence

$$\frac{d(R)}{dt} = n_{,} \frac{dR}{d(n_{,}t + \epsilon_{,})} = n_{,} \frac{dR}{d\epsilon_{,}}.$$

Again, to obtain $\frac{dR}{d\theta}$ and $\frac{dR}{dr}$ we observe, as before, that R is a function of $\epsilon_{,}$ and $\varpi_{,}$ solely because it is a function of r and θ;

$$\therefore \frac{dR}{d\epsilon_{,}} = \frac{dR}{d\theta}\frac{d\theta}{d\epsilon_{,}} + \frac{dR}{dr}\frac{dr}{d\epsilon_{,}},$$

$$\frac{dR}{d\varpi_{,}} = \frac{dR}{d\theta}\frac{d\theta}{d\varpi_{,}} + \frac{dR}{dr}\frac{dr}{d\varpi_{,}}.$$

Now by referring to the expansions of r and θ we have

$$\frac{d\theta}{d\epsilon_{,}} + \frac{d\theta}{d\varpi_{,}} = 1 \text{ and } \frac{dr}{d\epsilon_{,}} + \frac{dr}{d\varpi_{,}} = 0;$$

in consequence of these the above equations give by addition

$$\frac{dR}{d\theta} = \frac{dR}{d\epsilon_{,}} + \frac{dR}{d\varpi_{,}}.$$

361. Again, to obtain $\frac{dR}{dr}$ we observe that r is a function of $\epsilon_,$ solely because it is a function of θ; for $\epsilon_,$ does not occur in the equation $\frac{1}{r} = \frac{1 + e_, \cos(\theta - \varpi_,)}{a_,(1 - e_,^2)}$;

$$\therefore \frac{dr}{d\epsilon_,} = \frac{dr}{d\theta}\frac{d\theta}{d\epsilon_,} = \frac{r^2 e_, \sin(\theta - \varpi_,)}{a_,(1 - e_,^2)}\frac{d\theta}{d\epsilon_,}$$

$$\text{and } \frac{d\theta}{d\epsilon_,} = \frac{d\theta}{d(n_, t + \epsilon_,)} = \frac{1}{n_,}\frac{d\theta}{dt} = \frac{\sqrt{a_, \mu (1 - e_,^2)}}{n_, r^2} \text{ (Art 273.)}$$

$$= \frac{a_,^2 \sqrt{1 - e_,^2}}{r^2}.$$

Substitute these in the formula

$$\frac{dR}{d\epsilon_,} = \frac{dR}{d\theta}\frac{d\theta}{d\epsilon_,} + \frac{dR}{dr}\frac{dr}{d\epsilon_,}$$

transposing and dividing by $\frac{dr}{d\epsilon_,}$ we have

$$\frac{dR}{dr} = \frac{\sqrt{1 - e_,^2}}{a_, e_, \sin(\theta - \varpi_,)}\left\{\frac{dR}{d\epsilon_,} - \frac{a_,^2\sqrt{1 - e_,^2}}{r^2}\frac{dR}{d\theta}\right\}.$$

We shall find the following Proposition of use hereafter in reducing our formulæ.

Prop. *To obtain* $\frac{dR}{de_,}$ *in terms of* $\frac{dR}{d\epsilon_,}$ *and* $\frac{dR}{d\varpi_,}$.

362. Since R is a function of $e_,$ solely because it is a function of r and θ;

$$\therefore \frac{dR}{de_,} = \frac{dR}{d\theta}\frac{d\theta}{de_,} + \frac{dR}{dr}\frac{dr}{de_,}.$$

We must therefore calculate $\frac{d\theta}{de_,}$ and $\frac{dr}{de_,}$.

Now $\theta = n_, t + \epsilon_, + (2e_, - \frac{1}{4}e_,^3)\sin(n_, t + \epsilon_, - \varpi_,) + \ldots$

and $r = a_,\{1 + \frac{1}{2}e_,^2 - e_, \cos(n_, t + \epsilon_, - \varpi_,) - \ldots\ldots\}$, and from these we should obtain $\frac{d\theta}{de_{,}}$ and $\frac{dr}{de_{,}}$: but since we do not know the *law* of these series we wust refer to the functions from which they were developed, viz: (Arts. 273, 279.)

$$\frac{dt}{d\theta} = \frac{(1-e_{,}^2)^{\frac{3}{2}}}{n_{,}\{1+e_{,}\cos(\theta-\varpi_{,})\}^2}, \quad n_{,} = \sqrt{\frac{\mu}{a_{,}^3}}$$

$$\text{and } r = \frac{a_{,}(1-e_{,}^2)}{1+e_{,}\cos(\theta-\varpi_{,})},$$

θ is calculated by the first, and substituted in the value of r, and then r is expanded.

By integrating the first we shall have $t = \phi(\theta, e_{,})$;

$$\therefore\ dt = \frac{dt}{d\theta}d\theta + \frac{dt}{de_{,}}de_{,};$$

transposing and multiplying by $\frac{d\theta}{dt}$ we have

$$d\theta = \frac{d\theta}{dt}dt - \frac{dt}{de_{,}}\,\frac{d\theta}{dt}de_{,};$$

$$\therefore\ \frac{d\theta}{de_{,}} = -\frac{dt}{de_{,}}\,\frac{d\theta}{dt};$$

$$\text{Now } t = \frac{1}{n_{,}}\int\frac{(1-e_{,}^2)^{\frac{3}{2}}d\theta}{\{1+e_{,}\cos(\theta-\varpi_{,})\}^2};$$

$$\therefore\ \frac{dt}{de_{,}} = -\frac{1}{n_{,}}\int\frac{\sqrt{1-e_{,}^2}\,\{3e_{,}+(2+e_{,}^2)\cos(\theta-\varpi_{,})\}\,d\theta}{\{1+e_{,}\cos(\theta-\varpi_{,})\}^3};$$

$$= -\frac{\sqrt{1-e_{,}^2}}{n_{,}}\left\{\frac{\sin(\theta-\varpi_{,})}{\{1-e_{,}\cos(\theta-\varpi_{,})\}^2} + \frac{\sin(\theta-\varpi_{,})}{1+e_{,}\cos(\theta-\varpi_{,})}\right\}.*$$

* This integral is obtained in the following manner.

Assume $\int\frac{3e+(2+e^2)\cos x}{(1+e\cos x)^3}dx = \frac{A\sin x}{(1+e\cos x)^2} + \frac{B\sin x}{1+e\cos x} + \int\frac{C\,dx}{1+e\cos x}$;

this is evidently the *form* of the integral.

Differentiating

$$\text{Also}\quad \frac{d\theta}{dt} = \frac{n_{\prime}}{(1-e_{\prime}^2)^{\frac{3}{2}}}\{1+e_{\prime}\cos(\theta-\varpi_{\prime})\}^2;$$

$$\therefore\ \frac{d\theta}{de_{\prime}} = \frac{\sin(\theta-\varpi_{\prime})}{1-e_{\prime}^2}\{2+e_{\prime}\cos(\theta-\varpi_{\prime})\}.$$

Also, since the series for r is obtained by developing the function $\dfrac{a_{\prime}(1-e_{\prime}^2)}{1+e_{\prime}\cos(\theta-\varpi_{\prime})}$ after substituting for θ,

$$\therefore\ \frac{dr}{de_{\prime}} = \left(\frac{dr}{de_{\prime}}\right) + \frac{dr}{d\theta}\frac{d\theta}{de_{\prime}}$$

$$= \frac{a_{\prime}\{-2e_{\prime}-(1+e_{\prime}^2)\cos(\theta-\varpi_{\prime})\}}{\{1+e_{\prime}\cos(\theta-\varpi_{\prime})\}^2} + \frac{a_{\prime}e_{\prime}\sin^2(\theta-\varpi_{\prime})\{2+e_{\prime}\cos(\theta-\varpi_{\prime})\}}{\{1+e_{\prime}\cos(\theta-\varpi_{\prime})\}^2}$$

$$= -a_{\prime}\cos(\theta-\varpi_{\prime});$$

$$\therefore\ \frac{dR}{de_{\prime}} = \frac{dR}{d\theta}\frac{\sin(\theta-\varpi_{\prime})\{2+e_{\prime}\cos(\theta-\varpi_{\prime})\}}{1-e_{\prime}^2} - a_{\prime}\frac{dR}{dr}\cos(\theta-\varpi_{\prime}).$$

We now proceed to obtain the formulæ for calculating the variations of the elements.

PROP. *To calculate the variations of the mean distance, the eccentricity, and the longitude of the perihelion of the instantaneous orbit of the disturbed planet.*

363. Let $a_{\prime}$, $e_{\prime}$, $\varpi_{\prime}$ be these elements; the subscript accents indicating that the elements are functions of t. Then by Art. 354

Differentiating and multiplying by $(1+e\cos x)^3$

$$3e+(2+e^2)\cos x = A\{\cos x(1+e\cos x)+2e\sin^2 x\} + B(1+e\cos x)(\cos x+e) + C(1+e\cos x)^2.$$

Let $x=0$; $3e+(2+e^2) = A(1+e)+(B+C)(1+e)^2$

or $2+e = A+(B+C)(1+e)$.........(1)

$x=\frac{\pi}{2}$; $3e = 2eA+eB+C$ ....(2)

$x=\pi$; $3e-2-e^2 = -A(1-e)-(B-C)(1-e)^2$

or $2-e = A+(B-C)(1-e)$...............(3).

From these $A=1$, $B=1$, $C=0$.

Hence the integral in the text.

1. The mean distance $a_, = -\frac{\mu}{c}$; $\therefore \frac{1}{a_,} = -\frac{c}{\mu}$;

$$\therefore \frac{da_,}{dt} = \frac{a_,^2}{\mu}\frac{dc}{dt} = -\frac{2a_,^2}{\mu}\frac{d(R)}{dt} \text{ by Art. 356.}$$

$$= -\frac{2n_,a_,^2}{\mu}\frac{dR}{d\epsilon_,} \text{ by Art. 360.}$$

2. The eccentricity

$$e_, = \frac{1}{\mu}\sqrt{f^2+f_1^2+f_2^2} = \frac{1}{\mu}\sqrt{f_1^2+f_2^2} \text{ (Art. 356);}$$

$$\therefore \frac{de_,}{dt} = \frac{1}{\mu\sqrt{f_1^2+f_2^2}}\left\{f_1\frac{df_1}{dt}+f_2\frac{df_2}{dt}\right\}$$

$$= \frac{1}{\mu}\left\{\sin\varpi_,\frac{df_1}{dt}+\cos\varpi_,\frac{df_2}{dt}\right\}; \quad \because \tan\varpi_, = \frac{f_1}{f_2}.$$

But by Arts. 356, 358,

$$\frac{df_1}{dt} = \left\{\frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}\right\}\frac{dR}{d\theta} - h\left\{\frac{dR}{d\theta}\frac{\sin\theta}{r} - \frac{dR}{dr}\cos\theta\right\},$$

$$\left\{\text{but in small terms } h = \sqrt{a_,\mu(1-e_,^2)}, \text{ and } = r^2\frac{d\theta}{dt}\right\}$$

$$= \sqrt{a_,\mu(1-e_,^2)}\left\{\left(\frac{\cos\theta}{r^2}\frac{dr}{d\theta} - \frac{2\sin\theta}{r}\right)\frac{dR}{d\theta} + \cos\theta\frac{dR}{dr}\right\}$$

$$\frac{df_2}{dt} = -\left\{\frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}\right\}\frac{dR}{d\theta} - h\left\{\frac{dR}{d\theta}\frac{\cos\theta}{r} + \frac{dR}{dr}\sin\theta\right\}$$

$$= -\sqrt{a_,\mu(1-e_,^2)}\left\{\left(\frac{\sin\theta}{r^2}\frac{dr}{d\theta} + \frac{2\cos\theta}{r}\right)\frac{dR}{d\theta} + \sin\theta\frac{dR}{dr}\right\}$$

$$\therefore \frac{de_,}{dt} = \sqrt{\frac{a_,(1-e_,^2)}{\mu}}$$

$$\times\left\{-\left(\frac{\sin(\theta-\varpi_,)}{r^2}\frac{dr}{d\theta} + \frac{2\cos(\theta-\varpi_,)}{r}\right)\frac{dR}{d\theta} - \sin(\theta-\varpi_,)\frac{dR}{dr}\right\},$$

{putting $\frac{1}{r} = \frac{1 + e_{,} \cos(\theta - \varpi_{,})}{a_{,}(1 - e_{,}^2)}$ in small terms, and using the properties proved in Arts. 360, 361.}

$$= \sqrt{\frac{a_{,}(1 - e_{,}^2)}{\mu}}$$

$$\times \left\{ \frac{-e_{,}^2 \sin^2(\theta - \varpi_{,}) - e_{,}^2 \cos^2(\theta - \varpi_{,}) + 1}{a_{,} e_{,} (1 - e_{,}^2)} \frac{dR}{d\theta} - \frac{\sqrt{1 - e_{,}^2}}{a_{,} e_{,}} \frac{dR}{d\epsilon_{,}} \right\}$$

$$= \sqrt{\frac{1 - e_{,}^2}{a_{,} e_{,}^2 \mu}} \left(\frac{dR}{d\epsilon_{,}} + \frac{dR}{d\varpi_{,}} \right) - \frac{1 - e_{,}^2}{e_{,} \sqrt{\mu a}} \frac{dR}{d\epsilon_{,}}.$$

3. For the longitude of the perihelion $\tan \varpi_{,} = \frac{f_1}{f_2}$;

$$\therefore \frac{d\varpi_{,}}{dt} = \frac{1}{f_1^2 + f_2^2} \left\{ f_2 \frac{df_1}{dt} - f_1 \frac{df_2}{dt} \right\}$$

$$= \frac{1}{\mu e_{,}} \left\{ \cos \varpi_{,} \frac{df_1}{dt} - \sin \varpi_{,} \frac{df_2}{dt} \right\}$$

$$= \sqrt{\frac{a_{,}(1 - e_{,}^2)}{\mu e_{,}^2}}$$

$$\times \left\{ \left(\frac{\cos(\theta - \varpi_{,})}{r^2} \frac{dr}{d\theta} - \frac{2 \sin(\theta - \varpi_{,})}{r} \right) \frac{dR}{d\theta} + \cos(\theta - \varpi_{,}) \frac{dR}{dr} \right\}$$

$$= \sqrt{\frac{a_{,}(1 - e_{,}^2)}{\mu e_{,}^2}}$$

$$\times \left\{ - \frac{\sin(\theta - \varpi_{,}) \{2 + e_{,} \cos(\theta - \varpi_{,})\}}{a_{,}(1 - e_{,}^2)} \frac{dR}{d\theta} + \cos(\theta - \varpi_{,}) \frac{dR}{dr} \right\}$$

$$= - \sqrt{\frac{1 - e_{,}^2}{a_{,} \mu e_{,}^2}} \frac{dR}{de_{,}} \text{ by Art. 362.}$$

Prop. *To find the variations of the inclination and the longitude of the node.*

364. We have by Art. 354 the formulæ

$$\tan i_{,} = \frac{\sqrt{h_1^2 + h_2^2}}{h}, \quad \tan \Omega_{,} = -\frac{h_2}{h_1};$$

$$\therefore \tan i_{,} \sin \Omega_{,} = -\frac{h_2}{h}, \quad \tan i_{,} \cos \Omega_{,} = \frac{h_1}{h}.$$

Hence $\dfrac{d(\tan i_{,} \sin \Omega_{,})}{dt} = \dfrac{1}{h^2}\left\{h_2 \dfrac{dh}{dt} - h\dfrac{dh_2}{dt}\right\}.$

Substituting by the equations of Art. 355, and neglecting z and $\dfrac{dz}{dt}$ as being small,

$$\sin \Omega_{,} \frac{d \tan i_{,}}{dt} + \cos \Omega_{,} \tan i_{,} \frac{d\Omega_{,}}{dt} = \frac{y}{h} \frac{dR}{dz}.$$

In a similar manner by differentiating

$$\tan i_{,} \cos \Omega_{,} = \frac{h_1}{h}, \text{ we have}$$

$$\cos \Omega_{,} \frac{d \tan i_{,}}{dt} - \sin \Omega_{,} \tan i_{,} \frac{d\Omega_{,}}{dt} = \frac{x}{h} \frac{dR}{dz}.$$

Multiplying these equations by $\sin \Omega_{,}$ and $\cos \Omega_{,}$ respectively and adding

$$\frac{d \tan i_{,}}{dt} = \frac{1}{h} (y \sin \Omega_{,} + x \cos \Omega_{,}) \frac{dR}{dz}.$$

Multiplying by $\cos \Omega_{,}$, $\sin \Omega_{,}$ and subtracting

$$\tan i_{,} \frac{d\Omega_{,}}{dt} = \frac{1}{h} (y \cos \Omega_{,} - x \sin \Omega_{,}) \frac{dR}{dz}.$$

Now $\dfrac{dR}{di_{,}} = \dfrac{dR}{dx} \dfrac{dx}{di_{,}} + \dfrac{dR}{dy} \dfrac{dy}{di_{,}} + \dfrac{dR}{dz} \dfrac{dz}{di_{,}} = \dfrac{dR}{dz} \dfrac{dz}{di_{,}}$ nearly;

and similarly $\dfrac{dR}{d\Omega_{,}} = \dfrac{dR}{dz} \dfrac{dz}{d\Omega_{,}}$ nearly.

since for a given alteration in the inclination or longitude of the node the alteration in z is greater than in x or y.

Also $z = -\frac{h_2}{h}x - \frac{h_1}{h}y$ by Art. 353, equation (7).

$$= \tan i_, (x \sin \Omega_, - y \cos \Omega_,),$$

$$\therefore \frac{dz}{di_,} = \sec^2 i_, (x \sin \Omega_, - y \cos \Omega_,);$$

$$\frac{dz}{d\Omega_,} = \tan i_, (x \cos \Omega_, + y \sin \Omega_,).$$

Hence $\dfrac{d \tan i_,}{dt} = \dfrac{1}{h \tan i_,} \dfrac{dR}{d\Omega_,}$;

$$\frac{d\Omega_,}{dt} = -\frac{1}{h \tan i_,} \frac{dR}{di_,}, \text{ neglecting } \tan^2 i_, \ldots$$

Since the squares of $i_,$ are neglected and since $h = \dfrac{\mu\sqrt{1-e_,^2}}{n_, a_,}$ these may be written

$$\frac{di_,}{dt} = \frac{na}{\mu \sin i_, \sqrt{1-e_,^2}} \frac{dR}{d\Omega_,},$$

$$\frac{d\Omega_,}{dt} = -\frac{na}{\mu \sin i_, \sqrt{1-e_,^2}} \frac{dR}{di_,};$$

which agree with those given by M. Pontècoulant *Théorie Analytique du Système du Monde*, Tom. I. p. 330.

PROP. *To find the variation of the epoch.*

365. Now R is a function of $a_,$ $e_,$ $\varpi_,$ $n_,t + \epsilon_,$ $i_,$ $\Omega_,$, and since the instantaneous ellipse is an ellipse of curvature to the orbit described of the first order (Art. 350), it follows that the first differential coefficients of r and θ with respect to t will be the same in the real orbit and in the instantaneous ellipse. The same will be the case with the first differential coefficient of any function of r and θ, as R.

Now in the ellipse $\frac{d(R)}{dt} = n_{,} \frac{dR}{d\epsilon_{,}}$; and in the real orbit, since R is a function of the variable quantities $a_{,}$, $e_{,}$, $\varpi_{,}$, $n_{,}t + \epsilon_{,}$, $i_{,}$, and $\Omega_{,}$;

$$\therefore \frac{d(R)}{dt} = \frac{dR}{da_{,}}\frac{da_{,}}{dt} + \frac{dR}{de_{,}}\frac{de_{,}}{dt} + \frac{dR}{d\varpi_{,}}\frac{d\varpi_{,}}{dt}$$

$$+ \frac{dR}{d\epsilon_{,}}\left\{n_{,} + t\frac{dn_{,}}{dt} + \frac{d\epsilon_{,}}{dt}\right\} + \frac{dR}{di_{,}}\frac{di_{,}}{dt} + \frac{dR}{d\Omega_{,}}\frac{d\Omega_{,}}{dt}.$$

Equating these values of $\frac{d(R)}{dt}$ and substituting for $\frac{da_{,}}{dt}$, $\frac{de_{,}}{dt}$, $\frac{d\varpi_{,}}{dt}$, $\frac{dn_{,}}{dt}\left(= -\frac{3n_{,}}{2a_{,}}\frac{da_{,}}{dt}\right)$, $\frac{di_{,}}{dt}$, $\frac{d\Omega_{,}}{dt}$ the values found in Arts. 363, 364, and transposing $\frac{d\epsilon_{,}}{dt}$ and dividing by $\frac{dR}{d\epsilon_{,}}$, we have

$$\frac{d\epsilon_{,}}{dt} = -\frac{3n_{,}^2 a_{,}}{\mu}\frac{dR}{d\epsilon_{,}}t + \frac{2n_{,}a_{,}^2}{\mu}\frac{dR}{da_{,}} - \frac{n_{,}a_{,}}{\mu e_{,}}\left\{\sqrt{1-e_{,}^2} - (1-e_{,}^2)\right\}\frac{dR}{de_{,}}.$$

366. We shall now bring together the variations of the elliptic elements obtained in the last three Articles, and present them under one point of view.

(1) $$\frac{da_{,}}{dt} = -\frac{2a_{,}^2 n_{,}}{\mu}\frac{dR}{d\epsilon_{,}}.$$

(2) $$\frac{de_{,}}{dt} = \frac{n_{,}a_{,}\sqrt{1-e_{,}^2}}{\mu_{,}e_{,}}\left(\frac{dR}{d\epsilon_{,}} + \frac{dR}{d\varpi_{,}}\right) - \frac{n_{,}a_{,}(1-e_{,}^2)}{\mu e_{,}}\frac{dR}{d\epsilon_{,}}.$$

(3) $$\frac{d\varpi_{,}}{dt} = -\frac{n_{,}a_{,}\sqrt{1-e_{,}^2}}{\mu e_{,}}\frac{dR}{de_{,}}.$$

(4) $$\frac{d\epsilon_{,}}{dt} = -\frac{3n_{,}^2 a_{,}}{\mu}\frac{dR}{d\epsilon_{,}}t + \frac{2n_{,}a_{,}^2}{\mu}\frac{dR}{da_{,}}$$
$$- \frac{n_{,}a_{,}}{\mu e_{,}}\left\{\sqrt{1-e_{,}^2} - (1-e_{,}^2)\right\}\frac{dR}{de_{,}}.$$

(5) $$\frac{d \tan i_,}{dt} = \frac{n_, a_,}{\mu \tan i_, \sqrt{1 - e_,^2}} \frac{dR}{d\Omega_,}.$$

(6) $$\frac{d\Omega_,}{dt} = - \frac{n_, a_,}{\mu \tan i_, \sqrt{1 - e_,^2}} \frac{dR}{di_,}.$$

Before we can make use of these formulæ we must explain how R is to be developed.

PROP. *To explain the manner in which* R *is to be developed.*

367. If we recur to Art. 352 we see that

$$R = \frac{m'(xx' + yy' + zz')}{(x'^2 + y'^2 + z'^2)^{\frac{3}{2}}} - \frac{m'}{\sqrt{(x' - x)^2 + (y' - y)^2 + (z' - z)^2}}.$$

Let r and θ be the radius vector and longitude of m measured on the plane of xy as far as the node and then on the plane of m's orbit.

$r_,$ and $\theta_,$ the rad. vect. and longitude projected on plane xy.

$\Omega_,$ and $i_,$ the longitude of the node and inclination of m's orbit to the plane xy.

λ the latitude of m.

Then we have

$$x = r_, \cos \theta_,, \quad y = r_, \sin \theta_,, \quad z = r_, \tan \lambda = r_, \sin (\theta_, - \Omega_,) \tan i_,,$$

Similar expressions are true for m'.

Hence R

$$= \frac{m' \{r_, r_,' \cos(\theta_, - \theta_,') + zz'\}}{(r_,'^2 + z'^2)^{\frac{3}{2}}} - \frac{m'}{\sqrt{r_,^2 + r_,'^2 - 2r_, r_,' \cos(\theta_, - \theta_,') + (z - z')^2}}.$$

Now $r_, = r \cos \lambda = r \left\{1 - \frac{1}{2} \tan^2 \lambda\right\}$ very nearly

$$= r \left\{1 - \tfrac{1}{2} \tan^2 i_, \sin^2 (\theta_, - \Omega_,)\right\}$$

$$= r \left\{1 - \tfrac{1}{4} \tan^2 i_, + \tfrac{1}{4} \tan^2 i_, \cos 2 (\theta_, - \Omega_,)\right\}.$$

Also $\tan(\theta_, - \Omega_,) = \cos i_, \tan(\theta - \Omega_,)$;

$$\therefore\ \theta_, - \Omega_, = \theta - \Omega_, - \tan^2 \frac{i_,}{2} \sin 2(\theta - \Omega_,), \text{ (see Art. 274).}$$

Substitute in these the values of r and θ given by Art. 357, and we have

$$r_, = a_, \left\{1 + \tfrac{1}{2} e_,^2 - e_, \cos(n_,t + \epsilon_, - \varpi_,) - \tfrac{1}{2} e_,^2 \cos 2(n_,t + \epsilon - \varpi_,) \right.$$
$$\left. - \tfrac{1}{4} \tan^2 i_, + \tfrac{1}{4} \tan^2 i_, \cos 2(n_,t + \epsilon_, - \Omega_,) + \ldots \right\}$$
$$= a_, \{1 + u\}$$

and $\theta_, = n_,t + \epsilon_, + 2e_, \sin(n_,t + \epsilon_, - \varpi_,) + \frac{5}{4} e_,^2 \sin 2(n_,t + \epsilon_, - \varpi_,)$

$$- \tan^2 \tfrac{1}{2} i_, \sin 2(n_,t + \epsilon_, - \Omega_,) + \ldots$$
$$= n_,t + \epsilon_, + v \text{ suppose.}$$

Now let R' be the value of R when $a_,$ and $a_,'$ are put for $r_,$ and $r_,'$ and suppose $r_, = a_, (1 + u)$ and $r_,' = a_,' (1 + u')$ u and u' are small quantities because the orbits of the planets are nearly circular: then by Taylor's Theorem

$$R = R' + \frac{dR'}{da_,} a_, u + \frac{dR'}{da_,'} a_,' u' + \ldots$$

also $$R' = \frac{m' \{a_, a_,' \cos(\theta_, - \theta_,') + a_, a_,' \tan i_, \tan i_,' \sin(\theta_, - \Omega_,) \sin(\theta_,' - \Omega_,')\}}{\{a_,'^2 + a_,'^2 \tan^2 i_,' \sin^2(\theta_,' - \Omega_,')\}^{\frac{3}{2}}}$$

$$- \frac{m'}{\sqrt{a_,^2 + a_,'^2 - 2a_, a_,' \cos(\theta_, - \theta_,') + \{a_, \tan i_, \sin(\theta_, - \Omega_,) - a_,' \tan i_,' \sin(\theta_,' - \Omega_,')\}^2}}$$

$$= \frac{m' a_, \cos(\theta_, - \theta_,')}{a_,'^2} - \frac{m'}{\sqrt{a_,^2 + a_,'^2 - 2a_, a_,' \cos(\theta_, - \theta_,')}}$$

$$+ \frac{m' a_, \tan i_, \tan i_,' \sin(\theta_, - \Omega_,) \sin(\theta_,' - \Omega_,')}{a_,'^2}$$

$$- \frac{3m' a_, \tan^2 i_,' \sin^2(\theta_,' - \Omega_,') \cos(\theta_, - \theta_,')}{2a_,'^2}$$

$$+ \frac{m' \{a_, \tan i_, \sin(\theta_, - \Omega_,) - a_,' \tan i_,' \sin(\theta_,' - \Omega_,')\}^2}{2\{a_,^2 + a_,'^2 - 2a_, a_,' \cos(\theta_, - \theta_,')\}^{\frac{3}{2}}} + \ldots$$

$$\text{Let } \frac{1}{\sqrt{a_{\prime}^2 + a_{\prime}'^2 - 2a_{\prime}a_{\prime}' \cos(\theta_{\prime} - \theta_{\prime}')}}$$

$$= \tfrac{1}{2} C_0 + C_1 \cos(\theta_{\prime} - \theta_{\prime}') + C_2 \cos 2(\theta_{\prime} - \theta_{\prime}') + \ldots\ldots$$

$$\frac{1}{\{a_{\prime}^2 + a_{\prime}'^2 - 2a_{\prime}a_{\prime}' \cos(\theta_{\prime} - \theta_{\prime}')\}^{\frac{3}{2}}}$$

$$= \tfrac{1}{2} D_0 + D_1 \cos(\theta_{\prime} - \theta_{\prime}') + D_2 \cos 2(\theta_{\prime} - \theta_{\prime}') + \ldots\ldots$$

These coefficients should be calculated and then R' may be arranged in a series. When we have thus calculated R' we must find $\frac{dR'}{da_{\prime}}$, $\frac{dR'}{da_{\prime}'}$ ...... and substitute them in

$$R' + \frac{dR'}{da_{\prime}} a_{\prime} u + \frac{dR'}{da_{\prime}'} a_{\prime}' u' + \ldots\ldots\ldots$$

and we shall have R expressed in a series of terms depending on the time and the elements of the instantaneous orbits.

It is not our object to enter into the numerical calculation of the coefficients of the expansion of R: for this we refer the reader to M. Pontécoulant's *Theorie Analytique du Système du Monde*, Tom. I. p. 340, *Mécanique Céleste*, Tom. III. and Mr. Lubbock's Papers in the *Philos. Trans.* and *Astron. Trans.*

We proceed to demonstrate some Propositions relative to the general nature of the terms.

PROP. *To prove that the terms of* R *which depend on the mean anomalies* ($n_{\prime}t$ *and* $n_{\prime}'t$) *of the planets are of the form* $P \cos\{(pn_{\prime} - qn_{\prime}')t + Q\}$ *or* $P \cos\{(pn_{\prime} + qn_{\prime}')t + Q\}$, *where* P *is a function of the mean distances, eccentricities, and inclinations of the orbits, and* Q *is a function of the longitudes of the perihelia and nodes and of the epochs; and* p *and* q *are positive integers.*

368. We shall make use of the following elementary trigonometrical formulæ:

I. $\text{Cos}\, a \cos b = \frac{1}{2}\cos(a-b) + \frac{1}{2}\cos(a+b)$.

II. $\text{Sin}\, a \sin b = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b)$.

III. $\text{Sin}\, a \cos b = \frac{1}{2}\sin(a+b) + \frac{1}{2}\sin(a-b)$.

$$\text{Now } \theta_{,} - \theta_{,}' = (n_{,}t + \epsilon_{,}) - (n_{,}'t + \epsilon_{,}')$$

$$+(2e_{,}\ \ldots\ldots)\sin(n_{,}t + \epsilon_{,} - \varpi_{,}) + (\tfrac{5}{4}e_{,}^2 + \ldots)\sin 2(n_{,}t + \epsilon_{,} - \varpi_{,}) + \ldots$$

$$-(2e_{,}' + \ldots)\sin(n_{,}'t + \epsilon_{,}' - \varpi_{,}') + (\tfrac{5}{4}e_{,}'^2 + \ldots)\sin 2(n_{,}'t + \epsilon_{,}' - \varpi_{,}') + \ldots$$

$$-\tan^2\tfrac{1}{2}i_{,}\sin 2(n_{,}t + \epsilon_{,} - \Omega_{,}) + \ldots\ldots$$

$$+\tan^2\tfrac{1}{2}i_{,}'\sin 2(n_{,}'t + \epsilon_{,}' - \Omega_{,}') + \ldots\ldots$$

$$= (n_{,}t + \epsilon_{,}) - (n_{,}'t + \epsilon_{,}') + T \text{ suppose};$$

$$\therefore \cos k(\theta_{,} - \theta_{,}') = \cos\{k(n_{,}t + \epsilon_{,}) - k(n_{,}'t + \epsilon_{,}')\}\cos kT$$

$$- \sin\{k(n_{,}t + \epsilon_{,}) - k(n_{,}'t + \epsilon_{,}')\}\sin kT,$$

$$\text{and } \cos kT = 1 - \tfrac{1}{2}k^2T^2 + \ldots\ldots$$

$$\sin kT = kT - \frac{1}{1.2.3}k^3T^3 + \ldots\ldots$$

Now by formulæ II. and I. the even powers of T, and $\therefore \cos kT$, will involve only simple cosines; and by formulæ II. and III. the odd powers of T, and therefore $\sin kT$, only simple sines.

Hence by formulæ I. and II. the expansion of $\cos k(\theta_{,} - \theta_{,}')$ given by the above formula will contain only simple cosines.

In the same manner we might shew that $\sin(\theta_{,} - \Omega_{,})$ and $\sin(\theta_{,}' - \Omega_{,}')$ will equal a series of simple sines (with no constant term), and therefore by formula II. the squares or product of these will contain only simple cosines.

We see then that when the complete development of R' given in Art. 367. is worked out and arranged in a series, it will consist only of simple cosines.

Again by Art. 367. we see that u and u' consist of a series of terms involving the simple cosines of angles. Hence, by formula I. each of the quantities $u, u', u^2, uu', u'^2 \ldots.$ will consist of a series of simple cosines of angles formed by combining the arguments* of terms of u and u' in endless variety by addition and subtraction.

It follows, then, finally, that the series into which R is to be developed (Art. 367.) will by formula I. consist only of terms of the form

$$P\cos\{(pn_{,}-qn_{,}')t+Q\} \text{ and } P\cos\{(pn_{,}+qn_{,}')t+Q\},$$

p and q being positive integers, P a function of the mean distances, eccentricities, and inclinations of the instantaneous orbits; and Q a function of the longitudes of the perihelia and nodes and the epochs.

369. We have already frequently remarked, that the eccentricities and inclinations of the planets are so small, that their higher powers will be of almost imperceptible magnitude. It becomes important then to search for some means of determining the relative magnitude of P in reference to the argument $(pn_{,}-qn_{,}')t+Q$ for this will materially shorten the calculation of R, by pointing out at once those terms of the infinite series into which R is developed, which are of sufficient importance to be retained.

In the following Article we shall prove a principle which answers our purpose.

Prop. *The lowest dimension of the quantities* $e_{,}$, $e_{,}'$, *tan* $i_{,}$, *tan* $i_{,}'$ *in the coefficient of* $P\cos\{(pn_{,}-qn_{,}')t+Q\}$ *is of the order* $p \sim q$.

370. We have by Art. 367.

$$R = R' + \frac{dR'}{da_{,}}a_{,}u + \frac{dR'}{da_{,}'}a_{,}'u_{,}' + \dots\dots$$

(1) A remarkable law prevails in the expansions of u and u'. It is this (Art. 367). The number which multiplies $n_{,}t+\epsilon_{,}$ in the argument of any term in these expansions represents the dimensions in $e_{,}$, $e_{,}'$, tan $i_{,}$, tan $i_{,}'$ of the principal part of the coefficient of that term.

* In an expression $a\cos(pnt+q)$, the angle $pnt+q$ is called the *argument* of the term $a\cos(pnt+q)$.

Now the same holds good in any power of u or u'. Thus in u^2 a term $P \cos (pn_{,}t + P')$ can arise only in the following ways, partly from the multiplication of any two terms in u of which the arguments are $ln_{,}t+L$ and $mn_{,}t+M$, where $l+m=p$; and partly from such as have the arguments $l'n_{,}t+L$ and $m'n_{,}t+M$, where $l'-m'=p$. In the former case the dimension of the principal part of the coefficient will be $l+m=p$, in the latter it will be $l'+m'$ and this is greater than p. Hence the principal part of the coefficient of a term $P \cos (pn_{,}t + P')$ in u^2 will be of the dimension p.

The same is evidently true of u'^2, u^3, u'^3, ...

(2) In the product of any powers of u and u' as $u^{\alpha}u'^{\beta}$, the dimension is the *sum* of the multipliers of nt and $n't$.

For let us consider a term $N \cos \{(ln \pm l'n')\,t + M\}$. Now this must evidently have arisen from the multiplication of $\cos (lnt + L)$ and $\cos (l'n_{,}'t + L')$ in u^{α} and u'^{β} respectively. The coefficient of this is of the order $l + l'$.

(3) Let us next consider the law of the coefficients in $\cos k (\theta_{,} - \theta_{,}')$.

If we turn to Art. 368. and examine the expansions of $\cos kT$ and $\sin kT$, we shall find that the laws (1) and (2) hold equally in them. But in $\cos k (\theta_{,} - \theta_{,}')$, since it is equal to

$$\cos \{k (n_{,}t + \epsilon_{,}) - k (n_{,}'t + \epsilon_{,}')\} \cos kT$$

$$- \sin \{k (n_{,}t + \epsilon_{,}) - k (n_{,}'t + \epsilon_{,}')\} \sin kT,$$

the dimension of the coefficient of any term calculated by the laws (1), (2) will be higher or lower by $2k$ than it ought to be according as the argument is formed by addition or subtraction.

If, then, we turn to Art. 367. and examine the expression given for R we see that the laws (1), (2) just proved hold for R, if we leave out of consideration all the multipliers which are of the form $\cos k (\theta_{,} - \theta_{,}')$. Bearing this in mind we shall be able to prove our Proposition.

Any term $P \cos \{(pn_{,} - qn_{,}')\,t + Q\}$ in R has partly arisen from the multiplication of $\cos \{(kn_{,} - kn_{,}')\,t + Q'\}$ with $\cos \{[(p-k)\,n_{,} - (q-k)\,n_{,}']\,t + Q''\}$ and partly from multiplication with $\cos \{[(p+k)\,n_{,} - (q+k)\,n_{,}']\,t + Q'''\}$, and in no other way can it have been formed: k being any number of the series 0, 1, 2, 3,......

First: suppose k intermediate to p and q. Then the first of these cosines becomes

$$\cos\{[(p-k)n_{,}+(k-q)n_{,}']t+Q''\},$$

and the dimension of the principal coefficient of this and therefore of $\cos\{(pn_{,}-qn_{,}')t+Q\}$ is by law (2) equal to $(p\sim k)+(k\sim q)=p\sim q$, since k is intermediate to p and q.

Second: suppose k is not greater than the smaller of p and q. Then the dimension of the principal coefficient of

$$\cos\{[(p-k)n_{,}-(q-k)n_{,}']t+Q''\} \text{ is } p+q-2k:$$

and therefore the dimension of the principal coefficient of $\cos\{(pn'-qn_{,}')t+Q\}$ is the least value of which $p+q-2k$ is susceptible, and that is $p\sim q$.

Third: suppose k is not less than the greater of p and q. Then the dimension of the principal coefficient of

$$\cos\{[(k-p)n_{,}-(k-q)n_{,}']t-Q''\} \text{ is } 2k-p-q,$$

and therefore the dimension of the principal coefficient of $\cos\{(pn_{,}-qn_{,}')t+Q\}$ is the least value of $2k-p-q$, and this is $p\sim q$, as before.

Hence the Proposition is true.

Prop. *To prove that the principal coefficient of the term* $\mathrm{P}\cos\{(\mathrm{pn}_{,}+\mathrm{qn}_{,}')\mathrm{t}+\mathrm{Q}\}$ *in* R *is of the dimension* $\mathrm{p}+\mathrm{q}$ *in* $\mathrm{e}_{,}$, $\mathrm{e}_{,}'$, $\tan \mathrm{i}_{,}$, $\tan \mathrm{i}_{,}'$.

371. This term arises from the multiplication of such terms as $P'\cos\{(kn_{,}-kn_{,}')t+Q'\}$ with

$$P''\cos\{[(p-k)n_{,}+(q+k)n_{,}']t+Q''\}$$

$$\text{and } P''\cos\{[(p+k)n_{,}+(q-k)n_{,}']t+Q''\}.$$

Hence, in both cases, the dimension will be $p+q$, since $(p-k)+(q+k)$ and $(p+k)+(q-k)$ each equals $p+q$: see law (2) of Art. 370. We have here supposed k is not greater than p and q: but if k be greater than p or q it will be easily seen that the dimension will be greater than $p+q$. Hence the Proposition is true.

PROP. *To determine the part of* R *which is independent of the periodic terms.*

372. We have

$$R = R' + \frac{dR'}{da_{,}} a_{,} u + \frac{dR'}{da_{,}'} a_{,}' u'$$

$$+ \frac{d^2R'}{da_{,}^2} \frac{a_{,}^2 u^2}{2} + \frac{d^2R'}{da_{,} da_{,}'} a_{,} a_{,}' u u' + \frac{d^2R'}{da_{,}'^2} \frac{a_{,}'^2 u'^2}{2}$$

$$+ \ldots\ldots\ldots\ldots$$

We shall neglect small quantities of the third order; hence we need calculate the first differential coefficients of R' only to the first order: and in the second differential coefficients we may neglect all small quantities.

Let us turn to the expression for R' and that for u in Art. 367, and it will be seen (after reduction) that the constant part of R' is

$+ \dfrac{m' a_{,} e_{,} e_{,}'}{a_{,}'^2} \cos (\varpi_{,} - \varpi_{,}')$ {from the first term of R'}

$- \frac{1}{2} m' C_0 - m' e_{,} e_{,}' C_1 \cos (\varpi_{,} - \varpi_{,}')$ {from the second term of R'}

$+ \frac{1}{8} m' (a_{,}^2 \tan^2 i_{,} + a_{,}'^2 \tan^2 i_{,}') D_0$ {from the fifth term of R'}

$- \frac{1}{4} m' a_{,} a_{,}' \tan i_{,} \tan i_{,}' D_1 \cos (\Omega_{,} - \Omega_{,}')$ {from the fifth term of R'}.

The constant part of $\dfrac{dR'}{da_{,}} a_{,} u$ is

$+ \dfrac{m' a_{,} e_{,} e_{,}'}{2 a_{,}'^2} \cos (\varpi_{,} - \varpi_{,}')$ {from the first term of R'}

$- \dfrac{dC_0}{da_{,}} \dfrac{m' a_{,}}{4} (e_{,}^2 - \frac{1}{2} \tan^2 i_{,})$ {from the second term of R'}

$- \dfrac{dC_1}{da_{,}} \dfrac{m' a_{,} e_{,} e_{,}'}{2} \cos (\varpi_{,} - \varpi_{,}')$ {from the second term of R'}.

The constant part of $\frac{dR'}{da_{,}'} a_{,}' u'$ is

$$- \frac{m' a_{,} e_{,} e_{,}'}{a_{,}'^2} \cos (\varpi_{,} - \varpi_{,}') \qquad \{\text{from the first term of } R'\}$$

$$- \frac{dC_0}{da_{,}'} \frac{m' a_{,}'}{4} (e_{,}'^2 - \tfrac{1}{2} \tan^2 i_{,}') \quad \{\text{from the second term of } R'\}$$

$$- \frac{dC_1}{da_{,}'} \frac{m' a_{,}' e_{,} e_{,}'}{2} \cos (\varpi_{,} - \varpi_{,}') \; \{\text{from the second term of } R'\}.$$

The constant part of $\frac{d^2 R'}{da_{,}^2} \frac{a_{,}^2 u^2}{2}$ is

$$- \frac{d^2 C_0}{da_{,}^2} \frac{m' a_{,}^2 e_{,}^2}{8} \text{ (from the second term of } R').$$

The constant part of $\frac{d^2 R'}{da_{,} da_{,}'} a_{,} a_{,}' u u'$ is

$$- \frac{m' a_{,} e_{,} e_{,}'}{2 a_{,}'^2} \cos (\varpi_{,} - \varpi_{,}') \quad \{\text{from the first term of } R'\}$$

$$- \frac{d^2 C_1}{da_{,} da_{,}'} \frac{m' a_{,} a_{,}' e_{,} e_{,}'}{4} \cos (\varpi_{,} - \varpi_{,}') \{\text{from the second term of } R'\}.$$

The constant part of $\frac{d^2 R'}{da_{,}'^2} \frac{a_{,}'^2 u_{,}^2}{2}$ is

$$- \frac{d^2 C_0}{da_{,}'^2} \frac{m' a_{,}'^2 e_{,}'^2}{8} \text{ (from the second term of } R').$$

The part of R which is independent of periodic terms equals the sum of these parts. We shall call this sum F;

$$\therefore \; F = - \frac{m'}{2} C_0$$

$$+ \frac{m'}{8} \left(a'^2 D_0 + a_{,} \frac{dC_0}{da_{,}} \right) \tan^2 i_{,} + \frac{m'}{8} \left(a_{,}'^2 D_0 + a_{,}' \frac{dC_0}{da_{,}'} \right) \tan^2 i_{,}'$$

$$- \frac{m'}{4} a_{,} a_{,}' \tan i_{,} \tan i_{,}' D_1 \cos (\Omega_{,} - \Omega_{,}')$$

$$-\frac{m'}{4}\left(a_{,}\frac{dC_0}{da_{,}}+\tfrac{1}{2}a_{,}^2\frac{d^2C_0}{da_{,}^2}\right)e_{,}^2-\frac{m'}{4}\left(a_{,}'\frac{dC_0}{da_{,}'}+\tfrac{1}{2}a_{,}'^2\frac{d^2C_0}{da_{,}'^2}\right)e_{,}'^2$$

$$-\frac{m'}{4}\left(4C_1+2a_{,}\frac{dC_1}{da_{,}}+a_{,}a_{,}'\frac{d^2C_1}{da_{,}da_{,}'}+2a_{,}'\frac{dC_1}{da_{,}'}\right)e_{,}e_{,}'\cos(\varpi_{,}-\varpi_{,}').$$

Now $\frac{1}{2}C_0+C_1\cos(\theta_{,}-\theta_{,}')+\ldots=\{a_{,}^2+a_{,}'^2-2a_{,}a_{,}'\cos(\theta_{,}-\theta_{,}')\}^{-\frac{1}{2}}$;

$$\therefore\ \tfrac{1}{2}\frac{dC_0}{da_{,}}+\ldots=-\{\tfrac{1}{2}D_0+D_1\cos(\theta_{,}-\theta_{,}')+\ldots\}.\{a_{,}-a_{,}'\cos(\theta_{,}-\theta_{,}')\}$$

$$=-\left(\frac{a_{,}}{2}D_0-\frac{a_{,}'}{2}D_1\right)+\ldots\ldots$$

$$\therefore\ a_{,}^2D_0+a_{,}\frac{dC_0}{da_{,}}=a_{,}a_{,}'D_1.$$

Similarly, $a_{,}'^2D_0+a_{,}'\dfrac{dC_0}{da_{,}'}=a_{,}'a_{,}D_1.$

Wherefore putting the coefficients of the last three terms of F equal to B, B', C, we have

$$F=-\frac{m'}{2}C_0+\frac{m'a_{,}a_{,}'}{8}D_1(\tan^2 i_{,}+\tan^2 i_{,}')$$

$$-\frac{m'a_{,}a_{,}'}{4}\tan i_{,}\tan i_{,}'D_1\cos(\Omega_{,}-\Omega_{,}')$$

$$-m'Be_{,}^2-m'B'e_{,}'^2-m'Ce_{,}e_{,}'\cos(\varpi_{,}-\varpi_{,}')\ *$$

in which we observe that C is symmetrical with respect to $a_{,}$ and $a_{,}'$.

373. In Art. 366. we collected the formulæ for calculating the elements of the instantaneous ellipse at any time. Since the object of the present work is only to explain the theory and not to enter into the numerical calculations of the perturbations, we shall proceed to demonstrate a few of the most important and interesting results to which these equations conduct us.

* We might shew that $B=B'=\frac{1}{8}a_{,}a_{,}'D_1$; but there is no occasion for this in what follows.

PROP. *To shew that the effect of all the terms of* R *(after the first) upon the elements of the planetary orbits is periodical.*

374. Any term $P \cos\{(pn_, \pm qn_,')t + Q\}$ will produce a similar term in $\frac{dR}{da_,}$, $\frac{dR}{de_,}$ and $\frac{dR}{di_,}$; but a term of the form $P \sin\{(pn_, \pm qn_,')t + Q\}$ in $\frac{dR}{d\epsilon_,}$, $\frac{dR}{d\varpi_,}$ and $\frac{dR}{d\Omega_,}$: since Q is independent of $a_,$, $e_,$, $i_,$; and P is independent of $\epsilon_,$, $\varpi_,$, $\Omega_,$. If then these be substituted in the equations of Art. 366. and the integrations be effected, the elements $a_,$, $e_,$, $\varpi_,$, $i_,$, $\Omega_,$ will receive, in consequence, a term of the form

$$\frac{P}{pn_, \pm qn_,'} \begin{matrix}\cos\\ \sin\end{matrix} \{(pn_, \pm qn_,')t + Q\};$$

since the formula for $\frac{d\epsilon_,}{dt}$ contains a term multiplied by t, the element $\epsilon_,$ will receive, besides this, terms of the form (as may be shewn by integrating by parts)

$$\frac{Pt}{(pn_, \pm qn_,')} \cos\{(pn_, \pm qn_,')t + Q\}$$

$$+ \frac{P}{(pn_, \pm qn_,')^2} \sin\{(pn_, \pm qn_,')t + Q\}.$$

It follows, then, that after a period of time $= \frac{360}{pn_, \pm qn_,'}$, the perturbations of the elements, arising from the above term in R, will have gone through their changes.

These variations of the elements are therefore termed *Periodic Variations.*

It will be remarked that if $pn_, + qn_,'$ or $pn_, - qn_,'$ be a very small quantity the integration described in the last Article will increase the corresponding terms considerably: and therefore it may happen that terms in R, of which the coefficients are so small as to appear of no consequence may rise to importance by receiving in the process of integration a small divisor.

PROP. *To find what terms in the development of* R *will be much increased by the process of integration in determining the elliptic elements.*

375. By reference to the last Article we see that either *First*, $pn_{,} + qn_{,}'$ must be a small quantity: hence, since p and q are positive integers or zero, $n_{,}$ and $n_{,}'$ must be small. By reference to the first Table in Art. 391. we see that this is not the case with any of the planets.

Or *Secondly*: $pn_{,} \sim qn_{,}'$ must be small.

Hence p and q must be in the ratio $n_{,}' : n_{,}$ as nearly as possible. Now the lowest dimension of the coefficient in terms of small quantities is $p \sim q$, Art. 370. If, then, we can find two integers p and q nearly in the ratio $n_{,}' : n_{,}$ and having a small difference, the corresponding term of R will rise into importance by the integration. If we turn, now, to the first Table in Art. 391, and by continued fractions find the convergents which express the ratio of the values of $n_{,}$ for any two planets, and choose those of them which have a small difference between the numerator and denominator, we shall be able to detect the most important of the terms of the development of R.

For Jupiter and Saturn $n_{,} : n_{,}' :: 5 : 2$ nearly, and $5 - 2 = 3$: hence the dimension of the coefficient of a term $P \cos \{(2n_{,} - 5n_{,}')t + Q\}$ will be of the third order and will be divided by the small quantities $(2n_{,} - 5n_{,}')$ and $(2n_{,} - 5n_{,}')^2$.

For the Earth and Venus $n_{,} : n_{,}' :: 8 : 13$ nearly and $13 - 8 = 5$: hence the order of small quantities in the coefficient will be of the fifth degree and the argument

$$(13n_{,} - 8n_{,}')t + Q.$$

376. These two examples present very remarkable instances of the agreement of theory with observation.

The observations upon Jupiter and Saturn from the times of the Chinese and Arabian Astronomers down to the present day prove, that for ages the mean motions of these planets have been affected by an inequality of long period. This formed an apparent anomaly in the Planetary Theory till Laplace pointed out the real cause of the inequality, and rescued Newton's doctrine of Gravity from the reproach which had long attached to it in consequence of its inability to assign the cause of so remarkable a phenomenon. Laplace proved that the inequality depends upon the near commensurability of the mean motions of the planets (as explained in Art. 375), and succeeded in calculating its period and amount.

Mr Airy has discovered a similar inequality in the motion of the Earth and Venus. In the *Phil. Trans.* for 1832 he shews that it amounts to no more than a few seconds at its maximum, though its period is no less than 240 years. Mr Airy had detected an error in the solar tables, and this induced him to seek for the cause, which is so satisfactorily shewn to arise from the near commensurability of the mean motions of the Earth and Venus.

PROP. *To explain the difference between Periodic Variations and Secular Variations.*

377. In the last Proposition we have supposed the elements which are involved in the right hand side of the equations to be constant, while they are in fact functions of t. The only effect however which would result from this consideration would be that the *period* of the variations would be slightly altered.

But if we consider the effect of the first part of the expansion of R, which is independent of the periodic terms, and which we call F, and suppose the elements involved in F *constant*, it is evident, that by the integration of the equations of Art. 366. the elements will receive additions which continually increase or decrease with the time, unless in any instance the right hand side of the equation vanishes when F is put for R. If, however, we make a nearer approximation, and suppose that the elements in F are variable, and then integrate the equations of Art. 366, the integrals *may* give periodical values for the elements. If this be the case in any instance the variation is not called a *Periodic Variation*, though in fact it is periodical, but a *Secular Variation;* since it arises from a cause quite different from that, which produces the periodic variations. In short a periodic variation arises from the fact of R involving r and θ the co-ordinates of the planet disturbed: but a secular variation arises from the fact that the elements of the orbit vary. And since they vary very slowly, the period in which they perform their secular variations is of immense duration*. Perhaps the following observations may throw light upon this subject.

The magnitude of the forces which disturb the elliptic motion of the planets depends solely upon their relative positions, and not on their velocities and the directions of their motion. When therefore, after a lapse of years, the planets return to the same relative positions that they occupied at the commencement of that period, the disturbing forces and the perturbations in the places of the planets will have gone through a series of changes, compensating in one part of this period for the errors they have caused in some other part. The inequalities produced during this interval of time are termed Periodical Variations. But although the configuration of the planetary system may become the same, yet, as was before mentioned, the velocities and directions of the motion of the planets will not necessarily become the same also; the original and final orbits intersecting respectively in those points which the planets occupy at the beginning and end of the time, which the periodic variations have taken to go through their changes. The inequalities produced in this way are termed Secular Variations in consequence of their very slow variation.

We proceed now to the examination of the Secular Variations.

PROP. *To obtain the equations for calculating the Secular Variations of the elliptic elements of a planet's orbit.*

378. We must first find the differential coefficients of F (the part of R independent of the periodic terms) with respect to the elements: hence by Art. 372.

* The *periodic variation* of longest duration among those that are of sufficient importance to be calculated has a period equal to 929 years. But some of the *secular variations* have a period of 70000 and even more years.

$$\frac{dF}{d\epsilon_{,}} = 0,$$

$$\frac{dF}{d\varpi_{,}} = m' Ce_{,}e_{,}' \sin(\varpi_{,} - \varpi_{,}'),$$

$$\frac{dF}{de_{,}} = -2m' Be_{,} - m'Ce_{,}' \cos(\varpi_{,} - \varpi_{,}'),$$

$$\frac{dF}{d\Omega_{,}} = \frac{m'}{4} a_{,}a_{,}' \tan i_{,} \tan i_{,}' D_1 \sin(\Omega_{,} - \Omega_{,}'),$$

$$\frac{dF}{di_{,}} = m' \frac{a_{,}a_{,}'}{4} D_1 \tan i_{,} - \frac{m'}{4} a_{,}a_{,}' \tan i_{,}' D_1 \cos(\Omega_{,} - \Omega_{,}').$$

Substituting these in the equations of Art. 366.

$$\frac{da_{,}}{dt} = 0,$$

$$\frac{de_{,}}{dt} = \frac{n_{,}a_{,}m'Ce_{,}'}{\mu} \sin(\varpi_{,} - \varpi_{,}'),$$

$$\frac{d\varpi_{,}}{dt} = \frac{n_{,}a_{,}m'\sqrt{1-e_{,}^2}}{\mu e_{,}} \{2Be_{,} + Ce_{,}' \cos(\varpi_{,} - \varpi_{,}')\},$$

$$\frac{d \tan i_{,}}{dt} = \frac{n_{,}a_{,}m'a_{,}a_{,}' \tan i_{,}' D_1}{4\mu} \sin(\Omega_{,} - \Omega_{,}'),$$

$$\frac{d\Omega_{,}}{dt} = \frac{n_{,}a_{,}^2 a_{,}' m' D_1}{4\mu} \left\{-1 + \frac{\tan i_{,}'}{\tan i_{,}} \cos(\Omega_{,} - \Omega_{,}')\right\}.$$

We have retained the variable values of the elements on the right hand of these equations: but should it be necessary, we may use the constant values in approximating.

Prop. *To prove the stability of the mean distances of the planets from the Sun: and of their mean motions.*

379. By Art. 378 $\frac{da_{,}}{dt} = 0$; $\therefore a_{,} = \text{const.}$

This shews that the axis-major of any of the planets is susceptible of no secular variation; and will suffer no permanent change: the changes it undergoes in consequence of the mutual attraction of the planets are wholly periodical. The same is true of the mean motion $n_{,}$, since it $= \sqrt{\frac{\mu}{a_{,}^3}}$, and μ does not alter. We are hereby assured of the impossibility of any of the bodies of our system ever leaving it in consequence of the disturbances it may experience from the other bodies, and secures the general permanence of the whole by keeping the mean distances and periodic times perpetually fluctuating between certain limits (very restricted ones) which they can never exceed, nor fall short of.

Prop. *To prove the stability of the eccentricities of the planetary orbits.*

380. By Art. 378 we have

$$\frac{de_{,}}{dt} = \frac{n_{,}a_{,}m'Ce_{,}'}{\mu} \sin(\varpi_{,} - \varpi_{,}').$$

In the same way we should have

$$\frac{de_{,}'}{dt} = \frac{n_{,}'a_{,}'mCe_{,}}{\mu} \sin(\varpi_{,}' - \varpi_{,}).$$

Multiply these by $\frac{m}{n_{,}a_{,}} e_{,}$ and $\frac{m'}{n_{,}'a_{,}'} e_{,}'$ and add them

$$\therefore \frac{m}{n_{,}a_{,}} e_{,} \frac{de_{,}}{dt} + \frac{m'}{n_{,}'a_{,}'} e_{,}' \frac{de_{,}'}{dt} = 0$$

$$\therefore \frac{m}{n_{,}a_{,}} e_{,}^2 + \frac{m'}{n_{,}'a_{,}'} e_{,}'^2 = \text{constant}.$$

If we had considered three planets we should have had the following equations

$$\frac{de_{,}}{dt} = \frac{n_{,}a_{,}m'Ce_{,}'}{\mu}\sin(\varpi_{,} - \varpi_{,}') + \frac{n_{,}a_{,}m''C'e_{,}''}{\mu}\sin(\varpi_{,} - \varpi_{,}''),$$

$$\frac{de_{,}'}{dt} = \frac{n_{,}'a_{,}'mCe_{,}}{\mu}\sin(\varpi_{,}' - \varpi_{,}) + \frac{n_{,}'a_{,}'m''C'''e_{,}''}{\mu}\sin(\varpi_{,}' - \varpi_{,}''),$$

and

$$\frac{de_{,}''}{dt} = \frac{n_{,}''a_{,}''mC'e_{,}}{\mu}\sin(\varpi_{,}'' - \varpi_{,}) + \frac{n_{,}''a_{,}''m'C'''e_{,}'}{\mu}\sin(\varpi_{,}'' - \varpi_{,}').$$

These equations give

$$\frac{m}{n_{,}a_{,}}e_{,}\frac{de_{,}}{dt} + \frac{m'}{n_{,}'a_{,}'}e_{,}'\frac{de_{,}'}{dt} + \frac{m''}{n_{,}''a_{,}''}e_{,}''\frac{de_{,}''}{dt} = 0;$$

$$\therefore \frac{m}{n_{,}a_{,}}e_{,}^{2} + \frac{m'}{n_{,}'a_{,}'}e_{,}'^{2} + \frac{m''}{n_{,}''a_{,}''}e_{,}''^{2} = \text{constant}.$$

And the same formula would be true of any number of planets.

Now observation shews that the eccentricities of the orbits of the planets at present are very small indeed, with the exception of the Asteroids, the masses of which are very small. Hence the above constant must be small. Since, then, all the terms of the first side of the last equation are positive and their sum always equals a small constant it follows, that the terms are always small and the eccentricities are always small.

Hence the eccentricities of the orbits of the planets are confined within very restricted limits: and therefore the forms of the orbits are said to be stable.

381. The only quantities in the above equation, subject to a change of sign in applying it to a system of bodies, are the mean motions $n_{,}$, $n_{,}'$, $n_{,}''$, ...But observation shews that all the planets revolve round the Sun in the same direction: and consequently the terms are all positive.

PROP. *To prove the stability of the inclinations of the planets of the Solar System.*

382. By Art. 378 we have

$$\frac{d \tan i_{,}}{dt} = \frac{n_{,} a_{,}^{2} a_{,}' m' \tan i_{,}' D_1}{4\mu} \sin (\Omega_{,} - \Omega_{,}'),$$

in the same way

$$\frac{d \tan i_{,}'}{dt} = \frac{n_{,}' a_{,}'^{2} a_{,} m \tan i_{,} D_1}{4\mu} \sin (\Omega_{,}' - \Omega_{,}),$$

$$\therefore \frac{m}{n_{,} a_{,}} \tan i_{,} \frac{d \tan i_{,}}{dt} + \frac{m'}{n_{,}' a_{,}'} \tan i_{,}' \frac{d \tan i_{,}'}{dt} = 0;$$

$$\therefore \frac{m}{n_{,} a_{,}} \tan^2 i_{,} + \frac{m'}{n_{,}' a_{,}'} \tan^2 i_{,}' = \text{constant}.$$

The same equation would (as in the eccentricities) be true for any number of planets: and we see that the inclinations must always be small. The certainty of this fact depends, as before, upon the fact that the planets all revolve in the same direction; Art. 381.

383. We are thus led to the following remarkable conclusion: *The fact that the planets revolve about the Sun in the same direction ensures the stability of the planetary system.*

The converse of this would not necessarily be true, as we shall see in Arts. 385, 387: the numerical relations of the dimensions and positions of the orbits of the planets might be such as to ensure stability although they revolved in opposite directions. But the above is independent of particular numerical relations.

384. We have given the two foregoing Propositions because of the simplicity of their demonstrations as well as the beauty of the results. We shall, however, in the following Articles obtain formulæ for calculating the magnitude of the variations of the orbits in dimension and position.

PROP. *To find the secular variation of the eccentricity of the planetary orbits.*

385. By Art. 378 we have for the planet m,

$$\frac{de_{,}}{dt} = \frac{n_{,}a_{,}m'Ce_{,}'}{\mu} \sin(\varpi_{,} - \varpi_{,}');$$

$$\frac{d\varpi_{,}}{dt} = \frac{n_{,}a_{,}m'\sqrt{1-e_{,}^2}}{\mu e_{,}} \{2Be_{,} + Ce_{,}' \cos(\varpi_{,} - \varpi_{,}')\}.$$

And for the other planet m',

$$\frac{de_{,}'}{dt} = \frac{n_{,}'a_{,}'mCe_{,}}{\mu} \sin(\varpi_{,}' - \varpi_{,}),$$

$$\frac{d\varpi_{,}'}{dt} = \frac{n_{,}'a_{,}'m\sqrt{1-e_{,}'^2}}{\mu e_{,}'} \{2B'e_{,}' + Ce_{,} \cos(\varpi_{,}' - \varpi_{,})\};$$

observing that C is the same for m and m', (Art. 372).

To integrate these equations assume

$$r = e_{,} \sin\varpi_{,}, \quad s = e_{,} \cos\varpi_{,},$$

$$r' = e_{,}' \sin\varpi_{,}', \quad s' = e_{,}' \cos\varpi_{,}';$$

$$\therefore \frac{dr}{dt} = e_{,} \cos\varpi_{,} \frac{d\varpi_{,}}{dt} + \frac{de_{,}}{dt} \sin\varpi_{,}$$

$$= \frac{n_{,}a_{,}m'}{\mu} \{2Be_{,} \cos\varpi_{,} + Ce_{,}' \cos\varpi_{,}'\}$$

$$= \frac{n_{,}a_{,}m'}{\mu} \{2Bs + Cs'\},$$

$$\frac{ds}{dt} = -e_{,} \sin\varpi_{,} \frac{d\varpi_{,}}{dt} + \frac{de_{,}}{dt} \cos\varpi_{,}$$

$$= -\frac{n_{,}a_{,}m'}{\mu} \{2Br + Cr'\},$$

$$\frac{dr'}{dt} = \frac{n_{,}'a_{,}'m}{\mu} \{2B's' + Cs\},$$

$$\frac{ds'}{dt} = -\frac{n_{,}'a_{,}'m}{\mu} \{2B'r' + Cr\}.$$

These four are linear equations and their solutions are of the form

$$r = D\sin(gt+k) + E\sin(ht+l)$$
$$s = D\cos(gt+k) + E\cos(ht+l)$$
$$r' = D'\sin(gt+k) + E'\sin(ht+l)$$
$$s' = D'\cos(gt+k) + E'\cos(ht+l).$$

If we put these values in the differential equations we arrive at the four following conditions connecting the eight constants, four of which are consequently arbitrary and depend upon the configuration of the planetary system.

$$Dg = \frac{n_{,}a_{,}m'}{\mu}\{2BD + CD'\},$$

$$Eh = \frac{n_{,}a_{,}m'}{\mu}\{2BE + CE'\},$$

$$D'g = \frac{n_{,}'a_{,}'m}{\mu}\{2B'D' + CD\},$$

$$E'h = \frac{n_{,}'a_{,}'m}{\mu}\{2B'E' + CE\}.$$

By eliminating D' from the first and third of these, we have

$$\left\{g - \frac{2n_{,}a_{,}m'B}{\mu}\right\}\cdot\left\{g - \frac{2n_{,}'a_{,}'mB'}{\mu}\right\} = \frac{n_{,}n_{,}'a_{,}a_{,}'mm'C^2}{\mu^2};$$

$$\therefore\ g = \frac{n_{,}a_{,}m'B + n_{,}'a_{,}'mB'}{\mu} \pm \frac{1}{\mu}\sqrt{(n_{,}a_{,}m'B - n_{,}'a_{,}'mB')^2 + n_{,}n_{,}'a_{,}a_{,}'mm'C^2}.$$

In a similar way we might shew that h has the same values.

Now these values of g and h are possible when $n_{,}$ and $n_{,}'$ have the same sign; that is, when the planets revolve in the same direction about the Sun. But even if they do not revolve in the same direction and $n_{,}a_{,}m'B + n_{,}'a_{,}'mB'$ be not less than $\sqrt{n_{,}n_{,}'a_{,}a_{,}'m'mC}$, then g and h are still possible.

Now $e_{,}^2 = r^2 + s^2 = D^2 + E^2 + 2DE\cos\{(g-h)t + k - l\}$, and a similar expression is true for $e_{,}'^2$.

This shews that the eccentricity of m's orbit fluctuates between the limits $D+E$ and $D \sim E$. Hence the *form* of the orbit will be stable: the same is true of m''s orbit.

The values of D and E are very small in all the planets, this is shewn by observation.*

The periods of the changes in the eccentricities of the orbits of the two planets are the same in each, being $\frac{360^0}{g-h}$. In the case of Jupiter and Saturn this equals 70414 years! The greatest and least eccentricities which Jupiter's orbit can attain are 0.06036 and 0.02606, and those of Saturn 0.08409 and 0.01345; the maximum of each taking place at the time of the minimum of the other, and vice versâ.

PROP. *To find the secular variation of the longitude of the perihelion.*

386. By Art. 385, $\tan \varpi_{,} = \frac{r}{s}$

$$= \frac{D \sin (gt+k) + E \sin (ht+l)}{D \cos (gt+k) + E \cos (ht+l)}.$$

The maxima and minima values of $\varpi_{,}$, or the greatest deviations of the perihelion, from its mean place are found by the equation

$$gD^2 + hE^2 + DE(g+h) \cos \{(g-h)t + (k-l)\} = 0,$$

$$\text{or } \cos \{(g-h)t + (k-l)\} = -\frac{gD^2 + hE^2}{DE(g+h)}$$

which is obtained by equating to zero the differential coefficient of $\tan \varpi_{,}$.

* Sir John Herschel finds that

$$D = -0.01715, \quad E = 0.04321 \text{ for Jupiter,}$$

$$D' = 0.04877, \quad E' = 0.03532 \text{ for Saturn,}$$

$$g = 21''.9905, \quad h = 3''.5851, \quad k = 306^0\,34'\,40'', \quad l = 210^0\,16'\,40'',$$

t being the number of *years* since the year A.D. 1700. See Article *Physical Astronomy* in *Encyclop. Metrop.*

If this (disregarding the sign) be not greater than unity, the perihelion will vibrate: but if, as is the case with Jupiter and Saturn, this be greater than unity the longitude of the perihelion has no maximum or minimum and therefore the mean motion of the perihelion is continually in one direction.

Prop. *To find the secular variation of the inclination.*

387. By Art. 378. we have for m

$$\frac{d\tan i_{,}}{dt} = \frac{n_{,}a_{,}m'a_{,}a_{,}'\tan i_{,}'D_1}{4\mu}\sin(\Omega_{,} - \Omega_{,}')$$

$$\frac{d\Omega_{,}}{dt} = \frac{n_{,}a_{,}'a_{,}^2m'D_1}{4\mu}\left\{-1 + \frac{\tan i_{,}'}{\tan i_{,}}\cos(\Omega_{,} - \Omega_{,}')\right\},$$

and for the planet m'

$$\frac{d\tan i_{,}'}{dt} = \frac{n_{,}'a_{,}'m a_{,}a_{,}'a_{,}\tan i_{,}D_1}{4\mu}\sin(\Omega_{,}' - \Omega_{,})$$

$$\frac{d\Omega_{,}'}{dt} = \frac{n_{,}'a_{,}'^2a_{,}mD_1}{4\mu}\left\{-1 + \frac{\tan i_{,}}{\tan i_{,}'}\cos(\Omega_{,}' - \Omega_{,})\right\}.$$

To integrate these, assume

$$p = \tan i_{,}\sin\Omega_{,}, \quad q = \tan i_{,}\cos\Omega_{,}$$

$$p' = \tan i_{,}'\sin\Omega_{,}', \quad q' = \tan i_{,}'\cos\Omega_{,}';$$

$$\therefore \frac{dp}{dt} = \tan i_{,}\cos\Omega_{,}\frac{d\Omega_{,}}{dt} + \sin\Omega_{,}\frac{d\tan i_{,}}{dt}$$

$$= \frac{n_{,}a_{,}^2a_{,}'m'D_1}{4\mu}(q' - q)$$

$$\frac{dq}{dt} = \frac{n_{,}a_{,}^2a_{,}'m'D_1}{4\mu}(p - p'),$$

$$\frac{dp'}{dt} = \frac{n_{,}'a_{,}a_{,}'^2mD_1}{4\mu}(q - q'), \quad \frac{dq'}{dt} = \frac{n_{,}'a_{,}a_{,}'^2mD_1}{4\mu}(p' - p).$$

The integrals are of the form

$$p = G\sin(\alpha t+\gamma) + H\sin(\beta t+\delta), \quad q = G\cos(\alpha t+\gamma) + H\cos(\beta t+\delta),$$

$$p' = G'\sin(\alpha t+\gamma) + H'\sin(\beta t+\delta), \quad q' = G'\cos(\alpha t+\gamma) + H'\cos(\beta t+\delta).$$

Substituting in the differential equations we have

$$G\alpha = \frac{n_, a_,^2 a_,' m' D_1}{4\mu}(G' - G), \quad H\beta = \frac{n_, a_,^2 a_,' m' D_1}{4\mu}(H' - H),$$

$$G'\alpha = \frac{n_,' a_, a_,'^2 m D_1}{4\mu}(G - G'), \quad H'\beta = \frac{n_,' a_, a_,'^2 m D_1}{4\mu}(H - H').$$

Eliminating G from the first and third

$$\left(\alpha + \frac{n_, a_,^2 a_,' m' D_1}{4\mu}\right)\left(\alpha + \frac{n_,' a_, a_,'^2 m D_1}{4\mu}\right) = \frac{n_, n_,' a_,^3 a_,'^3 m m' D_1^2}{16\mu^2};$$

$$\therefore\ \alpha^2 + \frac{(n_, a_, m' + n_,' a_,' m)\, a_, a_,' D_1}{4\mu}\,\alpha = 0.$$

We should arrive at the same equation for β: hence

$$\alpha = -\frac{n_, a_, m' + n_,' a_,' m}{4\mu}\, a_, a_,' D_1 \text{ and } \beta = 0,$$

$$\tan^2 i_, = p^2 + q^2 = G^2 + H^2 + 2GH\cos\{\alpha t + \gamma - \delta\}.$$

Observation proves that G and H are very small for all the planets (except the Asteroids). Hence the tangent of inclination fluctuates between the small limits $G + H$ and $G \sim H$*.

The period of the changes in the inclination equals $\frac{360^0}{\alpha}$ years. In the case of Jupiter and Saturn the number of years is 50673! The maximum and minimum inclinations of Jupiter's orbit to the ecliptic are $2^0\ 2'\ 30''$ and $1^0\ 17'\ 10''$: and those of Saturn are $2^0\ 32'\ 40''$ and $0^0\ 47'$. The maximum of each takes place at the time of the minimum of the other, and vice versâ.

* Sir John Herschel shews that when Jupiter and Saturn are the two planets,

$G = -0.00661$, $H = -0.02905$ for Jupiter.

$G' = 0.01537$, $H' = 0.02905$ for Saturn.

$\alpha = -25''.5756$, $\gamma = 125^0\ 15'\ 40''$, $\delta = 103^0\ 38'\ 40''$,

t being the number of years since A.D. 1700.

PROP. *To find the secular variation of the longitude of the node.*

388. By Art. 378, we have

$$\tan \Omega_{,} = \frac{p}{q} = \frac{G \sin (at + \gamma) + H \sin \delta}{G \cos (at + \gamma) + H \cos \delta}.$$

When $\Omega_{,}$ attains a maximum or minimum value the differential coefficient of $\tan \Omega_{,}$ equals zero: hence

$$0 = aG^2 + GHa \cos (at + \gamma - \delta);$$

$$\therefore \cos (at + \gamma - \delta) = -\frac{G}{H}.$$

If this (disregarding the sign) be not greater than unity, then the node fluctuates, the period of its fluctuation being $\frac{360^0}{a}$ years. But if this be greater than unity then there cannot be any stationary positions; but the node continually moves in one direction.

In the case of Saturn and Jupiter the node oscillates, the extent of oscillation being about $13^0\ 9'\ 40''$ in Jupiter's orbit and $31^0\ 56'\ 20''$ in Saturn's on either side of their mean positions: the plane of the ecliptic being supposed immoveable.

389. The conclusions at which we have arrived in Arts. 379—383, with regard to the stability of the planetary system are of especial interest. In consequence of the changes in the elements we might have fancied that in the lapse of ages the orbits would undergo such alterations in their dimensions as to bring the planets into collision or hurry them into boundless space. But we are assured that this can never be the case, unless by the action of a resisting medium; since analysis shews us that the orbits will continually fluctuate within very small limits, never departing considerably from circles; and the inclinations of the orbits will never change much.

390. Our calculations have not included the square of the disturbing forces. But the same conclusions are found to hold when the approximation is carried so far as analysts have at present advanced: see the *Mécanique Céleste*, Liv. VI; Pon-

técoulant's *Système du Monde*, Tom. III; Plana's *Planetary Theory* in the *Memoirs of the Astronomical Society*, Vol. II.; also a Memoir by Professor Hansen of Seeberg, the title of this Memoir is *Untersuchung üeber die gegenseitigen Störungen des Jupiters und Saturns*. In this method the true longitude is computed by means of the elements corresponding to the *invariable ellipse at the time of the epoch*, taking a function of t, instead of t, which corrects for the perturbations. See M. Pontécoulant's remarks on this in the *Connaissance des Tems* for 1837. And lastly Mr Lubbock's papers in the Transactions of the Royal Society and of the Astronomical Society may be consulted.

Prop. *To shew how the masses of the planets may be discovered.*

391. There are in general two methods of determining the masses of the planets; either by observing the elongations of a satellite, when the planet is accompanied by a satellite; or by comparing the inequalities produced in their motion by their mutual action. The secular variations are best adapted to give the most exact results; but these are not yet known with sufficient accuracy to allow of this use. We are therefore obliged to recur to the periodic variations, and, by combining a vast number of observations, gather from them the most probable results. It is by these means that Astronomers have obtained the following results.

Mass of Sun	1
.... Mercury	$\frac{1}{1909706}$
.... Venus	$\frac{1}{401839}$
.... Earth	$\frac{1}{356354}$
.... Mars	$\frac{1}{2680337}$

Mass of Jupiter........ $\frac{1}{1053.924}$

.... Saturn......... $\frac{1}{3512}$

.... Herschel....... $\frac{1}{17918}$.

We have taken these from M. Pontécoulant, *Système du Monde*, Tom. III. p. 341. The following is the formula for calculating the mass when the planet has a satellite.

Let 1, M, m be the masses of the Sun, the planet, and the satellite: T, t the periodic times of the planet about the Sun, and the satellite about the planet: A, a the mean distances of the planet from the Sun, and the satellite from the planet. Hence by Art. 269,

$$T = \frac{2\pi A^{\frac{3}{2}}}{\sqrt{1+M}}, \quad t = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{M+m}}; \quad \therefore \frac{M+m}{1+M} = \frac{a^3}{A^3}\,\frac{T^2}{t^2};$$

therefore (if we neglect m) $M = \dfrac{1}{\dfrac{A^3}{a^3}\,\dfrac{t^2}{T^2} - 1}$.

In the case of Jupiter and his fourth satellite, we find by this formula $M = \frac{1}{1048.69}$: this is more properly the mass of Jupiter with that of his fourth satellite.

The first value of the mass of Jupiter determined by Laplace (*Méc. Cél.* Liv. VI. §. 21.) is $\frac{1}{1067.09}$, and is founded on the observed elongations of the satellites by Pound. These elongations have been lately observed with much greater accuracy by Mr Airy at the Observatory of the University of Cambridge, the result of his measures gives $\frac{1}{1048.69}$; *Astronomische Nachrichten*, Vol. X. p. 304. Nicolai makes the mass $\frac{1}{1053.924}$ by observing the perturbations of Juno. Encke makes it $\frac{1}{1050.117}$ by observing the motion of Vesta, and $\frac{1}{1054.4}$ by observations on the comet which bears his name. All these concur in proving that the mass of Jupiter assumed by Laplace is too small by about $\frac{1}{70}$th part. The observations of Bouvard, however, are at variance with this: he gives $\frac{1}{1070.5}$. The mass of the Earth may be determined as follows.

The attraction of the Earth on a body at its surface in the parallel of which the square of the sine of the latitude is $\frac{1}{3}$, is very nearly the same as if the Earth were condensed into its centre: as we shall see in *the Figure of the Earth* in a subsequent Chapter. Let $\sin^2 l = \frac{1}{3}$, $g =$ gravity in latitude l, b the mean radius of the Earth, 1 and E the masses of the Sun and Earth, T the length of the year, a the mean radius of the Earth's orbit: hence

$$g = \frac{E}{b^2} \text{ and } T = 2\pi a^{\frac{3}{2}}; \quad \therefore E = \frac{T^2 g}{4\pi^2} \frac{b^2}{a^3},$$

$$\frac{b}{a} = \sin \text{ Sun's parallax} = \sin 8''.7.$$

The mass of the Moon $= \frac{1}{74}$ nearly. *Méc. Cél.* Liv. VI. §. 44. But this is not yet very satisfactorily determined: we have seen no value deduced from the observations mentioned in Art. 348.

392. We extract the following Table from M. Pontécoulant's *Système du Monde*. These results are obtained by the methods mentioned in Art. 270.

Epoch is Jan. 1. 1800.	Mean Motions in a Year of 365¼ Days.	Mean Distance from Sun.	Eccentricities.	Longitudes of Epochs.	Longitudes of Perihelia.	Inclinations.	Longitudes of Ascending Node.
Mercury	5323416″.79	0.38709812	0.2055149	110° 13′ 17″.9	74° 21′ 41″	7° 00′ 9″	45° 57′ 39″
Venus	2106641 .52	0.72333230	0.0068531	145 56 52 .1	128 43 6	3 23 29	74 52 39
Earth	1295977 .35	1.00000000	0.0168536	100 23 32 .6	99 29 53	0 00 00	0 00 00
Mars	689051 .12	1.52369352	0.0933061	232 49 50 .5	332 23 40	1 50 6	48 00 26
Jupiter	109256 .29	5.20116636	0.0481621	81 52 10 .3	11 7 36	1 18 52	98 25 45
Saturn	43996 .72	9.53787090	0.0561505	123 5 29 .4	89 8 20	2 29 38	111 56 7
Herschel	15424 .54	19.18330500	0.0466108	173 30 16 .6	167 30 24	46 26	72 59 21

Table of Secular Inequalities of the Planets calculated for the beginning of the Year 1801.

	In the Eccentricity.	In the Long. of Perihelion.	In the Long. of the Node.	In the Inclin. of Orbit to Ecliptic.
Mercury	0.000003867	9′ 43″.5	– 13′ 22″	19″.8
Venus	0.000062711	4′ 28″	– 31′ 10″	4″.5
Earth	0.000041200	11″.9496		
Mars	0.000090176	26″.22	– 38′ 48″	1″.5
Jupiter	0.00015935	11′ 4″	– 26′ 17″	23″
Saturn	0.000312402	31′ 17″	– 37′ 54″	15′ 5″
Herschel	0.000025072	4′	– 59′ 57″	3″.7

To obtain equations for calculating the effect of a resisting medium upon a comet we must refer the reader to the *Mécanique Céleste*, and also to Mr Airy's translation of the dissertation on Encke's Comet in the *Astronomische Nachrichten*.

Also for a very interesting paper on the orbits of revolving double stars the reader is referred to Vol. v. of the *Memoirs of the Astronomical Society*, in which Sir John F. W. Herschel has treated the subject in a very original manner.

The following are Tables of the elements of the four small planets Vesta, Juno, Pallas, and Ceres: and of the four known periodical comets. The comet of Olbers has been observed only once, at the time of its return to the perihelion in 1815: the others have been observed in several successive revolutions. It must be remarked that the elements of the small planets given in the Table are not their *mean* values, but their values at the specified epoch.

Epoch 1831 July 23d. 0h. Mean Time at Berlin.	Mean Longitude.	Mean Anomaly.	Longitude of Perihelion.	Longitude of Asc. Node.	Inclination.	Eccentricity.	Mean Motion.	Mean Distance.	Period.
Vesta	$84^d 47^m 03^s$	$195^d 35^m 26^s$	$249^d 11^m 37^s$	$103^d 20^m 28^s$	$7^d 07^m 57^s$	0.0885601	$977^s.75540$	2.361484	$1325^d.5$
Juno	74 39 44	20 22 31	54 17 13	170 52 34	13 02 10	0.2555592	813 .52533	2.669464	1593 .1
Pallas	290 38 12	169 33 11	121 05 01	172 38 30	34 35 49	0.2419986	768 .54421	2.772631	1686 .3
Ceres	307 03 26	159 22 02	147 41 23	80 53 50	10 36 56	0.0767379	769 .26059	2.770907	1684 .7

Name of the Comet.	Period.	Time of Perihelion Passage.	Longitude of Perihelion on the Orbit.	Longitude of Asc. Node.	Inclination.	Eccentricity.	Mean Distance.
Halley's	76 years	Nov. 7, 1835	$304^d\ 31^m\ 43^s$	$55^d\ 30^m$	$17^d\ 44^m\ 24^s$	0.9675212	17.98705
Olbers's	74 years	Ap. 26, 1815	149 02	83 29	44 30	0.9313	17.7
Encke's	1204 days	Jan. 10, 1829	157 18 35	$334\ 24\ 15^s$	13 22 34	0.8446862	2.224346
Biela's	6.7 years	Nov. 27, 1832	109 56 45	248 12 24	13 13 13	0.751748	3.53683

CHAPTER VII.

MOTION OF A PARTICLE ON CURVES AND SURFACES. SIMPLE PENDULUM.

PROP. *A material particle moves on a curve in a vertical plane, and acted upon by gravity: required to determine the motion.*

393. Let A be the lowest point of the curve (fig. 94.) Ax the axis of x drawn vertically upwards: P the position of the body on the curve AP at the time t: $AM = x$, $MP = y$: let R be the pressure of the curve against the body, this acts in the normal line PG: M the mass of the body: then $\frac{R}{M}$ is the accelerating force resulting from the action of R (Art. 225): g the force of gravity.

Now the forces acting vertically are g downwards and $\frac{R}{M}\cos PGM$ or $\frac{R}{M}\frac{dy}{ds}$ upwards, the only horizontal force is $\frac{R}{M}\frac{dx}{ds}$.

Hence, attending to the *directions* of the forces, we have the following equations of motion:

$$\frac{d^2x}{dt^2} = -g + \frac{R}{M}\frac{dy}{ds}\ldots\ldots(1),\quad \frac{d^2y}{dt^2} = -\frac{R}{M}\frac{dx}{ds}\ldots\ldots(2).$$

Multiply these respectively by $2\frac{dx}{dt}$, $2\frac{dy}{dt}$ and add, then

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} + 2\frac{dy}{dt}\frac{d^2y}{dt^2} = -2g\frac{dx}{dt} + \frac{2R}{M}\left(\frac{dx}{dt}\frac{dy}{ds} - \frac{dy}{dt}\frac{dx}{ds}\right)$$

$$= -2g\frac{dx}{dt};$$

$$\therefore \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} = \text{const.} - 2gx,$$

$$\text{or } \frac{ds^2}{dt^2} = \text{const.} - 2gx.$$

At the commencement of the motion let $x = h$;

$$\therefore 0 = \text{const.} - 2gh;$$

$$\therefore \frac{ds^2}{dt^2} = 2g(h - x).$$

This expression shews that the velocity at any time is independent of the form of the curve on which the body moves; and depends solely on the vertical space through which it passes.

Extracting the square root and inverting the two sides of the equation

$$-\frac{dt}{ds} = \frac{1}{\sqrt{2g}} \frac{1}{\sqrt{h-x}}$$

the negative sign being taken because s diminishes as t increases (Note in page 208).

$$\therefore t = -\frac{1}{\sqrt{2g}} \int \frac{dx}{\sqrt{h-x}} \frac{ds}{dx}.$$

We must determine $\frac{ds}{dx}$ from the equation to the curve by the formula $\frac{ds}{dx} = \sqrt{1 + \frac{dy^2}{dx^2}}$, then by integration we shall know t in terms of x and therefore x in terms of t. In this manner, then, we shall know the velocity and position of the body at every assigned instant.

PROP. *To find the pressure upon the curve.*

394. The equations of motion being

$$\frac{d^2x}{dt^2} = -g + \frac{R}{M}\frac{dy}{ds}, \quad \frac{d^2y}{dt^2} = -\frac{R}{M}\frac{dx}{ds}$$

we multiply them respectively by $\frac{dy}{dt}$, $\frac{dx}{dt}$ and subtract;

$$\therefore \frac{dy}{dt}\frac{d^2x}{dt^2} - \frac{dx}{dt}\frac{d^2y}{dt^2} = -g\frac{dy}{dt} + \frac{R}{M}\left(\frac{dy}{ds}\frac{dy}{dt} + \frac{dx}{ds}\frac{dx}{dt}\right)$$

$$= -g\frac{dy}{dt} + \frac{R}{M}\frac{ds}{dt}, \quad \because \frac{dy^2}{ds^2} + \frac{dx^2}{ds^2} = 1.$$

Now if ρ be the radius of curvature of the curve on which the body moves at the point (xy), then by the Differential Calculus

$$\frac{1}{\rho} = \frac{\frac{dy}{dt}\frac{d^2x}{dt^2} - \frac{dx}{dt}\frac{d^2y}{dt^2}}{\frac{ds^3}{dt^3}}$$

t being a function of x and y, as is the case here:

$$\therefore \frac{R}{M} = g\frac{dy}{ds} + \frac{v^2}{\rho}; \quad v = \text{velocity}.$$

This expression shews that the pressure consists of two parts, one the part of the forces which act upon the body resolved along the normal, and the other the centrifugal force arising from the motion. (Art. 254.)

Prop. *A body moves on a cycloid, the axis of the cycloid being vertical: required to find the time of an oscillation and to shew that it is independent of the extent of the vibration.*

395. We have shewn that $\frac{ds^2}{dt^2} = 2g(h - x)$;

$$\therefore -\frac{dt}{ds} = \frac{1}{\sqrt{2g}}\frac{1}{\sqrt{h - x}},$$

the negative sign being taken because the arc decreases as the time increases.

Now the equation to the cycloid is

$$y = \sqrt{2ax - x^2} + a \operatorname{vers}^{-1}\frac{x}{a},$$

the lowest point being the origin;

$$\therefore \frac{dy}{dx} = \frac{a - x}{\sqrt{2ax - x^2}} + \frac{a}{\sqrt{2ax - x^2}} = \sqrt{\frac{2a - x}{x}};$$

$$\therefore \frac{ds}{dx} = \sqrt{1 + \frac{dy^2}{dx^2}} = \sqrt{\frac{2a}{x}}.$$

Hence $\dfrac{dt}{dx} = \dfrac{dt}{ds}\,\dfrac{ds}{dx} = -\sqrt{\dfrac{a}{g}}\,\dfrac{1}{\sqrt{hx - x^2}};$

$$\therefore t = C - \sqrt{\frac{a}{g}} \operatorname{vers}^{-1}\frac{2x}{h}$$

when $t = 0$, $x = h$; $\therefore 0 = C - \sqrt{\dfrac{a}{g}}\,\pi$

$$t = \sqrt{\frac{a}{g}}\left\{\pi - \operatorname{vers}^{-1}\frac{2x}{h}\right\}$$

and, whenever the body stops, the velocity, or $\dfrac{ds}{dt} = 0$; and therefore $x = h$, and the values of $\operatorname{vers}^{-1}\dfrac{2x}{h}$ when $x = h$ are

$$\pm\pi,\ \pm 3\pi,\ \pm 5\pi, \ldots\ldots$$

and therefore the values of t are

$$2\pi\sqrt{\frac{a}{g}},\quad 4\pi\sqrt{\frac{a}{g}},\quad 6\pi\sqrt{\frac{a}{g}}, \ldots\ldots$$

which shew that the body will oscillate backwards and forwards, the interval of time in which each oscillation is performed being $2\pi\sqrt{\dfrac{a}{g}}$.

This expression is independent of h and therefore points out the remarkable fact that however large the arc of vibration be the time of oscillation is the same in all.

For this reason the cycloid is called a Tautochronous Curve.*

PROP. *A particle moves on a circular arc acted upon by gravity: required the time of oscillating through a given portion of the arc.*

396. As before $\frac{ds^2}{dt^2} = 2g(h-x)$ and the equation to the circle from the lowest point is $y^2 = 2ax - x^2$;

$$\therefore \frac{dy}{dx} = \frac{a-x}{\sqrt{2ax-x^2}}, \quad \frac{ds}{dx} = \sqrt{1+\frac{dy^2}{dx^2}} = \frac{a}{\sqrt{2ax-x^2}};$$

* It may be interesting to ascertain whether there are any other tautochronous curves when gravity is the force acting.

We have $\frac{dt}{dx} = -\frac{1}{\sqrt{2g}} \frac{1}{\sqrt{h-x}} \frac{ds}{dx}$

$$= -\frac{1}{\sqrt{2g}} \frac{ds}{dx} \left\{ \frac{1}{h^{\frac{1}{2}}} + \frac{1}{2} \frac{x}{h^{\frac{3}{2}}} + \frac{1.3}{2.4} \frac{x^2}{h^{\frac{5}{2}}} + \dots\dots + \frac{1.3\dots(2n-1)}{2.4\dots 2n} \frac{x^n}{h^{\frac{2n+1}{2}}} + \dots\dots \right\}.$$

Now $\frac{ds}{dx}$ is independent of h: and consequently the integral of the general term

$$-\frac{1}{\sqrt{2g}} \cdot \frac{1.3\dots(2n-1)}{2.4\dots 2n} \frac{x^n}{h^{\frac{2n+1}{2}}} \frac{ds}{dx}$$

must be of the form $c \cdot \left(\frac{x}{h}\right)^{\frac{2n+1}{2}}$, c being a constant, in order that when taken between the limits $x=0$ and $x=h$ the result may be independent of h: then

$$\int x^n \frac{ds}{dx} dx = \frac{A}{2n+1} x^{\frac{2n+1}{2}}, \ A \text{ a constant};$$

$$\therefore \frac{ds}{dx} = \frac{A}{2} x^{-\frac{1}{2}},$$

$$\therefore s = A x^{\frac{1}{2}}$$

$$s^2 = A^2 x,$$

and this is the equation to the cycloid and therefore this is the only tautochronous curve for gravity.

$$\therefore \frac{dt}{dx} = -\frac{a}{\sqrt{2g}} \frac{1}{\sqrt{(h-x)(2ax-x^2)}}.$$

We are not able to integrate this function of x: it is reducible to one of the class called Elliptic Transcendents, the properties of which Legendre has discussed in his *Traité des Fonctions Elliptiques:* tables are given of the approximate values of the integral for given values of x*.

By means of series, however, the integral can be obtained approximately.

$$\frac{dt}{dx} = -\tfrac{1}{2}\sqrt{\frac{a}{g}} \frac{1}{\sqrt{hx-x^2}} \left(1 - \frac{x}{2a}\right)^{-\frac{1}{2}}$$

$$= -\tfrac{1}{2}\sqrt{\frac{a}{g}} \frac{1}{\sqrt{hx-x^2}}$$

$$\times \left\{1 + \tfrac{1}{2}\frac{x}{2a} + \frac{1.3}{2.4}\left(\frac{x}{2a}\right)^2 + \dots + \frac{1.3\dots(2n-1)}{2.4\dots 2n}\left(\frac{x}{2a}\right)^n + \dots\right\}.$$

Now $$\int \frac{x^n dx}{\sqrt{hx-x^2}} = \frac{2n-1}{2n} h \int \frac{x^{n-1} dx}{\sqrt{hx-x^2}} - \frac{x^{n-1}\sqrt{hx-x^2}}{n},$$

and between the limits $x = h$ and $x = 0$, we have

$$\int_h^0 \frac{x^n dx}{\sqrt{hx-x^2}} = \frac{2n-1}{2n} h \int_h^0 \frac{x^{n-1} dx}{\sqrt{hx-x^2}};$$

$$\therefore \int_h^0 \frac{x\,dx}{\sqrt{hx-x^2}} = \frac{h}{2} \operatorname{vers}^{-1} \frac{2x}{h} + \text{constant} = -\frac{\pi h}{2};$$

* Let $x = h \sin^2\theta$: then $\theta = \frac{\pi}{2}$ when $x = h$ or $t = 0$;

$$\therefore \int \frac{dx}{\sqrt{(h-x)(2ax-x^2)}} = \int \frac{2\sin\theta\cos\theta\, d\theta}{\sqrt{\cos^2\theta\,(2a - h\sin^2\theta)\sin^2\theta}}$$

$$= \frac{2}{\sqrt{2a}} \int \frac{d\theta}{\sqrt{1 - \frac{h}{2a}\sin^2\theta}};$$

which is an elliptic function of the first order.

$$\int_h^0 \frac{x^2\,dx}{\sqrt{hx - x^2}} = -\frac{1\,.\,3}{2\,.\,4}\pi h^2, \quad \int_h^0 \frac{x^3}{\sqrt{hx - x^2}} = -\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\pi h^3$$

and so on;

$$\therefore\ T = \frac{\pi}{2}\sqrt{\frac{a}{g}}$$

$$\times \left\{1 + \left(\frac{1}{2}\right)^2 \frac{h}{2a} + \left(\frac{1.3}{2.4}\right)^2 \left(\frac{h}{2a}\right)^2 + \ldots + \left(\frac{1.3\ldots(2n-1)}{2\,.\,4\ldots 2n}\right)^2 \left(\frac{h}{2a}\right)^n + \ldots\right\}.$$

When the arc of vibration is very small, then

$$T = \frac{\pi}{2}\sqrt{\frac{a}{g}}$$

and the time of an oscillation $= \pi\sqrt{\frac{a}{g}}$, which coincides with that in a cycloid, observing that the a in this case is four times the a in that.

The next approximation gives a correction of the time $= \frac{\pi}{2}\sqrt{\frac{a}{g}}\,\frac{h}{8a}$; and the ratio this bears to the time of oscillation $= \frac{h}{8a} = (\frac{1}{4}$ chord of $\frac{1}{2}$ angle of oscillation$)^2$.

Thus if the body oscillate on each side of the vertical through an angle of which the chord is $\frac{1}{10}$, the time of oscillation will be greater by a $\frac{1}{1600}$th part than that calculated by the formula $\pi\sqrt{\frac{a}{g}}$.

397. Instead of supposing the body to move on a curve, we may imagine it suspended by a string of invariable length, or a thin wire considered of no weight. In this case the instrument is called a *Pendulum*, and is of great importance in physical researches. For if l be the length of a pendulum oscillating in a second (or unit of time) then $\pi\sqrt{\frac{l}{g}} = 1$,

$$\text{and } g = \pi^2 l,$$

By this formula we may estimate the relative intensity of the Earth's attraction at different stations on the surface, above, or below it.

Prop. *A seconds pendulum is carried to the top of a mountain; required to find the height of the mountain by observing the change in the time of oscillation.*

398. Let r be the radius of the Earth, considered spherical; h the height of the mountain; l the length of the pendulum: the force of gravity on bodies outside of the Earth varies inversely as the square of the distance from the centre: hence $\frac{gr^2}{(r+h)^2}$ is gravity at the top of the mountain. Let n be the number of oscillations the pendulum makes in a day, or in $24 \times 60 \times 60$ seconds: then time of oscillation $= \frac{24 \times 60 \times 60}{n}$:

$$\therefore\ 1 = \pi\sqrt{\frac{l}{g}} \text{ and } \frac{24\times 60\times 60}{n} = \pi\sqrt{\frac{l\,(r+h)^2}{gr^2}} = \frac{\pi\,(r+h)}{r}\sqrt{\frac{l}{g}};$$

$$\therefore\ \frac{h}{r} = \frac{24 \times 60 \times 60}{n} - 1,$$

which gives the height of the mountain. For the sake of example suppose the pendulum loses 5″ a day:

$$\text{then } n = 24 \times 60 \times 60 - 5,$$

$$\frac{h}{r} = \left(1 - \frac{1}{24 \times 12 \times 60}\right)^{-1} - 1 = \frac{1}{24 \times 12 \times 60} \text{ nearly;}$$

$$\therefore\ h = \frac{4000}{24 \times 12 \times 60} = \tfrac{1}{4} \text{ mile nearly.}$$

Prop. *To find the depth of a mine by observing the change of oscillation in a seconds pendulum.*

399. The gravity in the interior of the Earth varies directly as the distance from the centre: if, then, h be the depth, $\frac{g\,(r-h)}{r}$ is gravity at the bottom of the mine:

$$\therefore\ 1 = \pi \sqrt{\frac{l}{g}},\quad \frac{24 \times 60 \times 60}{n} = \pi \sqrt{\frac{lr}{g(r-h)}};$$

$$\therefore\ 1 - \frac{h}{r} = \left(\frac{n}{24 \times 60 \times 60}\right)^2;$$

from which h can be found. If, as before, the pendulum lose $5''$ a day

$$\frac{h}{r} = 1 - \left(1 - \frac{1}{24 \times 60 \times 12}\right)^2 = \frac{1}{12 \times 60 \times 12} \text{ nearly};$$

$$\therefore\ h = \tfrac{1}{2} \text{ mile nearly}.$$

400. The results deduced by the pendulum, as far as we have at present explained its construction, would lead to erroneous conclusions; since we have supposed the rod supporting the *bob*, as the lower extremity is termed, to have no weight. We must leave the correction of this to a future part of the work, in which we shall shew that l must not be taken equal to the length of the pendulum; but some other expression which it is unnecessary to give here.

401. Owing to the remarkable property of the cycloid, that its evolute is an equal cycloid, we can easily make the bob of a flexible pendulum move in a cycloidal arc.

For let CA (fig. 95) be the pendulum when remaining at rest: PAP' the cycloid in which the bob is to move, the length of the axis being half that of the pendulum: CQ, CQ' the evolutes of PAP'. Now move the bob to the right, and let the upper portion of the pendulum bend round CQ and the other portion remain straight, touching CQ in Q. Then since CQ is the evolute of AP, the extremity of the pendulum will be in the curve AP: and by this contrivance the bob will be made to describe the cycloid PAP'.

This suggests the following means of correcting a common pendulum which makes small oscillations. Let a small portion of the upper extremity be flexible: (consisting of watch spring, &c.) and let it be suspended between two cycloidal cheeks, as in fig. 96. Then the small oscillations of the bob will be in a cycloid, and in the expression for the time of oscillation the correction depending on $\frac{h}{2a}$ is avoided: see Art. 396.

402. The following Table contains the results of experiments with a seconds pendulum on various parts of the Earth. It is extracted from the *Mécanique Céleste.*

Places.	Latitudes.	Lengths of a Seconds Pendulum.
Peru	$0^0.00$	0.99669
Porto Bello	10.61	0.99689
Pondicherry	13.25	0.99710
Jamaica	20.00	0.99745
Petit-Goave	20.50	0.99728
Cape of Good Hope	37.69	0.99877
Toulouse	48.44	0.99950
Vienna	53.57	0.99987
Paris	54.26	1.00000
Gotha	56.63	1.00006
London	57.22	1.00018
Petersburgh	64.72	1.00074
Arensgberg	66.60	1.00101
Ponoi	74.22	1.00137
Lapland	74.53	1.00148

403. Mr Airy, in a Paper which was read before the Philosophical Society of Cambridge in the year 1826, has reduced the usual theorems for the alteration in the time and extent of vibration produced by the difference between cycloidal and circular arcs, by the resistance of the air, by the friction at the point of suspension, and by other disturbing causes, to

a very general investigation which leads to results remarkable for their simplicity. Since the principle of the pendulum is of vast importance in physical researches we shall not scruple to introduce large extracts from this valuable communication.

PROP. *A pendulum is acted upon by a small disturbing force: required the alteration in the time and extent of its oscillations.*

404. We shall suppose that the undisturbed pendulum moves with its extremity in a cycloidal arc, since in this case the calculation is not approximate.

Let s be the distance of the pendulum at the time t from the lowest point of the cycloid, s being measured along the arc described, l the length of the pendulum. Then the resolved part of gravity along the tangent is $g\frac{dx}{ds}$, x being measured vertically upwards: and $s^2 = 2lx$ is the equation to the cycloid;

$$\therefore g\frac{dx}{ds} = \frac{g}{l}s.$$

Wherefore the equation of motion of the bob of the pendulum is

$$\frac{d^2s}{dt^2} = -\frac{g}{l}s, \text{ or if } n^2 = \frac{g}{l},$$

$$\frac{d^2s}{dt^2} + n^2 s = 0.$$

The solution of this equation is

$$s = a\sin(nt + b),$$

where a and b are arbitrary constant quantities depending on the length of the arc of vibration and the time of passing the lowest point.

The velocity at time $t = \frac{ds}{dt} = na\cos(nt + b)$.

We shall now suppose that f is a small disturbing accelerating force resolved along the tangent: the equation of motion then is

$$\frac{d^2 s}{dt^2} + n^2 s = f.$$

The solution of this equation we shall assume to be

$$s = a \sin (nt + b)$$

(conformably to the principle of the variation of parameters) a and b being considered unknown functions of t, which it is our business now to determine.

Since there are two functions a and b we may assume any relation between them that we please, since we have but one quantity (s) to determine. Let this assumption be that the velocity is still expressed by $na \cos (nt + b)$: the convenience of this we shall soon discover.

$$\text{Now } s = a \sin (nt + b);$$

$$\therefore \frac{ds}{dt} = na \cos (nt + b) + \frac{da}{dt} \sin (nt + b) + a \cos (nt + b) \frac{db}{dt},$$

$$\text{and } \therefore \frac{da}{dt} \sin (nt + b) + a \cos (nt + b) \frac{db}{dt} = 0,$$

this is the assumed relation between a and b.

$$\text{Again since } \frac{ds}{dt} = na \cos (nt + b);$$

$$\therefore \frac{d^2 s}{dt^2} = - n^2 a \sin (nt + b) + n \frac{da}{dt} \cos (nt + b) - na \sin (nt + b) \frac{db}{dt},$$

in this substitute for $\frac{d^2 s}{dt^2}$ its value;

$$\therefore n \frac{da}{dt} \cos (nt + b) - na \sin (nt + b) \frac{db}{dt} = f,$$

this is the second equation between a and b.

Eliminating successively $\frac{db}{dt}$ and $\frac{da}{dt}$ from these, we have

$$\frac{da}{dt} = \frac{f}{n} \cos (nt + b), \quad \frac{db}{dt} = -\frac{f}{na} \sin (nt + b).$$

If we could solve these equations we should have the complete determination of the motion. In few cases is this practicable: in all to which we shall have to apply the investigation an approximation is sufficient.

We suppose f to be a very small force. Hence the variable parts of a and b are of the same order of magnitude as f and consequently may be neglected on the right-hand side of the above equations if we agree to neglect the square and higher powers of f.

In order to find the alteration in the extent of vibration which takes place in one oscillation we must integrate $\frac{f}{n} \cos (nt + b)$ through the limits of t corresponding to one oscillation: that is from a value of t which gives $nt + b = \alpha$ to the value of t which gives $nt + b = \pi + \alpha$. Here α may be any quantity: in different cases we shall find it convenient to integrate between different limits.

$$\therefore \textit{ increase of arc of semi-vibration} = \frac{1}{n}\int f \cos (nt + b)\, dt$$

between the above-mentioned limits.

To find the alteration in the time of oscillation, let T, T' be the values of t at two successive arrivals of the pendulum at the lowest point; B, B' the values of b at these times. Then

$$nT + B = m \,.\, \pi, \quad nT' + B' = (m + 1) \,.\, \pi;$$

$$\therefore n(T' - T) + B' - B = \pi,$$

$$T' - T = \frac{\pi}{n} - \frac{1}{n}(B' - B).$$

$$\text{Now } B' - B = \int_T^{T'} \frac{db}{dt}\, dt = -\frac{1}{na}\int_T^{T'} f \sin (nt + b)\, dt$$

between the proper limits;

$\therefore$ the increase of time of oscillation $= \dfrac{1}{n^2 a} \displaystyle\int_T^{T'} f \sin(nt+b)\, dt$,

and *the proportionate increase of time of oscillation*

$$= \frac{1}{\pi n a} \int_T^{T'} f \sin(nt+b)\, dt.$$

If the circumstances are such that we must integrate through two vibrations, then

proportionate increase of time of osc. $= \dfrac{1}{2\pi n a} \displaystyle\int f \sin(nt+b)\, dt.$

These formulæ are convenient when f can be expressed in terms of t. If however f be expressed in terms of s, as is the case particularly in clock escapements, we must modify the formulæ

$$\frac{da}{ds} = \frac{da}{dt}\frac{dt}{ds} = \frac{1}{na\cos(nt+b)}\frac{da}{dt} = \frac{f}{n^2 a},$$

$$\text{and } \frac{db}{ds} = \frac{1}{na\cos(nt+b)}\frac{db}{dt}$$

$$= -\frac{f}{n^2 a^2}\tan(nt+b) = -\frac{f}{n^2 a^2}\frac{s}{\sqrt{a^2 - s^2}};$$

$\therefore$ *increase of arc of semi-vibration* $= \dfrac{1}{n^2 a} \displaystyle\int_0^s f ds$,

proportionate increase of the time of vib^n. $= \dfrac{1}{\pi n^2 a^2} \displaystyle\int_0^s \frac{f ds}{\sqrt{a^2 - s^2}}$.

We shall subjoin a variety of examples.

Ex. 1. *Instead of vibrating in a cycloid let the pendulum vibrate in a circle.*

Here the force $= g \sin \dfrac{s}{l} = \dfrac{gs}{l} - \dfrac{gs^3}{6l^3}$ nearly;

$$\therefore f = \frac{g}{6l^3} s^3 = \frac{ga^3}{6l^3} \sin^3(nt+b);$$

therefore proportionate increase in time of vibration

$$= \frac{ga^2}{6\pi n l^3} \int \sin^4 (nt + b)\, dt.$$

Now $\int \sin^4 (nt+b)\, dt = \frac{1}{8} \int \{3 - 4 \cos 2 (nt+b) + \cos 4 (nt+b)\} dt$

$$= \frac{1}{8} \left\{3t - \frac{2}{n} \sin 2 (nt + b) + \frac{1}{4n} \sin 4 (nt + b)\right\} + C$$

$$= \frac{3}{8} \frac{\pi}{n}, \text{ from } nt + b = 0 \text{ to } \pi;$$

$\therefore$ proportionate increase of time $= \frac{ga^2}{16 n^2 l^3} = \frac{a^2}{16 l^2}$ since $n^2 = \frac{g}{l}$.

The increase of arc of vib. $= \frac{ga^3}{6 n l^3} \int \cos (nt + b) \sin^3 (nt + b)\, dt$

$$= \frac{ga^3}{24 n^2 l^3} \sin^4 (nt + b) + C = 0 \text{ between the limits,}$$

as we might easily have foreseen.

Ex. 2. *Suppose the friction at the point of suspension to be constant.*

Here $f = -c$, since the friction *retards* the motion; and the motion is considered *from* the lowest point. It will be convenient to take the integrals during that time in which the friction acts in the same direction: that is, from the beginning of a vibration to its end, or from $nt + b = -\frac{\pi}{2}$ to $nt + b = \frac{\pi}{2}$;

$\therefore$ increase of arc $= -\frac{c}{n} \int \cos (nt + b)\, dt$

$$= -\frac{c}{n^2} \sin (nt + b) + C = -\frac{2c}{n^2},$$

proportionate increase of time $= -\frac{c}{\pi n a} \int \sin (nt + b)\, dt$

$$= \frac{c}{\pi n^2 a} \cos(nt + b) + C = 0,$$

between the limits $nt + b = -\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Ex. 3. *Suppose the resistance of the air to produce a force varying as the* m[th] *power of the velocity or* = k v[m], m *being any whole number.*

The velocity in moving from the lowest point

$$= \frac{ds}{dt} = na \cos(nt + b);$$

$$\therefore f = -kn^m a^m \cos^m (nt + b);$$

therefore increase of arc

$$= -kn^{m-1} a^m \int \cos^{m+1} (nt + b)\, dt \text{ from } nt + b = -\frac{\pi}{2} \text{ to } \frac{\pi}{2}$$

$$= -k\pi n^{m-2} a^m \frac{m(m-2) \ldots\ldots\ldots\ldots 1}{(m+1)(m-1) \ldots . 2} \quad (m \text{ odd})$$

$$= -2kn^{m-2} a^m \frac{m(m-2) \ldots\ldots\ldots\ldots 2}{(m+1)(m-1) \ldots . 3} \quad (m \text{ even}).$$

When $m = 2$ (the law usually taken) the decrease of the arc $= \frac{4ka^2}{3}$.

The proportionate increase of time of oscillation

$$= -\frac{k}{\pi} n^{m-1} a^{m-1} \int \cos^m (nt + b) \sin (nt + b)\, dt$$

$$= -\frac{kn^{m-2} a^{m-1}}{\pi (m+1)} \cos^{m+1} (nt + b) + C$$

$$= 0 \text{ between } nt + b = -\frac{\pi}{2} \text{ and } \frac{\pi}{2}$$

whether m be a positive integer or fraction.

Ex. 4. *Suppose the resistance of the air is expressed by any function of the velocity.*

Here $f = \phi(v)$ for the descent and $-\phi(v)$ for the ascent, and the increase of the arc of vibration

$$= \frac{1}{n^3 a} \int \phi(v) \frac{\cos(nt+b)}{\sin(nt+b)} dv = \frac{1}{n^3 a} \int \frac{v\phi(v)\,dv}{\sqrt{n^2a^2 - v^2}}$$

from $v = 0$ to $v = 0$ again. But it must be observed that from $v = 0$ to $v = na$ (that is, from $s = -a$ to $s = 0$) the radical must be taken with a negative sign, because $\sin(nt+b)$ is then negative. The increase of the arc is consequently

$$-\frac{1}{n^3 a} \int_0^{na} \frac{v\phi(v)\,dv}{\sqrt{n^2a^2 - v^2}} + \frac{1}{n^3 a} \int_{na}^{0} \frac{v\phi(v)\,dv}{\sqrt{n^2a^2 - v^2}},$$

$$\text{and therefore decrease} = \frac{2}{n^3 a} \int_0^{na} \frac{v\phi(v)\,dv}{\sqrt{n^3a^2 - v^2}}.$$

The proportionate increase of time of vibration

$$= -\frac{1}{\pi n a} \int \phi(v) \sin(nt+b)\,dt = \frac{1}{\pi n^3 a^2} \int \phi(v)\,dv$$

$$= \frac{1}{\pi n^3 a^2} \psi(v) = 0, \text{ from } v = 0 \text{ to } v = 0.$$

Hence a resistance which is constant, or which depends on the velocity, does not alter the time of vibration.

Ex. 5. *Let the resistance be that produced by a current of air moving in the plane of vibration with a velocity* V *greater than the greatest velocity of the pendulum: and varying as the square of their relative velocity.*

Here $\phi(v) = -k(V-v)^2$ when the pendulum moves in the direction of the current

$$\phi(v) = k(V+v)^2 \text{ when it moves in the opposite direction.}$$

By the formula in the last Example, when the pendulum moves in the direction of the current, the arc is increased by

$k\left(\frac{2V^2}{n^2} - \frac{Va\pi}{n} + \frac{4a^2}{3}\right)$ and when it returns the arc is diminished by $k\left(\frac{2V^2}{n^2} + \frac{Va\pi}{n} + \frac{4a^2}{3}\right)$.

The diminution in two vibrations $= \frac{2kVa\pi}{n}$. The time is unaffected.

Ex. 6. *Let a force* F *act through a very small space* x *at the distance* c *from the lowest point.*

The increase of the arc $= \frac{1}{n^2 a}\int_c^{c+x} F\,ds = \frac{Fx}{n^2 a}$ nearly.

The proportionate increase of the time of vibration

$$= \frac{1}{\pi n^2 a^2}\int_c^{c+x} \frac{Fs\,ds}{\sqrt{a^2 - s^2}},$$

if the general value of the integral be $\phi(s)$, then the proportionate increase of time $= \phi(c+x) - \phi(c) = \phi'(c)x$

$$= \frac{Fx}{\pi n^2 a^2}\,\frac{c}{\sqrt{a^2 - c^2}}.$$

If, then, an impulse be given when the pendulum is at its lowest point, $c = 0$ and the time of vibration is unaffected.

405. Since the preceding theory is applicable to every case in which a pendulum is acted on by small forces, it can be applied to determine the effect produced on the motion of the pendulum of a clock, or the balance of a watch, by the machinery which serves to maintain that motion.

If a pendulum vibrate uninfluenced by any external forces except that of gravity, the resistance of the air and the friction of the point of suspension gradually reduce the extent of vibration. But this diminution goes on very slowly. A pendulum suspended on knife edges has been observed to vibrate more than seven hours before its arc was reduced from two degrees to $\frac{1}{5}$th of a degree. In order to maintain vibrations of the same or nearly the same length (which for clocks is indispensable) a force must act on the pendulum: this force is generally given by the action of a tooth of the seconds wheel on the inclined surfaces of small arms or pallets carried by the pendulum: and the whole apparatus is called an *escapement.*

Now it appears from Examples 2, 3, 4 and 5 of the last Article, that the friction and the resistance of the air do not affect the time of vibration. The maintaining force, therefore, must be impressed in such a manner as not to alter the time of vibration. The escapements of clocks in general use may be divided into the three following classes: recoil escapements, dead-beat escapements, and the escapements in which the action of the wheels raises a small weight which by its descent accelerates the pendulum: this last is Cumming's escapement. A full discussion of these will be found in Mr Airy's communication. He comes to the conclusion that the dead-beat escapement is far superior to any other.

406. In this the wheel acts on the pallet for a small space near the middle of the vibration, and during the remainder of the vibration it has no effect except in producing a slight friction. The impact also at the beat does not tend to accelerate or retard the pendulum. Neglecting then the consideration of the friction, we have a constant force F, which begins to act when $x = -c$ and ceases when $x = c'$. Hence by Ex. 6. of last Article, proportionate increase of time

$$= \frac{F}{\pi n^2 a^2} \int_{-c}^{c'} \frac{s\,ds}{\sqrt{a^2 - s^2}} = \frac{F}{\pi n^2 a^2} \left\{\sqrt{a^2 - c^2} - \sqrt{a^2 - c'^2}\right\}$$

$$= \frac{F}{\pi n^2 a^2} \frac{c'^2 - c^2}{\sqrt{a^2-c^2} + \sqrt{a^2-c'^2}} = \frac{F}{2\pi n^2 a^3} (c'+c)(c'-c) \text{ nearly;}$$

an extremely small quantity, since c and c' are very small when compared with a, and $c' - c$ may be made almost as small as we please, though it cannot be made absolutely zero; for the wheel must be so adapted to the pallets, that when it is disengaged from one it may strike the other, not on the acting surface, but a little above it; that is, the instant of disengagement from a pallet must follow the instant at which the pendulum is in its middle position by a rather longer time than that by which the instant of beginning to act preceded it. Hence c' must be rather greater than c. But the difference may be made so small that the effect on the clock's rate shall be almost insensible. This escapement, then, approaches very nearly to absolute perfection: and in this respect theory and practice are in exact agreement.

Mr Airy suggests a construction (*Trans. Cam. Phil. Soc.* Vol. III. p. 125.) for a clock escapement similar in its principles to the best detached escapements of chronometers.

PROP. *To prove that the velocity of a particle moving on a smooth surface is independent of the path described, but depends solely on the co-ordinates of position.*

407. Let R be the normal pressure between the surface and particle at the time t, M the mass of the particle; $\alpha\beta\gamma$ the angles which the direction of R makes with the axes: then, X, Y, Z being the other forces acting on the particle, the equations of motion are

$$\frac{d^2x}{dt^2} = X + \frac{R}{M}\cos\alpha, \quad \frac{d^2y}{dt^2} = Y + \frac{R}{M}\cos\beta,$$

$$\frac{d^2z}{dt^2} = Z + \frac{R}{M}\cos\gamma.$$

Multiply these by $2\frac{dx}{dt}$, $2\frac{dy}{dt}$, $2\frac{dz}{dt}$ and add; then

$$\frac{d\,.v^2}{dt} = 2\left(X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt}\right)$$

$$+ \frac{2R}{M}\left(\frac{dx}{dt}\cos\alpha + \frac{dy}{dt}\cos\beta + \frac{dz}{dt}\cos\gamma\right).$$

But $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$ are the cosines of the angles which the tangent line to the curve described makes with the axes; hence

$$\frac{dx}{ds}\cos\alpha + \frac{dy}{ds}\cos\beta + \frac{dz}{ds}\cos\gamma$$

equals the cosine of the angle which this tangent makes with the normal, and therefore equals zero;

$$\therefore\ v^2 = 2\int(X dx + Y dy + Z dz),$$

and X, Y, Z being functions of x, y, z this expression when integrated will be a function of x, y, z, the co-ordinates of position, and does not depend on the path described.

PROP. *A particle moves in a spherical bowl acted on by gravity: required to determine the motion.*

408. The equations of motion are (z being vertical)

$$\frac{d^2x}{dt^2} = -\frac{R}{M}\cos\alpha,\quad \frac{d^2y}{dt^2} = -\frac{R}{M}\cos\beta,\quad \frac{d^2z}{dt^2} = g - \frac{R}{M}\cos\gamma,$$

also $x^2+y^2+z^2=a^2$ is the equation to the surface: in this case,

$$\cos\alpha = \frac{x}{a},\quad \cos\beta = \frac{y}{a},\quad \cos\gamma = \frac{z}{a},$$

then (as in last Article)

$$\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2} = C + 2gz.$$

Let V and k be the initial values of the velocity and of z: then

$$\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2} = V^2 - 2g(k - z),$$

$$\text{also } x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0;$$

$$\therefore\ x\frac{dy}{dt} - y\frac{dx}{dt} = \text{const.} = h,$$

$$\text{likewise } x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt} = 0,$$

By eliminating $\frac{dx}{dt}$ and $\frac{dy}{dt}$ from these, we have

$$t = \int\frac{a\,dz}{\sqrt{(a^2 - z^2)\{V^2 - 2g(k - z)\} - h^2}}.$$

This is an elliptic function, Art. 396. If this could be integrated, then z (and consequently x and y) is known in terms of t, and the motion is determined.

409. We may obtain approximate results by supposing the oscillations to be very small.

In this case, let θ be the angle that the radius drawn to the particle makes with the vertical, ψ the angle which the vertical plane in which θ is measured makes with the vertical plane through the centre of the sphere and the point of projection; let the velocity of projection $(V) = \beta\sqrt{ga}$, β being a small numerical quantity, the direction of V horizontal, α the initial value of θ; then

$$k = a - \tfrac{1}{2}a\alpha^2, \quad z = a - \tfrac{1}{2}a\theta^2, \quad h^2 = a^3 g\alpha^2\beta^2,$$

$$y = x\tan\psi, \quad x^2 + y^2 + z^2 = a^2;$$

$$\therefore \frac{dt}{d\theta} = -a\theta\frac{dt}{dz} = -\sqrt{\frac{a}{g}}\,\frac{\theta}{\sqrt{(\alpha^2-\theta^2)(\theta^2-\beta^2)}},$$

$$\frac{d\psi}{d\theta} = \frac{d\psi}{dt}\frac{dt}{d\theta} = \frac{1}{x^2+y^2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)\frac{dt}{d\theta} = -\frac{\alpha\beta}{\theta\sqrt{(\alpha^2-\theta^2)(\theta^2-\beta^2)}}.$$

The first of these equations gives

$$2t = -\sqrt{\frac{a}{g}}\int\frac{2d\,.\,\theta^2}{\sqrt{(\alpha^2-\beta^2)^2 - \{2\theta^2-(\alpha^2+\beta^2)\}^2}}$$

$$= \sqrt{\frac{a}{g}}\cos^{-1}\left\{\frac{2\theta^2-(\alpha^2+\beta^2)}{\alpha^2-\beta^2}\right\}, \text{ const.} = 0;$$

$$\therefore \theta^2 = \tfrac{1}{2}(\alpha^2+\beta^2) + \tfrac{1}{2}(\alpha^2-\beta^2)\cos 2\sqrt{\frac{g}{a}}\,t:$$

this shews that the pendulum makes isochronous oscillations in the moveable vertical plane: the extreme angles being α and β, and the time of oscillation being $\frac{\pi}{2}\sqrt{\frac{a}{g}}$, or half the time of oscillation when the plane of motion is constant.

Hence also $\frac{d\psi}{dt} = \sqrt{\frac{g}{a}} \frac{\alpha\beta}{\alpha^2 \cos^2 \sqrt{\frac{g}{a}}\,t + \beta^2 \sin^2 \sqrt{\frac{g}{a}}\,t}$;

$$\therefore\ \alpha \tan \psi = \beta \tan \sqrt{\frac{g}{a}}\,t,$$

from which the azimuth of the plane of oscillation is known at any time.

By substitution we have

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = (a^2 - z^2)\left(\frac{\cos^2\psi}{\alpha^2} + \frac{\sin^2\psi}{\beta^2}\right) = a^2\theta^2\left(\frac{\cos^2\psi}{\alpha^2} + \frac{\sin^2\psi}{\beta^2}\right),$$

and substituting for θ and ψ their values in terms of t,

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = a^2,$$

which shews that the projection of the path on a horizontal plane is an ellipse with its centre in the vertical radius of the sphere.

Cor. If $\alpha = \beta$, then $\theta^2 = \alpha^2$, $\psi = \sqrt{\frac{g}{a}}\,t$, $x^2 + y^2 = a^2\alpha^2$, and the pendulum describes a conical surface with a uniform motion.

Prop. *A particle moves on a curve surface, required to find the pressure at any instant.*

410. The equations of motion are

$$\frac{d^2x}{dt^2} = X + \frac{R}{M}\cos\alpha,\quad \frac{d^2y}{dt^2} = Y + \frac{R}{M}\cos\beta,\quad \frac{d^2z}{dt^2} = Z + \frac{R}{M}\cos\gamma.$$

Multiply by $\cos\alpha$, $\cos\beta$, $\cos\gamma$ respectively, and add, then

$$\frac{R}{M} = \frac{d^2x}{dt^2}\cos\alpha + \frac{d^2y}{dt^2}\cos\beta + \frac{d^2z}{dt^2}\cos\gamma$$
$$- \{X\cos\alpha + Y\cos\beta + Z\cos\gamma\}.$$

To calculate the former part suppose that the co-ordinate planes are so chosen, that, at the instant under consideration, the axis of z is the normal line at the point of contact of the particle: hence $\cos\alpha = 0$, $\cos\beta = 0$, $\cos\gamma = 1$, and this part becomes $\frac{d^2z}{dt^2}$.

Now z is a function of x and y: x and y are functions of t; hence

$$\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} + \frac{dz}{dy}\frac{dy}{dt},$$

$$\frac{d^2z}{dt^2} = \frac{d^2z}{dx^2}\frac{dx^2}{dt^2} + 2\frac{d^2z}{dxdy}\frac{dx}{dt}\frac{dy}{dt} + \frac{d^2z}{dy^2}\frac{dy^2}{dt^2} + \frac{dz}{dx}\frac{d^2x}{dt^2} + \frac{dz}{dy}\frac{d^2y}{dt^2}.$$

But $\frac{dz}{dx} = 0$, $\frac{dz}{dy} = 0$ as the axes are chosen.

Hence $$\frac{d^2z}{dt^2} = \frac{ds^2}{dt^2}\left\{\frac{d^2z}{dx^2}\frac{dx^2}{ds^2} + 2\frac{d^2z}{dxdy}\frac{dx}{ds}\frac{dy}{ds} + \frac{d^2z}{dy^2}\frac{dy^2}{ds^2}\right\}$$

$$= \frac{(\text{velocity})^2}{\text{radius of curvature}} = \frac{v^2}{\rho},$$

and the magnitude of this cannot depend upon the manner of fixing the axis; therefore, in general,

$$\frac{R}{M} = \frac{v^2}{\rho} - (X\cos\alpha + Y\cos\beta + Z\cos\gamma)$$

= centrifugal force − resolved part of the forces along the normal.

Prop. *A particle moves in a groove in the form of a curve of double curvature; required the pressure.*

411. The equations of motion are the same as in the last Article: $\alpha\beta\gamma$ being the angles which the direction of the pressure makes with the axes; this coincides with the radius of absolute curvature.

Let ρ be the radius, and $x_, y_, z_,$ the co-ordinates to the centre of curvature, then

$$x_, = x + \rho^2\frac{d^2x}{ds^2}, \quad y_, = y + \rho^2\frac{d^2y}{ds^2}, \quad z_, = z + \rho^2\frac{d^2z}{ds^2};$$

$$\therefore \cos \alpha = \frac{x - x_{\prime}}{\rho} = \rho \frac{d^2 x}{ds^2}, \quad \cos \beta = \rho \frac{d^2 y}{ds^2}, \quad \cos \gamma = \rho \frac{d^2 z}{ds^2};$$

$$\therefore \frac{R}{M} = \rho \left\{ \frac{d^2 x}{dt^2} \frac{d^2 x}{ds^2} + \frac{d^2 y}{dt^2} \frac{d^2 y}{ds^2} + \frac{d^2 z}{dt^2} \frac{d^2 z}{ds^2} \right\} - \{ X \cos \alpha + Y \cos \beta + Z \cos \gamma \},$$

the former part, by changing the independent variable to s (as in Art. 255), becomes

$$\frac{\left(\frac{d^2 x}{ds^2}\right)^2 + \left(\frac{d^2 y}{ds^2}\right)^2 + \left(\frac{d^2 z}{ds^2}\right)^2}{\frac{dt^2}{ds^2}} - \frac{\frac{d^2 t}{ds^2} \frac{d}{ds} \left\{ \frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} + \frac{dz^2}{ds^2} \right\}}{\frac{dt^3}{ds^3}}$$

$$= \frac{1}{\rho^2} (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma) \frac{ds^2}{dt^2} = \frac{1}{\rho^2} \frac{ds^2}{dt^2};$$

$$\therefore \frac{R}{M} = \frac{v^2}{\rho} - (X \cos \alpha + Y \cos \beta + Z \cos \gamma)$$

= centrifugal force − resolved part of the forces along the radius of absolute curvature.

CHAPTER VIII.

PROBLEMS ON THE MOTION OF BODIES CONSIDERED AS PARTICLES.

Prob. 1. A body is projected vertically upwards and the time between its leaving a given point and returning to it is given: find the velocity of projection, and the whole time of motion.

Prob. 2. Two bodies fall from two given points in space in the same vertical down two straight lines drawn to any point of a surface in the same time, find the form of the surface.

Prob. 3. A semi-cycloid is placed with its axis vertical and vertex downwards, and from different points in it a number of bodies are let fall at the same instant, each moving down the tangent at the point from which it sets out: prove that they will reach the involute (passing through the vertex) all at the same instant.

Prob. 4. From the top of a tower two bodies are projected with the same given velocity at different given angles of elevation, and they strike the horizon at the same place: find the height of the tower.

Prob. 5. A body acted upon by two central forces, each varying inversely as the square of a distance, is projected from a point between them towards one of the centres: required the velocity of projection that the body may just arrive at the neutral point of attraction and remain at rest there.

Prob. 6. A body, acted on by a force varying inversely as the fifth power of the distance, is projected in any direction with a velocity equal to that which would be acquired in falling from an infinite distance: find the orbit.

Prob. 7. A body, projected in a given direction with a given velocity and attracted towards a given centre of force, has its velocity at every point: the velocity in a circle at the same distance :: 1 : $\sqrt{2}$; find the orbit described, the position of the apse, the magnitude of its axis, and the law of force.

Prob. 8. Two bodies are connected by a string passing through a hole in a horizontal plane; one of them is projected in any direction in the horizontal plane, and the other descends vertically by the action of gravity: find the motion of the bodies, and the curve described on the plane.

Prob. 9. A body is projected in any direction from one extremity of a right line, each particle of which attracts it by a force proportional to the distance; prove that the body will pass through the other extremity.

Prob. 10. A body projected from a given point in a plane is attracted by forces $\frac{m}{x^3}$ in the direction of x, and $\frac{m'}{y^3}$ in the direction of y: prove that if the velocity and direction of projection be rightly assumed, it will describe a circle round the origin as centre, and find how the velocity varies in different parts of the orbit.

Prob. 11. A body, urged towards a plane by a force varying as the perpendicular distance from it, is projected at right angles to the plane from a given point in it with a given velocity: find what force must act at the same time on the body parallel to the plane, that it may move in a given parabola having its axis in the plane; and determine the circumstances of the motion.

Prob. 12. A body acted on by a force varying partly as the inverse cube and partly as the inverse fifth power of the distance is projected with the velocity which would be acquired in falling from infinity, at an angle with the distance the tangent of which $= \sqrt{2}$, the forces being equal at the point of projection; determine the motion.

Prob. 13. A body is projected from a point near a centre of force which varies inversely as the square of the distance, in a direction perpendicular to the line joining the point of projection with the centre of force, and so as to describe an ellipse about that centre: shew that the point of projection will coincide with the nearer or further apse according as the velocity of projection is greater or less than that with which a circle might be described at the same distance.

Prob. 14. If a force vary inversely as the 7^{th} power of the distance, and a body be projected from an apse with a velocity which is to the velocity in a circle at the same distance $:: 1 : \sqrt{3}$; find the polar equation to the curve described, and transform it to rectangular co-ordinates.

Prob. 15. If a body be projected about a centre of force varying inversely as the square of the distance with a velocity equal to n times the velocity in a circle at the same distance, and in a direction making an angle β with the distance; the angle α between the axis major and this distance may be determined from the equation

$$\tan(\alpha - \beta) = (1 \sim n^2)\tan\beta.$$

Prob. 16. If a be the mean distance of a planet from the Sun, and l the length of the line of nodes, then the time of the planet's passage (supposed undisturbed) from node to node through perihelion is

$$= \frac{a^{\frac{3}{2}}p}{\pi}\left\{\tan^{-1}\sqrt{\frac{l}{2a-l}} - \frac{l}{2a}\sqrt{\frac{2a-l}{l}}\right\}$$

where $p =$ the length of the year, and $1 =$ mean distance of the Earth from the Sun.

Prob. 17. If a body revolve in an ellipse round the focus prove, that a progressive motion of the apse will be the effect of any continual addition of force in the direction of the radius vector during the progress of the body from the further to the nearer apse, and point out the effect on the eccentricity.

Prob. 18. A body is acted on by two forces, one repulsive and varying as the distance from a given point, and the other constant and acting in parallel lines: determine the motion of the body.

Prob. 19. If a body can describe a given curve about one centre with one law of force, about another centre with another law of force and so for any number of centres, it is possible to project the body with such a velocity that it may describe the same curve under the action of all those forces.

Prob. 20. A body describes a parabola about a centre of force residing in a point in the circumference of a given ellipse, the foci of which are in the circumference of the parabola, the force varying inversely as the square of the distance: shew that the time of moving from one focus to the other is the same, at whatever point in the circumference of the ellipse the centre of force is placed.

Prob. 21. If P be a central force attracting a catenary, and p be the perpendicular on the tangent at any point from the centre of force; then, the force which would cause a body to revolve in the curve formed by the catenary varies as $P \div p$.

Prob. 22. A body P is projected with a given velocity $a\sqrt{\mu}$ in a direction perpendicular to its distance SA from a centre of force S, which itself moves uniformly with velocity V in the direction AS produced; the force varies as the distance: determine the equation to the orbit described, and shew that the motions of P and S are parallel when the co-ordinates of P measured from the original position of S are a and $(\frac{1}{2}\pi - 1)\,V$.

Prob. 23. If two equal bodies, which attract each other with forces varying inversely as the square of the distance, are constrained to move in two straight lines at right angles to each other; shew that they will arrive together at the point of intersection of the lines, from whatever points their motions commence: and having given their distance at the beginning of the motion, find the time to the point of intersection.

Prob. 24. The times of oscillation of a pendulum are observed at the Earth's surface, and at a given depth below the surface; find from these data the radius of the Earth, supposed spherical.

Prop. 25. If a pendulum oscillating in a small circular arc be acted upon, in addition to the force of gravity, by a small horizontal force (as the attraction of a mountain) in the plane in which it oscillates; having given the number of oscillations gained in a day, find the horizontal force.

Prob. 26. A body oscillates in a cycloid on an inclined plane, and the friction on the plane $= \mu$ times the pressure: shew that the friction will not affect the time of oscillation, and that the body will stop after it has oscillated a number of times $= \frac{a}{2l\mu}\tan\alpha - \frac{1}{2}$, where a is the original distance from the lowest point and α the inclination of the plane.

Prob. 27. A body acted on by gravity moves on the convex surface of a cycloid, the vertex of which is its highest point; the velocity at the highest point being $\sqrt{2gh}$, determine the point where it will leave the curve, and the latus rectum of the parabola afterwards described.

Prob. 28. A body moving on the interior surface of a vertical cylinder was projected with a given velocity, and goes round precisely n times before it begins to descend: find the direction of projection.

Prob. 29. A body acted on by a repulsive central force varying as the distance, moves in a groove of the form of an epicycloid, the pole of which is in the centre of force: prove that the oscillations are isochronous.

Prob. 30. If a body move in an elliptic groove uniformly, round two centres of force situated in the foci; prove that the forces at any point of the ellipse are equal, and inversely proportional to the square of the corresponding diameter.

Prob. 31. A body moves in a groove under the action of two centres of force each varying inversely as the distance, and of equal intensity at the same distance; the body is projected from the mid-point between the centres: prove that if the velocity be uniform the form of the groove is a lemniscate.

Prob. 32. A body attracted to two centres of force varying inversely as the square of the distance moves in a hyperbolic groove, of which the foci are the centres of force: required to find the pressure on the groove; and to shew that if the particle begin to move from a point where it is equally attracted by the two centres, the pressure on the groove is zero during the whole motion.

CHAPTER IX.

PRELIMINARY ANALYSIS.

412. We now enter upon the calculation of the motion of a rigid body.

In the following Chapters we shall repeatedly meet with the expressions

$$\Sigma . mx, \quad \Sigma . my, \quad \Sigma . mz,$$
$$\Sigma . mxy, \quad \Sigma . mxz, \quad \Sigma . myz,$$
$$\Sigma . mx^2, \quad \Sigma . my^2, \quad \Sigma . mz^2;$$

xyz being the co-ordinates to a particle m of a material system, and Σ being a symbol which represents that the sum of the quantities symmetrical with that before which it is placed is to be taken throughout the system.

It becomes important, then, to enquire whether the axes of co-ordinates may not be so chosen, as to simplify these expressions.

Prop. *The first three may be simplified.*

413. Let $\bar{x}, \bar{y}, \bar{z}$ be the co-ordinates of the centre of gravity of the system: and let M be the mass of the system. Then by p. 67, we have

$$\Sigma . mx = M\bar{x}, \quad \Sigma . my = M\bar{y}, \quad \Sigma . mz = M\bar{z}.$$

If it be allowable in any case to choose for one of the co-ordinate planes a plane passing through the centre of gravity, then, supposing this the plane of xy, we have $\bar{z} = 0$ and therefore $\Sigma . mz = 0$.

If it be allowable to choose for the axis of x a line passing through the centre of gravity, then $\bar{y} = 0$, $\bar{z} = 0$, and these give $\Sigma . my = 0$, $\Sigma . mz = 0$.

Lastly, if it be allowable to choose the origin at the centre of gravity, then $\bar{x}=0$, $\bar{y}=0$, $\bar{z}=0$; and therefore $\Sigma . mx=0$, $\Sigma . my=0$, $\Sigma . mz=0$.

414. The second set of expressions, viz.: $\Sigma . mxy$, $\Sigma . mxz$, $\Sigma . myz$ may be made to vanish by properly choosing the co-ordinates.

This simplification is so important that the axes which possess this property are called the *Principal Axes* of the system. They are likewise termed the Natural Axes of Rotation for a reason hereafter to be assigned: see Art. 439.

Before proceeding to find these axes we must prove the formulæ by which we pass from one system of axes to another.

PROP. *To prove the formulæ for the transformation of one system of rectangular co-ordinates to another, the origin remaining the same.*

415. Let Ax, Ay, Az be the original axes (fig. 97), $Ax_{\prime}$, $Ay_{\prime}$, $Az_{\prime}$ the new axes.

θ = inclination of plane $x_{\prime}y_{\prime}$ to plane xy.

ψ = the angular distance of the line of intersection of these planes from the axis of x; *i. e.* the angle NAx.

ϕ = the angular distance of axis of $x_{\prime}$ from this line of intersection; *i. e.* the angle $NAx_{\prime}$.

xyz, $x_{\prime}y_{\prime}z_{\prime}$ the co-ordinates to any point referred to the two systems of axes respectively.

r = the distance of this point from the origin.

Then the cosines of the angles which r makes with the axes of xyz, $x_{\prime}y_{\prime}z_{\prime}$ are respectively $\frac{x}{r}$, $\frac{y}{r}$, $\frac{z}{r}$; $\frac{x_{\prime}}{r}$, $\frac{y_{\prime}}{r}$, $\frac{z_{\prime}}{r}$. Hence

$$\frac{x}{r}=\frac{x_{\prime}}{r}\cos xx_{\prime}+\frac{y_{\prime}}{r}\cos xy_{\prime}+\frac{z_{\prime}}{r}\cos xz_{\prime}$$

$$\frac{y}{r}=\frac{x_{\prime}}{r}\cos yx_{\prime}+\frac{y_{\prime}}{r}\cos yy_{\prime}+\frac{z_{\prime}}{r}\cos yz_{\prime}$$

$$\frac{z}{r}=\frac{x_{\prime}}{r}\cos zx_{\prime}+\frac{y_{\prime}}{r}\cos zy_{\prime}+\frac{z_{\prime}}{r}\cos zz_{\prime}.$$

Let us now suppose all the points where the six axes meet a sphere of radius unity described about A to be joined by arcs of great circles; then we shall have by the formula for the cosine of the side of a spherical triangle in terms of the other sides and opposite angle

$$\cos xx_{,} = \cos\phi\cos\psi + \sin\phi\sin\psi\cos\theta$$

$$\cos xy_{,} = -\sin\phi\cos\psi + \cos\phi\sin\psi\cos\theta$$

$$\cos xz_{,} = -\sin\psi\sin\theta$$

$$\cos yx_{,} = -\cos\phi\sin\psi + \sin\phi\cos\psi\cos\theta$$

$$\cos yy_{,} = \sin\phi\sin\psi + \cos\phi\cos\psi\cos\theta$$

$$\cos yz_{,} = -\cos\psi\sin\theta$$

$$\cos zx_{,} = \sin\phi\sin\theta$$

$$\cos zy_{,} = \cos\phi\sin\theta$$

$$\cos zz_{,} = \cos\theta.$$

Hence by substitution

$$x = x_{,}(\cos\phi\cos\psi + \sin\phi\sin\psi\cos\theta) - y_{,}(\sin\phi\cos\psi - \cos\phi\sin\psi\cos\theta) - z_{,}\sin\psi\sin\theta$$

$$y = -x_{,}(\cos\phi\sin\psi - \sin\phi\cos\psi\cos\theta) + y_{,}(\sin\phi\sin\psi + \cos\phi\cos\psi\cos\theta) - z_{,}\cos\psi\sin\theta$$

$$z = x_{,}\sin\phi\sin\theta + y_{,}\cos\phi\sin\theta + z_{,}\cos\theta.$$

416. In the same manner we should find

$$x_{,} = x(\cos\phi\cos\psi + \sin\phi\sin\psi\cos\theta) - y(\cos\phi\sin\psi - \sin\phi\cos\psi\cos\theta) + z\sin\phi\sin\theta$$

$$y_{,} = -x(\sin\phi\cos\psi - \cos\phi\sin\psi\cos\theta) + y(\sin\phi\sin\psi + \cos\phi\cos\psi\cos\theta) + z\cos\phi\sin\theta$$

$$z_{,} = -x\sin\psi\sin\theta - y\cos\psi\sin\theta + z\cos\theta.$$

PROP. *To prove that in every body there is a system of rectangular axes, and in general only one system, which will satisfy the conditions* $\Sigma . m x_{,} y_{,} = 0$, $\Sigma . m x_{,} z_{,} = 0$, $\Sigma . m y_{,} z_{,} = 0$.

417. Substitute in the equations $\Sigma . m x_{,} y_{,} = 0$, $\Sigma . m x_{,} z_{,} = 0$, $\Sigma . m y_{,} z_{,} = 0$ the values of $x_{,} y_{,} z_{,}$ given in Art. 416, and putting

$$\Sigma . m (y^2 + z^2) = D, \quad \Sigma . m (x^2 + z^2) = E, \quad \Sigma . m (x^2 + y^2) = F,$$

$$\Sigma . myz = G, \quad \Sigma . mxz = H, \quad \Sigma . mxy = K;$$

we have $L \sin 2\phi + M \cos 2\phi = 0$ .... (1),

$$N \cos \phi - P \sin \phi = 0,$$

$$N \sin \phi + P \cos \phi = 0,$$

where L, M, N, P are certain functions of θ, ψ, D, E, F, G, H, and K; and are independent of ϕ.

The first of these equations gives ϕ when θ and ψ are known. By eliminating ϕ from the second and third we have $P = 0$, $N = 0$: or, if we replace P and N by their values, we have

$$\left.\begin{aligned} &\sin 2\theta \{D \sin^2 \psi - 2K \sin \psi \cos \psi + E \cos^2 \psi - F\} \\ &\qquad + 2 \cos 2\theta \{G \cos \psi + H \sin \psi\} = 0 \\ &\sin \theta \{(D - E) \sin \psi \cos \psi - K (\cos^2 \psi - \sin^2 \psi)\} \\ &\qquad - \cos \theta \{G \sin \psi - H \cos \psi\} = 0 \end{aligned}\right\} \text{......} \ (2).$$

Let $\tan \psi = u$, and $\therefore \sin \psi = \dfrac{u}{\sqrt{1 + u^2}}$, $\cos \psi = \dfrac{1}{\sqrt{1 + u^2}}$:

also let θ be eliminated from the above equations by the formula $(1 - \tan^2 \theta) \tan 2\theta = 2 \tan \theta$; and we have, after all reductions,

$$\{(D - E) u - K (1 - u^2)\} \{(GD - GF + HK) u - HE + HF + GK\}$$
$$- (Gu - H)^2 (Hu + G) = 0.$$

This equation, being a cubic, must give at least one real value of u, and therefore of ψ: and substituting this in one of the equations (2) we shall have the value of θ and then ϕ is known from equation (1).

We conclude, then, that we can always find a system of co-ordinate axes which will satisfy our conditions. But, not only so, there is in general only one such system; for although we might fancy that there could be three since the equation in u is a cubic yet this will be found not to be the case when it is remarked that this equation, which is to obtain the angle between the axis of x and the intersection of the planes x_ly_l and xy, ought likewise to give the angles which the axis of x makes with the intersections of the two other planes x_lz_l and y_lz_l with the plane xy. Hence all three roots of the cubic will be possible and serve to determine the three angles specified above.

Hence the Proposition is true.

Cor. 1. The equation in u becomes identical whenever, in any particular case, we have $G = 0$, $H = 0$, $K = 0$. In this case *every* system of rectangular axes is a system of principal axes; as is proved by these three equations: and for this reason the equation in u gives no result.

Cor. 2. Again, the equation in u is identical when $G=0$, $H=0$. In this case also there is an infinite number of systems of principal axes; but they must all have a common axis, since F does not vanish.

It will be seen that in most cases the difficulty of calculating the position of the principal axes in a body is great. But whenever we know one of them the other two are easily determined, as we shall now shew.

Prop. *To find the principle axes of a body when one of them is known.*

418. Let Az_l be the known principal axis, Ax_l, Ay_l the others making an angle ψ with the arbitrary axes Ax, Ay drawn at right angles to Az_l (fig. 98).

Let $x_ly_lz_l$, xyz be the co-ordinates to a particle m referred to these two systems of axes: then

$$x_l = x\cos\psi + y\sin\psi,\quad y_l = y\cos\psi - x\sin\psi.$$

Hence $\Sigma . mx_ly_l = 0$ gives

$$(\cos^2\psi - \sin^2\psi)\,\Sigma . mxy - \cos\psi\sin\psi\,\Sigma . m(x^2 - y^2) = 0;$$

$$\therefore \tan 2\psi = \frac{\sin 2\psi}{\cos 2\psi} = \frac{2 \sin \psi \cos \psi}{\cos^2 \psi - \sin^2 \psi}$$

$$= \frac{2 \Sigma . mxy}{\Sigma . m (x^2 - y^2)}.$$

Ex. 1. *One principal axis of a rectangular parallelogram of uniform thickness is perpendicular to its plane through the centre: required the other two.*

Let $2a$, $2b$ be the sides of the parallelogram: M its mass: the sides parallel to the plane xy, and the centre of the origin: then the mass of an element $= M \frac{dx\,dy}{4ab}$: and therefore

$$\Sigma . mxy = \int_{-a}^{a} \int_{-b}^{b} \frac{M}{4ab} xy\,dx\,dy = \frac{M}{4ab} \int_{-a}^{a} 0 . dx = 0,$$

$$\Sigma . m(x^2 - y^2) = \frac{M}{4ab} \int_{-a}^{a} \int_{-b}^{b} (x^2 - y^2)\,dx\,dy = \frac{M}{2ab} \int_{-a}^{a} (x^2 b - \tfrac{1}{3} b^3)$$

$$= \frac{M}{3ab} (a^3 b - b^3 a) = \frac{M}{3} (a^2 - b^2);$$

$\therefore \tan 2\psi = 0$; and $\therefore 2\psi = 0$ and 180^0, or $\psi = 0$ and 90^0, and the other two axes are parallel to the sides of the parallelogram.

Cor. If the parallelogram be a square then $a = b$ and $\tan 2\psi = \frac{0}{0}$: which shews that in this case any pair of axes x and y are principal axes.

Ex. 2. *One principal axis of an elliptic board being perpendicular to its plane through its centre; the other two coincide with the axes of the ellipse.*

419. The last three of the expressions in Art. 412, viz. $\Sigma . mx^2$, $\Sigma . my^2$, $\Sigma . mz^2$, do not admit of much simplification.

The sum of the products of the mass of each particle of the system and the square of its distance from any straight line is called the *Moment of Inertia* of the System about that line.

We proceed to prove certain Propositions connected with the Moment of Inertia.

PROP. *The moment of inertia of a system about any axis is equal to the moment of inertia about an axis through the centre of gravity and parallel to the former, together with the product of the mass of the system and the square of the distance between the two axes.*

420. Let the plane of the paper pass through the centre of gravity G of the system: and be perpendicular to the original axis and cut it in A (fig. 99): Ax, Ay the axes of x and y, and P the projection of any particle of the system m on the plane of the paper: x, y the co-ordinates of P from G; $\bar{x}\,\bar{y}$ the co-ordinates of G from A. Then the moment of inertia

$$= \Sigma \, . \, m \, AP^2 = \Sigma \, . \, m \, \{(\bar{x} + x)^2 + (\bar{y} + y)^2\}$$

$$= M(\bar{x}^2 + \bar{y}^2) + 2\bar{x}\Sigma \, . \, mx + 2\bar{y}\Sigma \, . \, my + \Sigma \, . \, m\,(x^2 + y^2)$$

$$= M(\bar{x}^2 + \bar{y}^2) + \Sigma \, . \, m\,(x^2 + y^2): \text{ see Art. 413.}$$

$= M \, . \, GA^2 +$ moment of inertia about an axis of which the projection is G.

421. We shall now calculate the moment of inertia in some particular cases.

Let k be such a quantity that the moment of inertia$=Mk^2$. Then it will be seen that k is the distance of the point at which we may suppose the whole mass collected so as not to alter the moment of inertia. This quantity k is called the ***Radius of Gyration.***

We shall always use the symbol k for this radius when the axis passes through the centre of gravity, and $k_{,}$ (with a subscript accent) when it does not.

EX. 1. *A physical line about an axis through its centre and perpendicular to its length.*

$2a =$ length ; $r =$ distance of any particle from the centre ;

$$\therefore \text{ mass of a length } dr = M\frac{dr}{2a};$$

$$\therefore \text{ moment of inertia, or } M.k^2 = \int_{-a}^{a} M\frac{r^2}{2a}dr = M\frac{a^2}{3};$$

$$\therefore k, \text{ or radius of gyration, } = \frac{a}{\sqrt{3}}.$$

If the axis of rotation be at a distance c from the centre of gravity and parallel to that used above, then

$$k_{,}^2 = \frac{a^3}{3} + c^2 \text{ by Art. 420.}$$

Ex. 2. *A circular body of uniform thickness and density about an axis through its centre and perpendicular to its plane.*

$$a = \text{radius}, \quad \angle BAP = \theta, \quad AP = r \text{ (fig. 25)};$$

$$\text{therefore element of the mass at } P = M\frac{dr \, . \, r d\theta}{\pi a^2};$$

$$\therefore Mk^2 = \int_0^a \int_0^{2\pi} M\frac{r^3}{\pi a^2} dr d\theta = \int_0^a 2M\frac{r^3}{a^2}dr = M\frac{a^2}{2};$$

$$\therefore k^2 = \tfrac{1}{2}a^2.$$

For an axis parallel to the above at a distance c,

$$k_{,}^2 = \tfrac{1}{2}a^2 + c^2 \text{ by Art. 420.}$$

Ex. 3. *The same body about an axis through its centre and in its plane.*

$$\text{Mass of element at } P = M\frac{dr r d\theta}{\pi a^2},$$

$$Mk^2 = \int_0^a \int_0^{2\pi} M\frac{r^3 \sin^2\theta}{\pi a^2} dr d\theta = \frac{M}{2\pi a^2}\int_0^a \int_0^{2\pi} r^3(1-\cos 2\theta) dr d\theta$$

$$= \frac{M}{a^2}\int_0^a r^3 dr = M\frac{a^2}{4};$$

$$\therefore k^2 = \tfrac{1}{4}a^2.$$

About an axis parallel to the above at a distance c,

$$k_{,}^2 = \tfrac{1}{4} a^2 + c^2.$$

Ex. 4. *A solid of revolution about any axis perpendicular to the axis of the solid.*

Let $DA'E$ be the given axis cutting the axis of the solid in A': let A' be the origin of co-ordinates (fig. 27): $PM = y$, $A'M = x$: $A'A = m$, $A'B = n$, V the volume of the solid;

$$\therefore \text{ mass of elementary section } PP' = M\frac{\pi y^2 dx}{V},$$

$$\text{mom. of iner. of this element about } PP' = \frac{M}{V}\pi y^2 dx \frac{y^2}{4} \text{ (Ex. 3.)},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; DA'E = \frac{M}{V}\pi y^2 dx\left(\frac{y^2}{4} + x^2\right)$$

(Art. 420);

$$\therefore Mk_{,}^2 = \int_m^n \frac{M}{V}\pi y^2\left(\frac{y^2}{4} + x^2\right) dx: \text{ but } V = \int_m^n \pi y^2 dx:$$

$$\therefore k_{,}^2 = \int_m^n \left(\tfrac{1}{4} y^4 + x^2 y^2\right) dx \div \int_m^n y^2 dx.$$

Ex. 5. *A sphere about a tangent.*

$$y^2 = 2ax - x^2;$$

$$\therefore k_{,}^2 = \int_0^{2a} \left(a^2 x^2 + ax^3 - \tfrac{3}{4} x^4\right) dx \div \int_0^{2a} (2ax - x^2)\, dx$$

$$= \left(\tfrac{8}{3} + 4 - \tfrac{24}{5}\right) a^5 \div \left(4 - \tfrac{8}{3}\right) a^3 = \tfrac{28}{15} a^2 \div \tfrac{4}{3} = \tfrac{7}{5} a^2.$$

$$\text{Also } k^2 = k_{,}^2 - a^2 = \tfrac{2}{5} a^2.$$

Prop. *To find the moment of inertia of a system referred to any axis.*

422. Let AC be the axis (fig. 100): P any particle m of the system: PM perpendicular to AC: Ax, Ay, Az the co-ordinate axes: x, y, z co-ordinates to P; α, β, γ the angles AC makes with the axes;

$$\therefore\ PM^2 = AP^2 \sin^2 PAC = AP^2 (1 - \cos^2 PAC),\ AP = r$$

$$= r^2 - r^2 \left(\frac{x}{r}\cos\alpha + \frac{y}{r}\cos\beta + \frac{z}{r}\cos\gamma\right)^2$$

$$= x^2 + y^2 + z^2 - (x\cos\alpha + y\cos\beta + z\cos\gamma)^2$$

$$= x^2\sin^2\alpha + y^2\sin^2\beta + z^2\sin^2\gamma$$
$$- 2xy\cos\alpha\cos\beta - 2xz\cos\alpha\cos\gamma - 2yz\cos\beta\cos\gamma.$$

Hence moment of inertia =

$$\sin^2\alpha\,\Sigma\,.\,mx^2 + \sin^2\beta\,\Sigma\,.\,my^2 + \sin^2\gamma\,\Sigma\,.\,mz^2$$
$$-2\cos\alpha\cos\beta\,\Sigma\,.\,mxy - 2\cos\alpha\cos\gamma\,\Sigma\,.\,mxz - 2\cos\beta\cos\gamma\,\Sigma\,.\,myz.$$

If the axes of co-ordinates be principal axes, then, accenting the co-ordinates in accordance with the notation of Art. 417,

$$Mk_{,}^2 = \sin^2\alpha\,\Sigma\,.\,mx_{,}^2 + \sin^2\beta\,\Sigma\,.\,my_{,}^2 + \sin^2\gamma\,\Sigma\,.\,mz_{,}^2.$$

Let A, B, C be the moments of inertia of the system about the principal axes;

$$\therefore\ A = \Sigma\,.\,m(y_{,}^2 + z_{,}^2),\quad B = \Sigma\,.\,m(x_{,}^2 + z_{,}^2),\quad C = \Sigma\,.\,m(x_{,}^2 + y_{,}^3),$$

then $\Sigma\,.\,mx_{,}^2 = \frac{1}{2}(B + C - A)$, $\Sigma\,.\,my_{,}^2 = \frac{1}{2}(A + C - B)$,

$$\Sigma\,.\,mz_{,}^2 = \tfrac{1}{2}(A + B - C);$$

$$\therefore\ Mk_{,}^2 = \tfrac{1}{2}A(\sin^2\beta_{,} + \sin^2\gamma_{,} - \sin^2\alpha_{,})$$
$$+ \tfrac{1}{2}B(\sin^2\alpha_{,} + \sin^2\gamma_{,} - \sin^2\beta_{,}) + \tfrac{1}{2}C(\sin^2\alpha_{,} + \sin^2\beta_{,} - \sin^2\gamma_{,})$$
$$= A\cos^2\alpha_{,} + B\cos^2\beta_{,} + C\cos^2\gamma_{,}.$$

PROP. *If* A *and* C *be the greatest and least principal moments, then every other moment of inertia is intermediate to these.*

423. For $Mk_{,}^2 = A - (A - B)\cos^2\beta_{,} - (A - C)\cos^2\gamma_{,}$

and also $= C + (A - C)\cos^2\alpha_{,} + (B - C)\cos^2\beta_{,}$

since $\cos^2\alpha_{,} + \cos^2\beta_{,} + \cos^2\gamma_{,} = 1$.

The first is evidently less than A, and the second greater than C.

PROP. *When two of the principal moments are equal to each other, the moments about all axes lying in any right cone described about the principal axis of unequal moment are the same.*

424. For let $B = C$: then

$$Mk_{,}^{2} = A\cos^2\alpha_{,} + B(\cos^2\beta_{,} + \cos^2\gamma_{,}) = A\cos^2\alpha_{,} + B\sin^2\alpha_{,},$$

and this is constant when $\alpha_{,}$ remains the same although $\beta_{,}$ and $\gamma_{,}$ may vary.

PROP. *If the three principal moments be equal to each other, every other moment is equal to these.*

425. For $Mk_{,}^{2} = A(\cos^2\alpha_{,} + \cos^2\beta_{,} + \cos^2\gamma_{,}) = A$.

PROP. *To find the points in a system with respect to which the principal moments are equal to each other.*

426. Let the centre of gravity be the origin, and the principal axes the axes of co-ordinates:

$x_{,}y_{,}z_{,}$ co-ordinates to any particle m,

$x_{,}'y_{,}'z_{,}'$ the point which gives the principal moments equal: then from this point the co-ordinates of m are

$$x_{,} - x_{,}', \quad y_{,} - y_{,}', \quad z_{,} - z_{,}';$$

$\therefore$ by Art. 417. $\Sigma . m(x_{,} - x_{,}')(y_{,} - y_{,}') = 0$,

$\Sigma . m(x_{,} - x_{,}')(z_{,} - z_{,}') = 0$, and $\Sigma . m(y_{,} - y_{,}')(z_{,} - z_{,}') = 0$.

Observing the origin and axes we have chosen, we see that these conditions become,

$$Mx_{,}'y_{,}' = 0, \quad Mx_{,}'z_{,}' = 0, \quad My_{,}'z_{,}' = 0;$$

$\therefore$ two of $x_{,}'y_{,}'z_{,}'$ must $= 0$.

Suppose $y_{,}' = 0$, $z_{,}' = 0$ and then $x_{,}'$ remains indeterminate.

Hence by Art. 420,

moment about axis parallel to $x_,$ through $(x_,'y_,'z_,') = A$

.... $y_,$ .... $= B + Mx_,'^2$

.... $z_,$ .... $= C + Mx_,'^2$,

and these by hypothesis are all the same;

$$\therefore\ B = C, \text{ and } x_,'^2 = \frac{A - B}{M}.$$

Hence we derive the following corollaries.

1. If all the moments about the principal axes through the centre of gravity be unequal, there is no point in the system with respect to which the moments are equal.

2. If two of them be equal and the moment of the unequal one be the greatest, there are two points equally distant from the centre of gravity and on the axis of the greatest moment corresponding to which the moments are all equal.

3. When the principal moments are all equal, $x_,' = 0$, and there is no point but the centre of gravity with respect to which the moments are all equal.

CHAPTER X.

MOTION OF A RIGID BODY ACTED ON BY FORCES OF FINITE INTENSITY.

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427. In considering the equilibrium of a rigid body (Art. 27) we stated, that, in consequence of our ignorance of the nature and laws of the forces by which the molecules are held together, we are unable to deduce the conditions of equilibrium of a body from those of a single particle. By the aid, however, of the principle of the transmission of force through a body (Art. 28) we deduced certain relations which the impressed forces, that act upon the body when in equilibrium must satisfy independently of the molecular forces. It is evident that the system of molecular forces are themselves in equilibrium independently of the other forces which act upon the body.

In considering the motion of a rigid body we fall upon the same difficulty. We know nothing of the laws of the molecular forces, and consequently cannot calculate the motion of the body by calculating the motion of its molecules separately. But we may surmount this in the manner we overcame the difficulty just mentioned.

Let mX, mY, mZ be the *impressed* moving forces which act upon the particle m, not including the molecular forces which act upon this particle. Let xyz be the co-ordinates to m at the time t: then $m\dfrac{d^2x}{dt^2}$, $m\dfrac{d^2y}{dt^2}$, $m\dfrac{d^2z}{dt^2}$ are the *effective* moving forces of m (Art. 211).

Now by the first of the general principles enunciated in Art. 226, the forces

$$m\left(X-\frac{d^2x}{dt^2}\right),\quad m\left(Y-\frac{d^2y}{dt^2}\right),\quad m\left(Z-\frac{d^2z}{dt^2}\right)$$

acting on m parallel to the axes of x, y, z respectively, and similar forces acting on all the other particles ought, together with the molecular forces by which the particles of the body act upon each other, to satisfy the equations of equilibrium of forces acting on a rigid body.

But the molecular forces are of themselves in equilibrium, since the molecules retain their relative situations during the motion.

Hence the forces $m\left(X - \dfrac{d^2x}{dt^2}\right)$, $m\left(Y - \dfrac{d^2y}{dt^2}\right)$, $m\left(Z - \dfrac{d^2z}{dt^2}\right)$ acting on m and similar forces acting on the other particles of the body ought to satisfy the six equations of equilibrium of forces acting on a rigid body, given in Art. 65. Wherefore we have the six equations of motion

$$\Sigma . m\left(X - \frac{d^2x}{dt^2}\right) = 0, \quad \Sigma . m\left(Y - \frac{d^2y}{dt^2}\right) = 0, \quad \Sigma . m\left(Z - \frac{d^2z}{dt^2}\right) = 0,$$

$$\Sigma . m\left\{y\left(Z - \frac{d^2z}{dt^2}\right) - z\left(Y - \frac{d^2y}{dt^2}\right)\right\} = 0,$$

$$\Sigma . m\left\{z\left(X - \frac{d^2x}{dt^2}\right) - x\left(Z - \frac{d^2z}{dt^2}\right)\right\} = 0,$$

$$\Sigma . m\left\{x\left(Y - \frac{d^2y}{dt^2}\right) - y\left(X - \frac{d^2x}{dt^2}\right)\right\} = 0.$$

By these six equations we shall be able to calculate the motion of a rigid body acted on by any forces of finite intensity. They lead immediately to two Principles, one of which enables us to calculate the motion of *translation* of the body in space; and the other the motion of *rotation*.

PROP. *The motion of the centre of gravity of a body moving free in space and acted on by any forces is the same as if all the forces were applied at the centre of gravity parallel to their former directions.*

428. By the first three equations of Art. 427,

$$\Sigma . m\left(X - \frac{d^2x}{dt^2}\right) = 0, \quad \Sigma . m\left(Y - \frac{d^2y}{dt^2}\right) = 0, \quad \Sigma . m\left(Z - \frac{d^2z}{dt^2}\right) = 0.$$

Let $\bar{x}\,\bar{y}\,\bar{z}$ be the co-ordinates to the centre of gravity,

$x'y'z'$ ....m from the centre of gravity;

$$\therefore\ x = \bar{x} + x',\ y = \bar{y} + y',\ z = \bar{z} + z'.$$

Now $\Sigma\,.\,mx' = 0,\ \Sigma\,.\,my' = 0,\ \Sigma\,.\,mz' = 0$ (Art. 413).

Hence, substituting for xyz, the above equations give, M being the whole mass of the body,

$$\frac{d^2\bar{x}}{dt^2} = \frac{\Sigma\,.\,mX}{M},\quad \frac{d^2\bar{y}}{dt^2} = \frac{\Sigma\,.\,mY}{M},\quad \frac{d^2\bar{z}}{dt^2} = \frac{\Sigma\,.\,mZ}{M}:$$

and these are the equations we should obtain for the motion of the centre of gravity supposing the forces all applied at that point. Hence the Proposition is proved.

Prop. *The motion of rotation of a body acted on by any forces and moving freely is the same as if the centre of gravity were fixed and the same forces acted.*

429. The last three of the equations of Art. 427 are

$$\Sigma\,.\,m\left\{y\left(Z - \frac{d^2z}{dt^2}\right) - z\left(Y - \frac{d^2y}{dt^2}\right)\right\} = 0,$$

$$\Sigma\,.\,m\left\{z\left(X - \frac{d^2x}{dt^2}\right) - x\left(Z - \frac{d^2z}{dt^2}\right)\right\} = 0,$$

$$\Sigma\,.\,m\left\{x\left(Y - \frac{d^2y}{dt^2}\right) - y\left(X - \frac{d^2x}{dt^2}\right)\right\} = 0.$$

Now let $\bar{x},\ \bar{y},\ \bar{z}$ be the co-ordinates to the centre of gravity, and let (as before) $x = \bar{x} + x',\ y = \bar{y} + y',\ z = \bar{z} + z'$.

Let these be put in the above equations, observing that $\Sigma\,.\,mx' = 0,\ \Sigma\,.\,my' = 0,\ \Sigma\,.\,mz' = 0$ (Art. 413), and that therefore the differential coefficients of these with respect to t vanish; also bearing in mind the equations of last Article we have after all reductions,

$$\Sigma . m \left\{ y' \left(Z - \frac{d^2 z'}{dt^2} \right) - z' \left(Y - \frac{d^2 y'}{dt^2} \right) \right\} = 0,$$

$$\Sigma . m \left\{ z' \left(X - \frac{d^2 x'}{dt^2} \right) - x' \left(Z - \frac{d^2 z'}{dt^2} \right) \right\} = 0,$$

$$\Sigma . m \left\{ x' \left(Y - \frac{d^2 y'}{dt^2} \right) - y' \left(X - \frac{d^2 x'}{dt^2} \right) \right\} = 0.$$

But these are precisely the equations we should have obtained on the supposition that the centre of gravity were fixed, and that point taken as the origin of moments. Hence the Proposition is true.

430. From the first of the Principles demonstrated in the last two Articles we gather, that all the calculations we have made of the motion of a material particle will be true also of the centre of gravity of a rigid body. It remains then to ascertain the motion of the other parts of the body relative to the centre of gravity: and this the latter Principle enables us to accomplish, as we shall shew in the following Chapters. We shall consider the motion of rotation of a body first about any fixed axis, either passing through the centre of gravity or not, and lastly about a fixed point.

CHAPTER XI.

MOTION OF A RIGID BODY ABOUT A FIXED AXIS: FINITE FORCES.

PROP. *To calculate the angular accelerating force of a rigid body moving about a fixed axis, and acted on by any given forces.*

431. Let the fixed axis be taken as the axis of z, and let $x\,y$ be the co-ordinates to the projection of a particle m on the plane $x\,y$: also let r be the distance of m from the axis of rotation and θ the angle r makes with the plane $z\,x$: then $x = r\cos\theta,\ y = r\sin\theta$.

Now by Art. 68, we are to take only the last of the equations of Art 427;

$$\therefore\ \Sigma\,.\,m\left\{x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right\} = \Sigma\,.\,m\,(xY - yX).$$

$$\text{But } \frac{dx}{dt} = -r\sin\theta\,\frac{d\theta}{dt},\quad \frac{dy}{dt} = r\cos\theta\,\frac{d\theta}{dt};$$

$$\therefore\ x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = \frac{d}{dt}\left\{x\frac{dy}{dt} - y\frac{dx}{dt}\right\} = \frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = r^2\frac{d^2\theta}{dt^2}.$$

$$\text{Hence } \Sigma\,.\,mr^2\frac{d^2\theta}{dt^2} = \Sigma\,.\,m\,(xY - yX),$$

or, since $\frac{d^2\theta}{dt^2}$ is the same for every particle,

$$\frac{d^2\theta}{dt^2} = \frac{\Sigma\,.\,m\,(xY - yX)}{\Sigma\,.\,mr^2}$$

$$= \frac{\text{moment of the forces about the axis}}{\text{moment of inertia about the axis}}.$$

By integrating this equation we shall know the angle through which the body has revolved in a given time; and shall consequently be able to determine the position of the body at any instant.

PROP. *A body moves about a fixed horizontal axis acted on by gravity only: required to determine the time of a small oscillation.*

432. Let ABC (fig. 101.) be a section of the body made by the plane of the paper passing through the centre of gravity G and cutting the axis of rotation perpendicularly in C; P the projection of any particle m on this plane; CX vertical; GH perpendicular to CX; $CG = h$; $CP=r$; $PCX=\theta'$; $GCX=\theta$.

Then by Art. 431, $\dfrac{d^2\theta}{dt^2} = \dfrac{\text{moment of forces}}{\text{moment of inertia}}$

$$= -\frac{\Sigma . mgr \sin\theta'}{\Sigma . mr^2} = -\frac{Mgh\sin\theta}{M(k^2+h^2)} = -\frac{gh}{k^2+h^2}\sin\theta. \text{ Arts. 413, 420.}$$

If we put $\dfrac{k^2+h^2}{h} = l$, multiply by $2\dfrac{d\theta}{dt}$ and integrate;

$\therefore \dfrac{d\theta^2}{dt^2} = \dfrac{2g}{l}\cos\theta + \text{const.} = \dfrac{g}{l}(a^2-\theta^2)$, neglecting θ^3.... and supposing $\theta = a$ at first;

$$\therefore \text{time of oscillation} = -\sqrt{\frac{l}{g}}\int_{-a}^{a}\frac{d\theta}{\sqrt{a^2-\theta^2}} = \pi\sqrt{\frac{l}{g}}.$$

Hence the body will move as if collected in a material point at a distance $\dfrac{k^2+h^2}{h}$ from the axis. Take $CO = \dfrac{k^2+h^2}{h}$ in the line CG produced: then O is called the *centre of oscillation*: and $\dfrac{k^2+h^2}{h}$ is called the length of the *isochronous simple pendulum*, the body itself being denominated, in contradistinction, a *compound pendulum*. The point C is called the *centre of suspension*.

PROP. *The centres of oscillation and suspension are reciprocal: that is, if the body be suspended on an axis through* O *parallel to that through* C, *then* C *will be the centre of oscillation.*

433. For let l' be the length of the simple pendulum in this case; then

$$l' = \frac{k^2 + OG^2}{OG} = \frac{k^2}{l-h} + l - h$$

$$= \frac{lh - h^2}{l-h} + l - h \text{ (Art. 432.)} = l.$$

From which the truth of the Proposition is evident.

PROP. *To determine the length of the seconds pendulum experimentally.*

434. We have already shewn (Art. 396.) that if l be the length of a simple pendulum, that is, a pendulum consisting of a single particle suspended by a string without weight, t the duration of each oscillation and g the force of gravity, then

$$t = \pi \sqrt{\frac{l}{g}}.$$

But it is impossible to form a pendulum which may, with due regard to accuracy, be considered a simple pendulum. It becomes necessary, then, to measure the distance between the centres of suspension and oscillation (see Art. 432). The practical difficulties in the way of determining the latter point were considerable, and such as greatly to endanger the accuracy of the result, before Captain Kater removed the sources of difficulty by using the property of the compound pendulum proved in Art. 433, namely, that the centres of oscillation and suspension are reciprocal. We proceed to explain this.

Let AB be the pendulum (fig. 102); C the point of suspension; F a weight which may be shifted from one position to another on the pendulum: O the centre of oscillation of the pendulum including F.

The position of O is first found pretty accurately by making the pendulum oscillate about C and O till the times of oscillation are nearly the same. Knife edges are then fixed at C and O, and the weight F, which is placed near the middle point between C and O, is shifted till it is found that the time of oscillation about C and O is exactly the same. It remains only to measure CO and observe the time of oscillation. For the details of the experiment we refer the reader to the *Philosophical Transactions* for 1818. If t be the time of oscillation in seconds and $CO = l$, then, since the length of the simple pendulum varies as the square of the time of oscillation, the length of the seconds pendulum $= \dfrac{l}{t^2}$.

Prop. *To calculate the effect produced on the pendulum by shifting* F.

435. Let l' be the length of the simple pendulum when F is removed: $M(1+n)$ and M the masses of the pendulum with and without F, n being a small fraction: let l be the length of the simple pendulum when F is so situated that the times of oscillation about C and O are the same: and let L and L' be the lengths when the pendulum oscillates about C and O, the weight F being then at a distance x from C: and let $\frac{1}{2}l + \delta$ be the value of x when $L = l$. Then, by Art. 432,

$$l = \frac{\text{square of rad. of gyration about axis of suspension}}{\text{dist. of centre of gravity from same axis}}$$

$$= \frac{Ml'h + Mn\left(\frac{1}{2}l + \delta\right)^2}{Mh + Mn\left(\frac{1}{2}l + \delta\right)} = \frac{l'h + n\left(\frac{1}{2}l + \delta\right)^2}{h + n\left(\frac{1}{2}l + \delta\right)};$$

$$\therefore\ l' = l + \frac{n}{h}\left(\tfrac{1}{4}l^2 - \delta^2\right).$$

$$\text{Also } L = \frac{l'h + nx^2}{h + nx};$$

$$\therefore\ \frac{dL}{dx} = \frac{n^2x^2 + 2nhx - nl'h}{(h+nx)^2} = \frac{n^2(x-\alpha)(x+\beta)}{(h+nx)^2},$$

$$\text{where } \alpha = \frac{1}{n}\left\{\sqrt{h^2 + nl'h} - h\right\}$$

$$= \frac{h}{n}\left\{\frac{nl'}{2h} - \frac{n^2l'^2}{8h^2}\right\} \text{neg}^{\text{g}}.\, n^2 \ldots = \frac{l'}{2} - \frac{nl'^2}{8h}$$

$$= \frac{l}{2} - \frac{n\delta^2}{2h}$$

β is a positive quantity.

Let $CD = DO = \frac{1}{2}l$: and take $CP = \alpha$. Then if F be below P (that is, x greater than α), the time of oscillation about C will increase or decrease according as F is shifted from or towards C, since dL and dx have the same sign: the contrary will be the case when F is placed above P.

In like manner if the pendulum be suspended from O, we have a point Q, the distance of which from O equals $\frac{1}{2}l - \frac{n\delta^2}{2h'}$ (h' being the distance of the centre of gravity from O), such that when F is beyond Q from O the time of oscillation about O is increased or diminished according as F is moved further from O or nearer to it; and *vice versâ*.

Since $DP = \frac{n\delta^2}{2h}$, and $DQ = \frac{n\delta^2}{2h'}$, and these are both less than δ (δ being by hypothesis a very small quantity), it follows that F cannot be between P and Q when the times of oscillation about C and O are the same.

PROP. *To shew that if the axes of suspension be equal cylinders rolling on horizontal plates, instead of knife edges, the length of the simple pendulum still equals the distance of the axes.*

436. Let AB be the pendulum (fig. 103); G its centre of gravity, O its centre of oscillation, CDE the semi-cylindrical axis of suspension, C being the point of contact with the horizontal plane of support when the pendulum hangs in its position of rest: P the point of contact at the time t, when the pendulum oscillates; $CM = x$, $MG = y$, the co-ordinates to G, θ the angle CG makes with the vertical, R the pressure at P, F the friction on the plane of support, $CG = h$, M the mass of the pendulum, $CO = l$, $k =$ rad. of gyration about G, $a =$ rad. of the axis at C.

Now by Art. 428 the motion of the centre of gravity is the same as if all the forces were applied at that point;

$$\therefore \frac{d^2x}{dt^2} = -\frac{F}{M} \dots (1), \qquad \frac{d^2y}{dt^2} = g - \frac{R}{M} \dots (2).$$

Also by Art. 429 the motion of rotation is the same as if G were fixed; hence by Art. 431

$$\frac{d^2\theta}{dt^2} = \frac{Fy - R(a+h)\sin\theta}{Mk^2} \dots (3)$$

we have here three equations and five unknown quantities R, F, x, y, θ: we must seek, then, two relations connecting x, y, θ: these are

$$x = PM - PC = (a+h)\sin\theta - a\theta \dots (4)$$

$$y = (a+h)\cos\theta - a \dots (5).$$

By equations (1) (2) (3) we have

$$k^2\frac{d^2\theta}{dt^2} + y\frac{d^2x}{dt^2} + (a+h)\sin\theta\left(g - \frac{d^2y}{dt^2}\right) = 0;$$

differentiating (4) and (5) we have

$$\frac{dx}{d\theta} = (a+h)\cos\theta - a = y, \quad \frac{dy}{d\theta} = -(a+h)\sin\theta.$$

Hence our last equation becomes

$$k^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + \frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2} + (a+h)g\sin\theta\frac{d\theta}{dt} = 0;$$

$$\therefore k^2\frac{d\theta^2}{dt^2} + \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} = C + 2(a+h)g\cos\theta$$

when $\theta = \alpha$, velocity $= 0$;

$$\therefore k^2\frac{d\theta^2}{dt^2} + \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} = 2(a+h)g(\cos\theta - \cos\alpha);$$

$$\therefore \frac{d\theta^2}{dt^2}\left\{k^2 + (a+h)^2 + a^2 - 2a(a+h)\cos\theta\right\} = 2(a+h)g(\cos\theta - \cos\alpha)$$

$$\frac{d\theta^2}{dt^2} = \frac{(a+h)g(\alpha^2 - \theta^2)}{k^2 + h^2},$$

neglecting powers of α and θ higher than the square.

$$\therefore \frac{dt}{d\theta} = -\sqrt{\frac{k^2+h^2}{(a+h)g}} \frac{1}{\sqrt{a^2-\theta^2}},$$

$$t = \sqrt{\frac{k^2+h^2}{(a+h)g}} \cos^{-1}\frac{\theta}{a}, \text{ const.} = 0;$$

$$\therefore \text{ time of oscillation} = \pi \sqrt{\frac{k^2+h^2}{(a+h)g}}.$$

Also if b be the radius of the axis at O, and if $CO = m$, then

$$\text{time of oscillation about } O = \pi \sqrt{\frac{k^2+(m-h)^2}{(b+m-h)g}},$$

and these times being equal, we have

$$\frac{k^2+h^2}{a+h} = \frac{k^2+(m-h)^2}{b+m-h} = l;$$

$$\therefore l(a+h) - h^2 = (k^2 =) bl + (m-h)l - (m-h)^2;$$

$$\therefore l = \frac{(m-h)^2 - h^2}{m-2h+b-a} = \frac{m(m-2h)}{m-2h+b-a}.$$

If $b = a$, $l = m$; that is, the length of the simple pendulum equals the distance between the axes, when the cylinders are of equal radii.

437. Mr Lubbock has calculated, in a Paper read before the Royal Society in 1830, the errors in the length of the simple pendulum corresponding to given deviations of the knife edges. It is there shewn that a small deviation of one of the knife edges in azimuth is quite insensible: but that this is not the case for a small deviation in altitude: a deviation of one degree increases by 3 the vibrations of a seconds pendulum in 24 hours. A deviation from horizontality in the agate planes has a still greater influence: for a deviation in horizontality of 10′ increases by about 6 the vibrations in 24 hours.

Prop. *When a body moves about a fixed axis, required to find the pressure upon the axis at any instant.*

438. We shall suppose that the axis is fixed at two given points: let the axis of rotation be the axis of z, and let a and a' be the distances of the fixed points from the origin: let P, P' be the pressures at these points, $\alpha\beta\gamma$ and $\alpha'\beta'\gamma'$ the angles which their directions make with the axes of xyz respectively: X, Y, Z the impressed accelerating forces of the particle m, the co-ordinates of which are xyz at the time t; and therefore $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$ the effective accelerating forces of m: but since the *angular* accelerating force about the axis of rotation is calculated in Art. 432, we shall transform these effective forces as follows. Let f be the effective angular accelerating force, ω the angular velocity of the body at the time t, r the distance of m from the axis of rotation, θ the angle which r makes with the plane zx; then $x = r\cos\theta$, and $y = r\sin\theta$; differentiating twice with respect to t and observing that r does not vary with the time, and then replacing x and y, we have

$$\frac{d^2x}{dt^2} = -yf - x\omega^2, \quad \frac{d^2y}{dt^2} = xf - y\omega^2.$$

Then the moving forces $m(X+yf+x\omega^2)$, $m(Y-xf+y\omega^2)$, mZ acting parallel to the axes on the particle m, and similar forces acting on all the other particles, together with the pressures P, P' on the two fixed points of the axis ought to be in equilibrium at the time t, according to the first Principle of Art. 226. Hence by Art. 65,

$$P\cos\alpha + P'\cos\alpha' + \Sigma \,.\, m(X + yf + x\omega^2) = 0$$

$$P\cos\beta + P'\cos\beta' + \Sigma \,.\, m(Y - xf + y\omega^2) = 0$$

$$P\cos\gamma + P'\cos\gamma' + \Sigma \,.\, mZ = 0$$

$$-P\cos\beta \,.\, a - P'\cos\beta' \,.\, a' + \Sigma \,.\, m\{Zy - (Y - xf + y\omega^2)z\} = 0$$

$$P\cos\alpha \,.\, a + P'\cos\alpha' \,.\, a' + \Sigma \,.\, m\{(X + yf + x\omega^2)z - Zx\} = 0$$

$$\Sigma \,.\, m\{(Y - xf + y\omega^2)x - (X + yf + x\omega^2)y\} = 0.$$

These equations may generally be much simplified in applying them to any particular case, as we shall see in the

Chapter of Problems on this subject. The first, second, fourth, and fifth equations determine the four quantities $P\cos\alpha$, $P\cos\beta$, $P'\cos\alpha'$, $P'\cos\beta'$; from which the pressures perpendicular to the axis may be obtained. The third equation is the only equation which contains $P\cos\gamma$ and $P'\cos\gamma'$, and it shews that these quantities are indeterminate but that their sum must $= -\Sigma . mZ$. Lastly, the sixth equation is independent of the pressures, and, in short, determines the motion as calculated in Art. 432: this is easily seen, since the equation by reduction becomes

$$f . \Sigma . m(x^2 + y^2) = \Sigma . m(Yx - Xy).$$

The following Proposition is an application of these equations.

PROP. *The principal axes through the centre of gravity are* permanent axes, *when the body is not acted on by any forces.*

439. An axis is said to be *permanent* when the body permanently revolves about it when it is not fixed.

Let us suppose the body moves about a *fixed* principal axis. Since no forces act upon the body it follows that X, Y, Z each vanish, hence the equations of last Article become (since the sixth gives $f = 0$)

$$P\cos\alpha + P'\cos\alpha' + \omega^2\Sigma . mx = 0$$

$$P\cos\beta + P'\cos\beta' + \omega^2\Sigma . my = 0$$

$$P\cos\gamma + P'\cos\gamma' = 0$$

$$-Pa\cos\beta - P'a'\cos\beta' - \omega^2\Sigma . myz = 0$$

$$Pa\cos\alpha + P'a'\cos\alpha' + \omega^2\Sigma . mxz = 0.$$

Since the axis of z passes through the centre of gravity, therefore $\Sigma . mx = 0$, $\Sigma . my = 0$ (Art. 413): also if the other two principal axes $x_{,}y_{,}$ make each an angle ϕ with the axes of xy respectively at the time t, we have

$$x = x_{,}\cos\phi + y_{,}\sin\phi, \text{ and } y = y_{,}\cos\phi - x_{,}\sin\phi, \quad z = z_{,};$$

$$\therefore \Sigma . mxz = \cos\phi\,\Sigma . mx_{,}z_{,} + \sin\phi\,\Sigma . my_{,}z_{,} = 0;$$

so also $\Sigma . myz = 0$.

Hence the equations become

$$P\cos\alpha + P'\cos\alpha' = 0,\quad P\cos\beta + P'\cos\beta' = 0,\quad P\cos\gamma + P'\cos\gamma' = 0,$$

$$Pa\cos\beta + P'a'\cos\beta' = 0,\quad Pa\cos\alpha + P'a'\cos\alpha' = 0,$$

these give $P = 0$ and $P' = 0$. Hence there is no pressure on the fixed axis, and therefore it would not move if the body were to rotate about it when it is not fixed.

CHAPTER XII.

MOTION OF A RIGID BODY ABOUT A FIXED POINT: FINITE FORCES.

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440. In calculating the motion of a rigid body about a fixed point it is found most convenient to transform the equations of motion so as to contain angular co-ordinates and angular velocities.

Let the axes of co-ordinates be drawn through the fixed point: and suppose that $\omega' \omega'' \omega'''$ are three angular velocities such that if they were simultaneously impressed upon the body about the axes xyz respectively at the expiration of the time t, the motion of the body shall be what it actually is; then these are called the angular velocities of the body about the axes at that instant.

We shall always estimate those angular velocities positive which make the body revolve from the axis of x to the axis of y about z; from y to z about x; and from z to x about y: and those negative which act in the opposite directions.

When the axes of co-ordinates are principal axes we shall use $\omega_1 \omega_2 \omega_3$ for $\omega' \omega'' \omega'''$.

Prop. *To find the linear velocities, parallel to the axes of co-ordinates, of any particle of the body in terms of the angular velocities about the axes.*

441. Let xyz be the co-ordinates to particle m at P (fig. 104): draw PM perpendicular to the axis of x: PN perpendicular to plane xy: then at the time t the velocity of m about the axis of $x = \omega' PM$: resolving this parallel to the axes of y and z and reckoning those linear velocities positive which tend *from* the origin, and *vice versâ*, we have

vel. of m arising from ω' parallel to $y = -\omega' PM \sin PMN = -\omega' z$

.... $z = \omega' PM \cos PMN = \omega' y$,

also velocity of m arising from ω'' parallel to $x = \omega'' z$

.... $z = -\omega'' x$,

velocity of m arising from ω''' parallel to $x = -\omega''' y$

.... $y = \omega''' x$.

Adding together those velocities which are parallel to the same axes, we have

velocity of m parallel to $x = \omega'' z - \omega''' y$,

.... $y = \omega''' x - \omega' z$,

.... $z = \omega' y - \omega'' x$.

If m be at rest at the instant of expiration of the time t these expressions vanish; the third is a necessary consequence of the other two.

Hence $x = \dfrac{\omega'}{\omega'''} z$, $y = \dfrac{\omega''}{\omega'''} z$ are the equations to a straight line through the fixed point which is at rest at the instant under consideration.

This line is called the *Axis of Instantaneous Rotation.*

Prop. *To find the position of the instantaneous axis at any instant.*

442. Let $\alpha\beta\gamma$ be the angles which this line makes with the axes of xyz at the proposed instant: then by fig. 104,

$$\cos\alpha = \frac{AM}{AP} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{\omega'}{\sqrt{\omega'^2+\omega''^2+\omega'''^2}},$$

$$\cos\beta = \frac{MN}{AP} = \frac{y}{\sqrt{x^2+y^2+z^2}} = \frac{\omega''}{\sqrt{\omega'^2+\omega''^2+\omega'''^2}};$$

$$\cos\gamma = \frac{PN}{AP} = \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{\omega'''}{\sqrt{\omega'^2+\omega''^2+\omega'''^2}}.$$

By means of these we shall know the position at any instant when $\omega' \omega'' \omega'''$ are known.

PROP. *To find the angular velocity of the body about the instantaneous axis.*

443. Let ω be the required angular velocity: r the distance of the particle m from the origin: then the distance of this particle from the instantaneous axis

$$= r \sin (\angle \text{ between } r \text{ and inst. axis}) = r\sqrt{1 - \cos^2 (\text{same } \angle)}$$

$$= \sqrt{x^2 + y^2 + z^2 - (x \cos\alpha + y \cos\beta + z \cos\gamma)^2};$$

$\therefore$ the velocity of $m = \omega\sqrt{x^2+y^2+z^2-(x\cos\alpha+y\cos\beta+z\cos\gamma)^2}$.

But by Art. 441 the whole velocity

$$= \sqrt{(\omega'' z - \omega''' y)^2 + (\omega''' x - \omega' z)^2 + (\omega' y - \omega'' x)^2}.$$

Let us substitute for $\omega' \omega'' \omega'''$ in terms of $\alpha\beta\gamma$ by Art. 442, then whole velocity =

$$\sqrt{\omega'^2+\omega''^2+\omega'''^2}\sqrt{(z\cos\beta-y\cos\gamma)^2+(x\cos\gamma-z\cos\alpha)^2+(y\cos\alpha-x\cos\beta)^2}$$

$$= \sqrt{\omega'^2+\omega''^2+\omega'''^2}\sqrt{x^2+y^2+z^2-(x\cos\alpha+y\cos\beta+z\cos\gamma)^2}.$$

Hence by equating these expressions,

$$\omega = \sqrt{\omega'^2 + \omega''^2 + \omega'''^2},$$

this is the angular velocity required.

444. COR. If a body revolve about an axis with an angular velocity ω, then the resolved part of this about another axis inclined to the former at an angle α

$$= (\omega' = \sqrt{\omega'^2 + \omega''^2 + \omega'''^2} \cos\alpha =)\ \omega\cos\alpha.$$

PROP. *To find the inclinations of the instantaneous axis to the principal axes.*

445. Let $\alpha_{,}\beta_{,}\gamma_{,}$ be the angles the instantaneous axis makes with the principal axes, and $\omega_1\omega_2\omega_3$ the angular velocities about the principal axes.

$$\therefore \cos\alpha_{,} = \cos\alpha\cos x_{,}x + \cos\beta\cos x_{,}y + \cos\gamma\cos x_{,}z$$

$$= \frac{\omega'}{\omega}\cos x_{,}x + \frac{\omega''}{\omega}\cos x_{,}y + \frac{\omega'''}{\omega}\cos x_{,}z, \text{ (Arts. 443, 444.)}$$

$= \dfrac{\omega_1}{\omega}$, since by resolving the angular velocities $\omega'\omega''\omega'''$ about the axis of $x_{,}$ we have by Art. 444,

$$\omega_1 = \omega'\cos x_{,}x + \omega''\cos x_{,}y + \omega'''\cos x_{,}z.$$

$$\text{Similarly } \cos\beta_{,} = \frac{\omega_2}{\omega}, \quad \cos\gamma_{,} = \frac{\omega_3}{\omega}.$$

Cor. Also $\omega'^2 + \omega''^2 + \omega'''^2 = \omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2$.

Prop. *To obtain equations for calculating the angular velocities about the principal axes at any instant.*

446. Let Ax, Ay, Az be the axes of co-ordinates fixed in space; A being the fixed point of the body;

$Ax_{,}$, $Ay_{,}$, $Az_{,}$ the principal axes in the body.

Then the three equations of rotatory motion are by Art. 427.

$$\Sigma . m\left\{y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right\} = \Sigma . m\left\{yZ - zY\right\} = L \text{ suppose}$$

$$\Sigma . m\left\{z\frac{d^2x}{dt^2} - x\frac{d^2z}{dt^2}\right\} = \Sigma . m\left\{zX - xZ\right\} = M \ldots\ldots.$$

$$\Sigma . m\left\{x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right\} = \Sigma . m\left\{xY - yX\right\} = N \ldots\ldots.$$

Now by Art. 441.

$$\frac{dx}{dt} = \omega''z - \omega'''y, \quad \frac{dy}{dt} = \omega'''x - \omega'z, \quad \frac{dz}{dt} = \omega'y - \omega''x.$$

By differentiating these with respect to t

$$\frac{d^2x}{dt^2} = \omega''\frac{dz}{dt} - \omega'''\frac{dy}{dt} + \frac{d\omega''}{dt}z - \frac{d\omega'''}{dt}y$$

$$= -(\omega''^2 + \omega'''^2)\,x + \omega'\omega''\,y + \omega'\omega'''\,z + \frac{d\omega''}{dt}\,z - \frac{d\omega'''}{dt}\,y,$$

so $$\frac{d^2y}{dt^2} = -(\omega'^2 + \omega'''^2)\,y + \omega''\omega'\,x + \omega''\omega'''\,z + \frac{d\omega'''}{dt}\,x - \frac{d\omega'}{dt}\,z;$$

$$\frac{d^2z}{dt^2} = -(\omega'^2 + \omega''^2)\,z + \omega'''\omega'\,x + \omega'''\omega''\,y + \frac{d\omega'}{dt}\,y - \frac{d\omega''}{dt}\,x.$$

Hence $$\Sigma\,.\,m\left\{y\,\frac{d^2z}{dt^2} - z\,\frac{d^2y}{dt^2}\right\} = (\omega'''^2 - \omega''^2)\,\Sigma\,.\,myz$$

$$+ \left(\omega'''\omega' - \frac{d\omega''}{dt}\right)\Sigma\,.\,mxy - \left(\omega''\omega' + \frac{d\omega'''}{dt}\right)\Sigma\,.\,mxz$$

$$+ \omega'''\omega''\,\Sigma\,.\,my^2 - \omega''\omega'''\,\Sigma\,.\,mz^2 + \frac{d\omega'}{dt}\,\Sigma\,.\,m\,(y^2 + z^2) = L \;\ldots\; (1).$$

Now suppose the fixed axes Ax, Ay, Az were so chosen that at the instant of expiration of the time t the principal axes should coincide with them. Then *at this instant*

$$\Sigma.mxy = 0,\quad \Sigma.mxz = 0,\quad \Sigma.myz = 0: \text{ also } \omega' = \omega_1,\ \omega'' = \omega_2,\ \omega''' = \omega_3;$$

and likewise $\frac{d\omega'}{dt} = \frac{d\omega_1}{dt}$, for the changes in the two angular velocities ω' and ω_1 during a given small time after the axis of $x_,$ coincides with the axis of x will differ only by a quantity which depends upon the angle passed through by the axis of $x_,$ during that given small time: the difference between ω' and ω_1 will therefore be an infinitessimal of the second order and therefore their differential coefficients will be equal. Hence equation (1) becomes *at this instant*

$$\omega_2\omega_3\,\Sigma\,.\,m\,(y_,^2 - z_,^2) + \frac{d\omega_1}{dt}\,\Sigma\,.\,m\,(y_,^2 + z_,^2) = L_,$$

the letters with subscript accents having reference to the principal axes.

Now this equation is independent of the epoch from which the time is measured: it is also independent of the angles which the principal axes make with the fixed axes in space. It follows, then, that this equation will hold for *every instant* of the time t; and is therefore generally true.

Now $\Sigma . m(y_{,}^2 + z_{,}^2) = A$; and $\Sigma . m(y_{,}^2 - z_{,}^2) = C - B$;

$$\left.\begin{aligned} \therefore\ A\frac{d\omega_1}{dt} + (C-B)\,\omega_2\omega_3 &= L_{,} \\ \text{similarly}\ B\frac{d\omega_2}{dt} + (A-C)\,\omega_1\omega_3 &= M_{,} \\ C\frac{d\omega_3}{dt} + (B-A)\,\omega_1\omega_2 &= N_{,} \end{aligned}\right\} \ldots\ldots\ldots (2).$$

By means of these three equations the three quantities $\omega_1\omega_2\omega_3$ must be determined.

PROP. *To determine the position of the body in space when the angular velocities about the principal axes are known.*

447. We consider, as before, those angular velocities positive which tend to turn the body from the axis of $x_{,}$ to the axis $y_{,}$ about $z_{,}$, from $y_{,}$ to $z_{,}$ about $x_{,}$, and from $z_{,}$ to $x_{,}$ about $y_{,}$.

Also by Art. 444 an angular velocity is resolved about any new axis by multiplying it by the cosine of the angle between the axes.

Now the position of the principal axes of the body at the time t, is determined by the values of θ, ϕ, ψ, these angles being measured as explained in Art. 415: it follows, then, that

$\omega_1\ \omega_2\ \omega_3$ must be functions of θ, ϕ, ψ, and $\dfrac{d\theta}{dt}$, $\dfrac{d\phi}{dt}$, $\dfrac{d\psi}{dt}$.

The resolved parts of $\dfrac{d\theta}{dt}$ about the axes of $x_{,}\,y_{,}\,z_{,}$ are

$$\frac{d\theta}{dt}\cos\phi, \quad -\frac{d\theta}{dt}\sin\phi, \quad 0,$$

the resolved parts of $\frac{d\phi}{dt}$ about these axes are

$$0, \qquad 0, \qquad \frac{d\phi}{dt},$$

and the resolved parts of $\frac{d\psi}{dt}$ about these axes are

$$-\frac{d\psi}{dt}\cos zx_{,}, \quad -\frac{d\psi}{dt}\cos zy_{,}, \quad -\frac{d\psi}{dt}\cos zz_{,},$$

or $-\frac{d\psi}{dt}\sin\phi\sin\theta, \quad -\frac{d\psi}{dt}\cos\phi\sin\theta, \quad -\frac{d\psi}{dt}\cos\theta$ (see Art. 415).

Hence, adding those about the same axes,

$$\omega_1 = \frac{d\theta}{dt}\cos\phi - \frac{d\psi}{dt}\sin\phi\sin\theta,$$

$$\omega_2 = -\frac{d\theta}{dt}\sin\phi - \frac{d\psi}{dt}\cos\phi\sin\theta,$$

$$\omega_3 = \frac{d\phi}{dt} - \frac{d\psi}{dt}\cos\theta.$$

In these we must substitute the values of $\omega_1\ \omega_2\ \omega_3$ obtained by integrating the equations in Art. 446, and we shall find θ, ϕ, ψ, and so determine the position of the principal axes, and consequently of the body, at any proposed instant.

448. Cor. 1. By the above equations we obtain

$$\frac{d\theta}{dt} = \omega_1\cos\phi - \omega_2\sin\phi$$

$$\sin\theta\frac{d\psi}{dt} = -\omega_1\sin\phi - \omega_2\cos\phi$$

$$\frac{d\phi}{dt} = \omega_3 - \frac{\cos\theta}{\sin\theta}(\omega_1\sin\phi + \omega_2\cos\phi).$$

449. Cor. 2. When θ is very small these become

$$\frac{d\theta}{dt} = \omega_1\cos\phi - \omega_2\sin\phi$$

$$\frac{d\psi}{dt} = -\frac{\omega_1}{\dot\theta}\sin\phi - \frac{\omega_2}{\theta}\cos\phi,$$

$$\frac{d\phi}{dt} = \omega_3 - \frac{\omega_1}{\dot\theta}\sin\phi - \frac{\omega_2}{\theta}\cos\phi.$$

PROP. *A body revolves about its centre of gravity acted on by no forces but such as pass through that point: required to integrate the equations of motion.*

450. The equations (2) of Art. 446 become in this case

$$A\frac{d\omega_1}{dt} + (C-B)\,\omega_2\,\omega_3 = 0,$$

$$B\frac{d\omega_2}{dt} + (A-C)\,\omega_1\,\omega_3 = 0,$$

$$C\frac{d\omega_3}{dt} + (B-A)\,\omega_1\,\omega_2 = 0,$$

the principal axes being drawn through the centre of gravity.

Multiply these equations by $\omega_1\ \omega_2\ \omega_3$ respectively and add; then

$$A\omega_1\frac{d\omega_1}{dt} + B\,\omega_2\frac{d\omega_2}{dt} + C\,\omega_3\frac{d\omega_3}{dt} = 0$$

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = \text{constant} = h^2.$$

Again multiply the equations by $A\omega_1$, $B\omega_2$, $C\omega_3$, and add;

$$\therefore\ A^2\omega_1\frac{d\omega_1}{dt} + B^2\omega_2\frac{d\omega_2}{dt} + C^2\omega_3\frac{d\omega_3}{dt} = 0;$$

$$\therefore\ A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = \text{constant} = k^2.$$

Eliminating ω_3^2 from these two equations, we have

$$A(A-C)\,\omega_1^2 + B(B-C)\,\omega_2^2 = k^2 - Ch^2;$$

$$\therefore\ \omega_2^2 = \frac{1}{B(B-C)}\left\{k^2 - Ch^2 - A(A-C)\,\omega_1^2\right\},$$

$$\text{and } \omega_3^2 = \frac{1}{C(C-B)}\left\{k^2 - Bh^2 - A(A-B)\,\omega_1^2\right\}.$$

Hence the first of the equations of motion gives

$$\frac{d\omega_1}{dt} + \sqrt{\frac{(A-C)(A-B)}{BC}\left\{\omega_1^2 - \frac{k^2 - Ch^2}{A(A-C)}\right\}\left\{\frac{k^2 - Bh^2}{A(A-B)} - \omega_1^2\right\}} = 0$$

the integral of this equation, which in the general case cannot be found, will give ω_1 in terms of t and then ω_2 and ω_3 will be known.

Knowing $\omega_1 \omega_2 \omega_3$ the position of the body at any time is determined by integrating the equations of Art. 447.

Prop. *When the body is acted on by no forces except such as pass through the origin, there exists a plane to which it may be referred, which plane is invariable in position.*

451. Let

abc be the cosines of the angles which x makes with $x_, y_, z_,$
$a'b'c'$ .... y ....
$a''b''c''$ .... z ....

We shall now seek the values of the differential coefficients of these with respect to t.

Let the planes yz, $y_,z_,$ cut in the line AK (fig. 105): then this line is perpendicular to the plane $xAx_,$: let AI be the instantaneous axis: describe a sphere about A of radius unity and cutting the axes of co-ordinates, AK, and AI in the points marked in the figure.

Then $\frac{d \,.\, xAx_,}{dt}$ = the angular velocity about AK

$$= \omega \cos IAK, \text{ (Art. 444.)}$$

$$= \omega (\cos \alpha_, \cos Kx_, + \cos \beta_, \cos Ky_, + \cos \gamma_, \cos Kz_,)$$

$$= \omega (\cos \beta_, \sin Kz_, + \cos \gamma_, \cos Kz_,), \quad \because \; Kx_, = 90^0;$$

$$\therefore \frac{da}{dt} = \frac{d \cos xx_,}{dt} = -\omega_2 \sin xx_, \sin Kz_, - \omega_3 \sin xx_, \cos Kz_,$$

$$= -\omega_2 \sin xx_, \sin Kx_,z_, - \omega_3 \sin xx_, \cos Kx_,z_,$$

$$= -\omega_2 \sin xx_, \cos z_,x_,x + \omega_3 \sin xx_, \cos y_,x_,x$$

$$= -\omega_2 \cos xz_, + \omega_3 \cos xy_, = -\omega_2 c + \omega_3 b.$$

Similarly we should obtain

$$\frac{db}{dt} = \frac{d \cos x y_{,}}{dt} = -\omega_3 \cos x x_{,} + \omega_1 \cos x z_{,} = -\omega_3 a + \omega_1 c,$$

$$\frac{dc}{dt} = \frac{d \cos x z_{,}}{dt} = -\omega_1 \cos x y_{,} + \omega_2 \cos x x_{,} = -\omega_1 b + \omega_2 a, \text{ and so on.}$$

Now multiply the equations (2) of Art. 446 by a, b, c respectively and add,

$$\therefore A\left\{a \frac{d\omega_1}{dt} + \omega_1 (b\omega_3 - c\omega_2)\right\} + B\left\{b \frac{d\omega_2}{dt} + \omega_2 (c\omega_1 - a\omega_3)\right\}$$
$$+ C\left\{c \frac{d\omega_3}{dt} + \omega_3 (a\omega_2 - b\omega_3)\right\} = 0,$$

$$\text{or } A\left\{a \frac{d\omega_1}{dt} + \omega_1 \frac{da}{dt}\right\} + B\left\{b \frac{d\omega_2}{dt} + \omega_2 \frac{db}{dt}\right\} + C\left\{c \frac{d\omega_3}{dt} + \omega_3 \frac{dc}{dt}\right\} = 0;$$

$$\therefore Aa\omega_1 + Bb\omega_2 + Cc\omega_3 = \text{constant} = l.$$

Similarly, $Aa'\omega_1 + Bb'\omega_2 + Cc'\omega_3 = l'$,

$$Aa''\omega_1 + Bb''\omega_2 + Cc''\omega_3 = l''.$$

Add the squares of these together; observing that since the angle between any two axes of the same system of co-ordinates equals a right angle, therefore

$$ab + a'b' + a''b'' = 0, \quad ac + a'c' + a''c'' = 0, \quad bc + b'c' + b''c'' = 0;$$

and we have

$$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = l^2 + l'^2 + l''^2 = k^2 \text{ by Art. 450.}$$

Hence if we draw a line AI' making angles with the fixed axes of which the cosines are

$$\frac{Aa\omega_1 + Bb\omega_2 + Cc\omega_3}{\sqrt{A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2}}, \quad \frac{Aa'\omega_1 + Bb'\omega_2 + Cc'\omega_3}{\sqrt{A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2}},$$
$$\frac{Aa''\omega_1 + Bb''\omega_2 + Cc''\omega_3}{\sqrt{A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2}},$$

this line, and therefore the plane perpendicular to it, will remain invariable during the whole motion. For this reason it is called the *Invariable Plane*. In a future Chapter we shall speak more of this plane.

452. Cor. 1.

$$\cos. I'Ax_{,} = a \cos I'Ax + a' \cos I'Ay + a'' \cos I'Az$$
$$= \frac{A\omega_1}{k},$$

$$\text{also } \cos I'Ay_{,} = \frac{B\omega_2}{k}, \quad \cos I'Az_{,} = \frac{C\omega_3}{k}.$$

453. Cor. 2. If the invariable plane be taken for the plane of xy then

$$\frac{A\omega_1}{k} = \cos zx_{,} = \sin \phi \sin \theta : \text{ Art. 415.}$$

$$\frac{B\omega_2}{k} = \cos zy_{,} = \cos \phi \sin \theta,$$

$$\frac{C\omega_3}{k} = \cos zz_{,} = \cos \theta.$$

The equation in Art. 450 for finding ω_1 can be integrated when the principal moments are equal: and also when two only of them are equal. We shall investigate these cases.

Prop. *To find the motion of a rigid body about a fixed point when its principal moments are equal to each other: the forces all passing through the centre of gravity fixed.*

454. In this case $B = C = A$: and the equations of Art. 450 give

$$\omega_1 = \text{constant}, \quad \omega_2 = \text{constant}, \quad \omega_3 = \text{constant};$$

and therefore the instantaneous axis remains fixed in the body: see Art 445.

Since every axis is a principal axis (Art. 425); let the axis of $z_{,}$ coincide with the instantaneous axis.

$$\therefore \omega_1 = 0, \quad \omega_2 = 0, \quad \omega_3 = \text{constant} = n \text{ suppose.}$$

Hence the equations of Art. 447 become

$$0 = \frac{d\theta}{dt} \cos \phi - \frac{d\psi}{dt} \sin \phi \sin \theta,$$

$$0 = -\frac{d\theta}{dt} \sin \phi - \frac{d\psi}{dt} \cos \phi \sin \theta,$$

$$n = \frac{d\phi}{dt} - \frac{d\psi}{dt} \cos \theta.$$

By the first and second equations, we have

$$\frac{d\theta}{dt} = 0, \quad \frac{d\psi}{dt} = 0; \quad \therefore \theta \text{ and } \psi \text{ are constant.}$$

Also by the third equation $\frac{d\phi}{dt} = n$; $\therefore \phi = nt + \text{const.}$: this shews that the body revolves about a fixed axis: hence the instantaneous axis is not only fixed in the body, but also in space. The position of this axis and the magnitude of the angular velocity depend upon the circumstances of projection.

PROP. *When* A = B *required to determine the motion.*

455. The equations (2) of Art. 446 become

$$A \frac{d\omega_1}{dt} + (C - A)\, \omega_2 \omega_3 = 0,$$

$$A \frac{d\omega_2}{dt} + (A - C)\, \omega_1 \omega_3 = 0,$$

$$C \frac{d\omega_3}{dt} = 0;$$

$\therefore \omega_3 = \text{const.} = n$ suppose. Also by differentiating the first equations we have

$$A \frac{d^2 \omega_1}{dt^2} + n(C - A) \frac{d\omega_2}{dt} = 0,$$

and therefore by this and the second equation

$$\frac{d^2 \omega_1}{dt^2} + \left(\frac{C - A}{A}\right)^2 n^2 \omega_1 = 0;$$

$$\therefore\ \omega_1 = e \cos\left\{\frac{C-A}{A} nt + f\right\};$$

$$\therefore\ \omega_2 = -\frac{A}{C-A}\frac{1}{n}\frac{d\omega_1}{dt} = e \sin\left\{\frac{C-A}{A} nt + f\right\},$$

where e and f are constants to be determined by the circumstances of the motion at some given time.

The angular velocity (ω) about the instantaneous axis

$= \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}$ (Art. 443.) $= \sqrt{n^2 + e^2}$, and is constant.

We shall now substitute the values of $\omega_1\, \omega_2\, \omega_3$ in the equations of Art. 447: and put $\frac{C-A}{A} n = m$;

$$\therefore\ \frac{d\theta}{dt}\cos\phi - \frac{d\psi}{dt}\sin\phi\sin\theta = e\cos(mt+f)$$

$$-\frac{d\theta}{dt}\sin\phi - \frac{d\psi}{dt}\cos\phi\sin\theta = e\sin(mt+f)$$

$$\frac{d\phi}{dt} \qquad - \frac{d\psi}{dt}\cos\theta \qquad = n.$$

Let us take the Invariable Plane for the plane of xy: then by Art. 453.

$$\sin\phi\sin\theta = \frac{A}{k}\omega_1 = \frac{Ae}{k}\cos(mt+f),$$

$$\cos\phi\sin\theta = \frac{A}{k}\omega_2 = \frac{Ae}{k}\sin(mt+f),$$

$$\cos\theta \qquad = \frac{C}{k}\omega_3 = \frac{Cn}{k};$$

$$\therefore\ \tan\phi = \cot(mt+f),$$

$$\therefore\ \phi = \frac{\pi}{2} - mt - f;\ \text{ and }\ \frac{d\phi}{dt} = -m.$$

Also, since $\cos\theta = \frac{Cn}{k}$, θ is constant.

$$\text{And } \frac{d\psi}{dt} = \frac{1}{\cos\theta}\left(\frac{d\phi}{dt} - n\right) = -\frac{m+n}{n}\frac{k}{C} = -\frac{k}{A}.$$

Hence the body revolves uniformly about the principal axis $Az_{,}$; while the line of nodes (that is, the line of intersection of the planes xy and $x_{,}y_{,}$) revolves uniformly on the plane xy.

Also $\cos IAN$

$= \cos\alpha_{,} \cos\phi + \cos\beta_{,} \cos(\frac{1}{2}\pi + \phi) + \cos\gamma_{,} \cos\frac{1}{2}\pi$, see fig. 105,

$$= \frac{\omega_1}{\omega}\cos\phi - \frac{\omega_2}{\omega}\sin\phi = 0;$$

and therefore the instantaneous axis of rotation is always perpendicular to the line of nodes.

Again $\cos IAz_{,} = \frac{\omega_3}{\omega} = \frac{n}{\sqrt{n^2+e^2}}$; and therefore the instantaneous axis always makes the same angle with the axis of $z_{,}$.

We shall now obtain the arbitrary constants from the circumstances when $t = 0$.

Since any axis in the plane $x_{,}y_{,}$ is a principal axis (Art. 424), let the axis of $x_{,}$ be so chosen, that when $t = 0$ it coincides with the line of intersection of the planes $IAz_{,}$ and $x_{,}y_{,}$.

Let ω_0 be the angular velocity about the instantaneous axis when $t = 0$, and let $\angle IAz_{,} = \delta$; therefore when $t = 0$,

$$\omega_1 = \omega_0 \cos Ix_{,} = \omega_0 \sin\delta,$$

$$\omega_2 = \omega_0 \cos Iy_{,} = 0,$$

$$\omega_3 = \omega_0 \cos Iz_{,} = \omega_0 \cos\delta,$$

and consequently by the general values of $\omega_1\, \omega_2\, \omega_3$ we have

$$e\cos f = \omega_0 \sin\delta, \quad e\sin f = 0, \quad n = \omega_0\cos\delta;$$

$$\therefore f = 0, \quad e = \omega_0\sin\delta, \quad n = \omega_0\cos\delta,$$

$$k = \sqrt{A^2(\omega_1^2+\omega_2^2) + C\omega_3^2} \text{ (Art. 450.)} = \omega_0\sqrt{A^2\sin^2\delta + C^2\cos^2\delta}.$$

$$\text{Hence } \phi = \frac{\pi}{2} - \frac{C-A}{A}\,\omega_0 \cos\delta \,.\, t,$$

$$\frac{d\psi}{dt} = -\frac{\omega_0}{A}\sqrt{A^2 \sin^2\delta + C^2\cos^2\delta},$$

$$\cos\theta = \frac{C\cos\delta}{\sqrt{A^2\sin^2\delta + C^2\cos^2\delta}}; \quad \text{or} \quad \frac{\tan\theta}{\tan\delta} = \frac{A}{C}.$$

456. These formulæ lead to the following geometrical construction, fig. 106. Let the axes of co-ordinates $z_{,}z$ and the instantaneous axis cut a sphere of radius unity described about A in the points $z_{,}zI$ respectively.

$$\text{Then } \frac{d\psi}{dt} \div \frac{d\phi}{dt} = \frac{\sqrt{A^2\sin^2\delta + C^2\cos^2\delta}}{(C-A)\cos\delta} = \frac{C}{(C-A)\cos\theta} = \frac{\sin\delta}{\sin(\delta-\theta)}$$

$= \dfrac{\sin Iz_{,}}{\sin Iz}$, since AI is perpendicular to AN and makes a constant angle with Az, and is consequently always in the plane $zAz_{,}$.

About $z_{,}$ and z describe two small circles on the sphere of which the radii measured on the sphere are δ and $\delta - \theta$: then these circles touch in the point I, where the instantaneous axis meets the sphere.

Suppose I' and I'' are the points in these circles which were in contact when $t = 0$.

Then, since the angular velocities about Az and $Az_{,}$ are uniform and equal to $\dfrac{d\psi}{dt}$ and $\dfrac{d\phi}{dt}$, we have

$$\text{arc } I'I = \text{angle } I'zI \sin Iz = t\frac{d\psi}{dt}\sin Iz$$

$$= t\frac{d\phi}{dt}\sin Iz_{,} = \text{angle } I''z_{,}I \sin Iz_{,} = \text{arc } I''I,$$

wherefore the motion of the body may be described by making the circle $z_{,}I$ *roll* with its internal surface on the fixed circle zI.

Cor. If C be less than A, then I will be between z and $z_{,}$, and the external circumference of the circle $z_{,}I$ would roll on the circle Iz.

457. If we observe the apparent motion of the stars night after night we remark, that they all seem to move in parallel circles about the star named (on that account) the Pole Star. This proves that the axis about which the Earth revolves points towards the Pole Star, and never deviates from that direction by an angle appreciable by ordinary observation. Also geodetic measurements and other calculations for ascertaining the Figure of the Earth shew, that this axis of rotation coincides (so far as the approximation is carried) with the geometrical axis of the spheroidal form of the Earth's Surface. Theory shews that there is a necessary connexion between these two facts which are apparently independent of each other. This we proceed to prove.

PROP. *Suppose the Earth revolves about an axis nearly coinciding with one of its principal axes at any given time: required to find the motion, all external forces being neglected.*

458. Let the axis of $z_{\prime}$ be that near which the instantaneous axis lies at the given time t. Now the sine of the angle which these two axes make with each other $= \sqrt{\dfrac{\omega_1^2+\omega_2^2}{\omega_1^2+\omega_2^2+\omega_3^2}}$ (Art. 445), and this is small by hypothesis: hence $\omega_1^2+\omega_2^2$ is small, and ω_1 and ω_2 are small: and the equations (2) of Art. 446 give

$$A\frac{d\omega_1}{dt}+(C-B)\,\omega_2\omega_3=0,$$

$$B\frac{d\omega_2}{dt}+(A-C)\,\omega_1\omega_3=0,$$

$$C\frac{d\omega_3}{dt}+(B-A)\,\omega_1\omega_2=0,$$

then neglecting the product of ω_1 and ω_2, the last equation gives $\omega_3 = \text{constant} = n$: and the others give

$$A\frac{d\omega_1}{dt}+(C-B)\,\omega_2 n=0,$$

$$B\frac{d\omega_2}{dt} + (A - C)\,\omega_1 n = 0,$$

$$\therefore\ \frac{d^2\omega_1}{dt^2} + \frac{(A-C)(B-C)}{AB}\,n^2\omega_1 = 0;$$

$$\therefore\ \omega_1 = e\sin\left\{\sqrt{\frac{(A-C)(B-C)}{AB}}\,nt + f\right\},$$

e and f being constants which depend upon the circumstances at any given time;

$$\therefore\ \omega_2 = \frac{A}{(B-C)\,n}\,\frac{d\omega_1}{dt}$$

$$= e\sqrt{\frac{A}{B}\,\frac{A-C}{B-C}}\cos\left\{\sqrt{\frac{(A-C)(B-C)}{AB}}\,nt + f\right\},$$

and since $\omega_1^2 + \omega_2^2$ is small at the given time, e is small: and since e is constant it shews that ω_1 and ω_2 are *always* small so long as $(A - C)(B - C)$ is positive.

If however $(A - C)(B - C)$ be negative, then the trigonometrical expressions for $\omega_1\omega_2$ must be replaced by exponentials, and consequently they will not remain small.

From this we gather that if a body revolve at any time about an axis coinciding nearly with the principal axis of greatest or least moment, the axis of rotation will always nearly coincide with that principal axis. But if the axis be that of mean moment the instantaneous axis of rotation will deviate more and more from that principal axis till it approaches the principal axis of either greatest or least moment.

COR. If the instantaneous axis actually coincide with a principal axis at first, then $e = 0$, and ω_1 and ω_2 each vanish. Hence any principal axis is a permanent axis (see Art. 439).

If, however, the slightest cause tend to make the instantaneous axis of rotation deviate from the principal axis, the rotatory motion may be said to be *stable* or *unstable* according as the principal axis in question is not or is the *mean* principal axis.

This points out an admirable adaptation in the laws of nature: that the motion of rotation which causes the heavenly bodies to *bulge* at their equators, in so doing, gives them such a figure as to insure the stability of their rotation.

We shall now consider the action of the Sun and Moon on the rotatory motion of the Earth.

PROP. *To obtain equations for calculating the rotatory motion of the Earth when acted on by the Sun and Moon.*

459. The equations of motion referred to the principal axes are by Art. 446,

$$A\frac{d\omega_1}{dt} + (C - B)\,\omega_2\,\omega_3 = L_{\prime}$$

$$B\frac{d\omega_2}{dt} + (A - C)\,\omega_1\,\omega_3 = M_{\prime}$$

$$C\frac{d\omega_3}{dt} + (B - A)\,\omega_1\,\omega_2 = N_{\prime}.$$

To calculate $L_{\prime} M_{\prime} N_{\prime}$ let S be the mass of one of the disturbing bodies: $x_{\prime} y_{\prime} z_{\prime}$ the co-ordinates of the centre of S; $x_{\prime}' y_{\prime}' z_{\prime}'$ the co-ordinates to any particle m of the Earth's mass referred to the principal axes: $r^2 = x_{\prime}^2 + y_{\prime}^2 + z_{\prime}^2$.

Then the difference of the attractions of S on the particle (m), and the centre of the Earth (which we here suppose fixed, see Art. 429.) resolved parallel to the axes $y_{\prime}$ and $z_{\prime}$ and estimated positive in directions *from* the origin, are

$$\frac{S(y_{\prime} - y_{\prime}')}{\{(x_{\prime} - x_{\prime}')^2 + (y_{\prime} - y_{\prime}')^2 + (z_{\prime} - z_{\prime}')^2\}^{\frac{3}{2}}} - \frac{Sy_{\prime}}{r^3}, \quad = Y_{\prime} \text{ suppose}$$

$$\frac{S(z_{\prime} - z_{\prime}')}{\{(x_{\prime} - x_{\prime}')^2 + (y_{\prime} - y_{\prime}')^2 + (z_{\prime} - z_{\prime}')^2\}^{\frac{3}{2}}} - \frac{Sz_{\prime}}{r^3}, \quad = Z_{\prime} \text{ suppose.}$$

Hence $L_{\prime} = \Sigma . m\,(y_{\prime}' Z_{\prime} - z_{\prime}' Y_{\prime})$, see Art. 446.

$$= S\Sigma . m\left\{\frac{z_{\prime}y_{\prime}' - y_{\prime}z_{\prime}'}{\{r^2 - 2(x_{\prime}x_{\prime}' + y_{\prime}y_{\prime}' + z_{\prime}z_{\prime}') + (x_{\prime}'^2 + y_{\prime}'^2 + z_{\prime}'^2)\}^{\frac{3}{2}}} - \frac{z_{\prime}y_{\prime}' - y_{\prime}z_{\prime}'}{r^3}\right\}$$

$$= \frac{S}{r^3}\Sigma . m(z_{\prime}y_{\prime}' - y_{\prime}z_{\prime}')\left\{\left(1 - \frac{2(x_{\prime}x_{\prime}' + y_{\prime}y_{\prime}' + z_{\prime}z_{\prime}') - (x_{\prime}'^2 + y_{\prime}'^2 + z_{\prime}'^2)}{r^2}\right)^{-\frac{3}{2}} - 1\right\}$$

$$= \frac{3S}{r^5} \Sigma . m (z_, y_,' - y_, z_,')(x_, x_,' + y_, y_,' + z_, z_,'),$$

neglecting the cubes of very small quantities,

$$= \frac{3S}{r^5} \Sigma . m \{ (y_,'^2 - z_,'^2) z_, y_, - (y_,^2 - z_,^2) z_,' y_,' + z_, x_, y_,' x_,' - y_, x_, z_,' x_,' \}$$

$$= \frac{3S}{r^5} z_, y_, \Sigma . m (y_,'^2 - z_,'^2)$$ by the property of principal axes,

$$\therefore \; L_, = \frac{3S}{r^5} z_, y_, (C - B).$$

In the same manner we should find

$$M_, = \frac{3S}{r^5} x_, z_, (A - C),$$

$$N_, = \frac{3S}{r^5} x_, y_, (B - A).$$

Hence the equations of motion become

$$A \frac{d\omega_1}{dt} + (C - B)\, \omega_2 \omega_3 = \frac{3S}{r^5} y_, z_, (C - B),$$

$$B \frac{d\omega_2}{dt} + (A - C)\, \omega_1 \omega_3 = \frac{3S}{r^5} x_, z_, (A - C),$$

$$C \frac{d\omega_3}{dt} + (B - A)\, \omega_1 \omega_2 = \frac{3S}{r^5} x_, y_, (B - A).$$

In these equations the disturbing body is supposed to be at a very great distance, as is the case with the Sun and Moon; but it is remarkable that they are very nearly correct even when the attracting body is very near the Earth, supposing the Earth's figure to be spheroidal. For a demonstration of this we refer the reader to the *Mécanique Céleste.* Liv. V. Chap. I. §. 3.

It will be observed that we have taken account of only one disturbing body S in these equations: but since the perturbations are small and the equations in $\omega_1 \omega_2 \omega_3$ linear, we may calculate the effects of the disturbing bodies singly and add them together, Art. 288.

PROP. *To prove that the velocity of rotation of the Earth, and consequently the length of the mean day, is not altered by the action of the Sun and Moon, very small quantities being neglected.*

460. If we neglect the disturbing forces and suppose the figure of the Earth to be one of revolution and not differing much from a sphere, that is, $B = A$, and each of these nearly $= C$, the difference being of the order of the ellipticity of the terrestrial spheroid; then in this case the equations of the last Article give, for a first approximation, $\omega_3 = \text{const.} = n$, ω_1 and ω_2 very small quantities. These values may be put in the small terms of our equations in order to obtain a nearer approximation.

If we multiply the three equations of last Article by $\omega_1 \omega_2 \omega_3$ and divide them by A, B, C respectively and add them together, we have

$$\frac{d(\omega_1^2 + \omega_2^2 + \omega_3^2)}{dt} + 2\omega_1\omega_2\omega_3 \left\{\frac{C-B}{A} + \frac{A-C}{B} + \frac{B-A}{C}\right\}$$

$$= \frac{6S}{r^3}\left\{\frac{y_, z_,}{r^2}\,\frac{C-B}{A}\,\omega_1 + \frac{x_, z_,}{r^2}\,\frac{A-C}{B}\,\omega_2 + \frac{x_, y_,}{r^2}\,\frac{B-A}{C}\,\omega_3\right\}.$$

Now $\omega_1 \omega_2$ are each extremely small; $\dfrac{C-B}{A}$, $\dfrac{A-C}{B}$ are of the order of the ellipticity of the Earth; and $\dfrac{B-A}{C}$ is extremely small, since if we suppose the Earth a figure of revolution this expression vanishes; also $\dfrac{S}{r^3}$ is very small, because S varies as the cube of the radius of the body S.

Hence if we neglect extremely small quantities

$$\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} = \text{const.} = n,$$

since the *mean* values of $\omega_1 \omega_2 \omega_3$ are 0, 0, n.

Hence the angular velocity of the Earth is constant, and the length of the mean day is not affected by the action of the Sun and Moon, when we neglect inappreciable quantities.

A full discussion of this important question will be found in the *Méc. Céleste*, Liv. V. Chap. I. §. 8, 9: and also in the *Mémoires de l'Académie Royale des Sciences de l'Institut de France*: Vol. VII. p. 199.

Astronomical observations shew in a remarkable degree that the length of the mean day has been invariable for a long period of time. We proceed to explain how this result is obtained from observations.

PROP. *To shew from observations made on eclipses that the length of the mean day has been invariable for a great length of time.*

461. Let us take for the unit of time the length of any day at the present epoch: and suppose the day has been decreasing by a parts. Let n be the mean angular motion of the Moon on the day which is taken for the unit of time; then n is the number of degrees through which the Moon moves on that day: and $n(1+a)$, $n(1+2a)$,...... are the angles described by the Moon during the days preceding that day in order: and the angle described during t days $= nt + \frac{1}{2}na(t-1)t$, and if t be very large this angle $= nt + \frac{1}{2}nat^2$. Let n' be the mean motion of the Sun on the day of which the length is the unit of time, then the angle described by the Sun in the t days now elapsed $= n't + \frac{1}{2}n'at^2$, and the difference of longitude in the Sun and Moon being λ now, was $= \lambda + (n'-n)t + \frac{1}{2}(n'-n)at^2$ at the distance of t days from the present time.

Let δ be the error made in calculating the difference of longitudes of the Sun and Moon at a distance of t days on the supposition of the invariability of the length of the day: then $\frac{1}{2}(n'-n)at^2 = \delta$.

Now the values of δ have been calculated in the *Connaissance des Tems* of 1800, from 27 eclipses observed by the Chaldees, the Greeks, and the Arabs. The greatest value of δ corresponds to an eclipse observed in the year B.C. 382:

for this $\delta = -27'.41''$. For the most ancient eclipse $\delta = 2''$; this eclipse being observed by the Chaldees in the year B.C. 720.

Let i be the number of centuries in t days: then $t = 36525\,i$. By the mean of modern observations on the Sun and Moon it is found that $(n' - n)\,36525 = 445268^0$: for the most ancient eclipse $i = 25.56$;

$$\therefore\ \delta = \tfrac{1}{2}\,(36525)\,.\,(25.56)^2\,a \times 445268^0.$$

Now if the day be shorter by a ten-millionth part than at the epoch of the most ancient eclipse on record, then

$$(36525)\,(25.56)\,a = 0.0000001\,;$$

$$\therefore\ \delta = \tfrac{1}{2}\,(25.56)\,.\,(0.0000001)\,.\,(445268)^0$$
$$= 34',$$

a value which renders an eclipse impossible, since the sum of the greatest semi-diameters of the Sun and Moon does not exceed half a degree.

From this we learn that the length of the day has not changed even by a hundred and fifteenth part of a second of time during the last 2556 years. M. Poisson's *Traité de Mécanique*, Seconde Edition, Tom. II. p. 196—200.

462. By comparing the observed north-polar distances of stars made at epochs distant from each other Bradley shewed that the point in the heavens to which the Earth's axis of rotation is directed is not stationary, although for periods of time not very long this deviation, as we remarked in Art. 457, is not perceptible. It becomes an interesting question, then, to ascertain the cause of this perturbation. Since we neglected the action of the Sun and Moon in the calculation of Art. 458, we may readily conjecture that the action of these bodies is the cause required. This we proceed to demonstrate.

PROP. *To determine the position of the axis of rotation of the Earth at any given time, the action of the Sun being considered; and the figure of the Earth being taken to be one of revolution.*

463. We shall refer the disturbing body S to the ecliptic. Let the plane of the ecliptic be the plane of XY: the axis of X being drawn through the first point of Aries, which is *moveable;* the centre of the Earth the origin of co-ordinates; $x_{,} y_{,} z_{,}$ parallel to the principal axes,

$\theta =$ the angle between the equator and ecliptic,
or the angle between the axes of $z_{,}$ and Z.

$\phi =$ the right ascension of the axis of $x_{,}$,
or the angle between the axes of $x_{,}$ and X.

$l =$ longitude of the Sun,

$r =$ distance of Sun from the Earth's centre;

then by Spherical Trigonometry,

$$\frac{x_{,}}{r} = \cos\phi \cos l + \sin\phi \sin l \cos\theta,$$

$$\frac{y_{,}}{r} = -\sin\phi \cos l + \cos\phi \sin l \cos\theta,$$

$$\frac{z_{,}}{r} = -\sin l \sin\theta;$$

$$\therefore \frac{y_{,} z_{,}}{r^2} = \tfrac{1}{2} \sin 2l \sin\theta \sin\phi - \tfrac{1}{2} \sin^2 l \sin 2\theta \cos\phi,$$

$$\frac{x_{,} z_{,}}{r^2} = -\tfrac{1}{2} \sin 2l \sin\theta \cos\phi - \tfrac{1}{2} \sin^2 l \sin 2\theta \sin\phi,$$

substituting these in the equations of motion of Art. 459, and putting

$$P = \frac{3S}{2r^3} \sin 2l \sin\theta, \quad \text{and} \quad P' = \frac{3S}{2r^3} \sin^2 l \sin 2\theta,$$

we obtain, since $B = A$,

$$\left.\begin{aligned} \frac{d\omega_1}{dt} + \frac{C-A}{A}\omega_2\omega_3 &= \frac{C-A}{A}(P\sin\phi - P'\cos\phi) \\ \frac{d\omega_2}{dt} - \frac{C-A}{A}\omega_1\omega_3 &= \frac{C-A}{A}(P\cos\phi + P'\sin\phi) \\ \frac{d\omega_3}{dt} &= 0. \end{aligned}\right\} \dots (1).$$

The third equation gives $\omega_3 = \text{constant} = n$; and therefore $\phi = nt + \text{small terms}$ (see Art. 447).

Let the time be measured from the epoch when the Sun was in Aries: then $l = n't$. Since $B = A$ any axis in the plane $x_, y_,$ is a principal axis: let the axis of $x_,$ be so chosen, that when $t = 0$ it passed through Aries: then $\phi = nt$ neglecting small terms;

$$\therefore\ P = \frac{3S}{2r^3}\sin\theta\sin 2n't;\ P' = \frac{3S}{4r^3}\sin 2\theta\,(1 - \cos 2n't).$$

We shall neglect the variations of the inclination (θ) of the equator and ecliptic in calculating small terms.

Since the equations (1) are linear we may take one term only of P and P' in the calculation: let $k\sin it$ and $k'\cos it$ be corresponding terms: then i admits of two values 0 and $2n'$. Considering these terms we have

$$\frac{d\omega_1}{dt} + \frac{C-A}{A}n\omega_2 = \frac{C-A}{2A}\{(k-k')\cos(n-i)t - (k+k')\cos(n+i)t\},$$

$$\frac{d\omega_2}{dt} - \frac{C-A}{A}n\omega_1 = \frac{C-A}{2A}\{-(k-k')\sin(n-i)t + (k+k')\sin(n+i)t\}.$$

To solve these differentiate the first with respect to t and eliminate $\frac{d\omega_2}{dt}$ by the second;

$$\therefore\ \frac{d^2\omega_1}{dt^2} + \left(\frac{C-A}{A}\right)^2 n^2\omega_1 = \frac{C-A}{2A}\left\{(k-k')\left(\frac{C-A}{A}n - n + i\right)\sin(n-i)t\right.$$

$$\left. - (k+k')\left(\frac{C-A}{A}n - n - i\right)\sin(n+i)t\right\}.$$

The integral of this is of the form

$$\omega_1 = C_1 \cos\left(\frac{C-A}{A} nt + C_2\right) + M \sin(n-i)t + N \sin(n+i)t,$$

where C_1 and C_2 are constants independent of the disturbing forces; and M and N are to be found by putting this value for ω_1 in the differential equation, we find that

$$2M = \frac{(k-k')(C-A)\{Cn-(2n-i)A\}}{(C-A)^2 n^2 - A^2(n-i)^2},$$

$$2N = -\frac{(k+k')(C-A)\{Cn-(2n+i)A\}}{(C-A)^2 n^2 - A^2(n+i)^2}.$$

Now the only values of $\frac{i}{n}$ are 0 and $\frac{2n'}{n}$ $\left(= \frac{2}{365}\right.$, since there are 365 days in a year): hence by neglecting $\frac{i}{n}$ in the small terms, we have

$$M = \frac{k-k'}{2n}\frac{C-A}{C}, \quad N = -\frac{k+k'}{2n}\frac{C-A}{C}.$$

Now when there are no disturbing forces $\omega_1 = 0$, and consequently $C_1 = 0$;

$$\therefore \ \omega_1 = M \sin(n-i)t + N \sin(n+i)t,$$

$$\omega_2 = M \cos(n-i)t + N \cos(n+i)t.$$

Returning to the axes fixed in space and choosing the plane of the ecliptic for the plane of xy we must put the values of ω_1 and ω_2 in the equations (Art. 448.)

$$\frac{d\theta}{dt} = \omega_1 \cos\phi - \omega_2 \sin\phi,$$

$$\sin\theta \frac{d\psi}{dt} = -\omega_1 \sin\phi - \omega_2 \cos\phi,$$

$$\frac{d\phi}{dt} = n - \cot\theta\,(\omega_1 \sin\phi + \omega_2 \cos\phi),$$

in which θ is the obliquity of the ecliptic; ϕ the right ascension of a fixed terrestrial meridian; ψ the longitude of Aries measured in a retrograde direction: fig. 97.

Since $\dfrac{d\psi}{dt}$ vanishes when there are no disturbing forces;

$$\therefore \phi = nt \text{ for a first approximation;}$$

$$\therefore \frac{d\theta}{dt} = \omega_1 \cos nt - \omega_2 \sin nt$$

$$= (N - M) \sin it = -\frac{C-A}{nC} . k \sin it,$$

$$\sin\theta \frac{d\psi}{dt} = -\omega_1 \sin nt - \omega_2 \cos nt,$$

$$= -(M+N) \cos it = \frac{C-A}{nC} . k' \cos it,$$

and by replacing $k \sin it$ and $k' \cos it$ by P and P', of which they have been the representatives,

$$\frac{d\theta}{dt} = -\frac{C-A}{nC} P = -\frac{3S}{2r^3}\,\frac{C-A}{nC} \sin\theta \sin 2n't,$$

$$\frac{d\psi}{dt} = \frac{C-A}{nC\sin\theta} P' = \frac{3S}{2r^3}\,\frac{C-A}{nC} \cos\theta\,(1 - \cos 2n't),$$

integrating these equations and putting $\sqrt{\dfrac{S}{r^3}} = n'$, the mean motion of the Sun,

$$\theta = I + \frac{3n'}{4n}\,\frac{C-A}{C} \sin I \cos 2n't,$$

$$\psi = \frac{3n'^2}{2n}\,\frac{C-A}{C} \cos I . t - \frac{3n'}{4n}\,\frac{C-A}{C} \cos I \sin 2n't,$$

I being the mean value of θ: and the axis of x being so chosen that $t = 0$ when Aries was in that axis.

We should obtain analogous expressions for the perturbation of the Earth's axis by the Moon.

464. The first of these expressions shews that the obliquity of the ecliptic fluctuates; but preserves its mean value equal to the value it would have if there were no disturbing forces.

The second shews that the first point of Aries, or the vernal equinox, has on the whole a retrograde motion on the ecliptic, though at the same time it is subject to a small oscillatory motion.

The steady retrograde motion is called *the Precession of the Equinoxes*; the solar precession (i. e. the precession caused by the Sun) equals $\frac{3n'^2}{2n} \frac{C-A}{C} \cos I$ in a unit of time. This precessional motion causes the pole of the Earth to describe a small circle about the pole of the ecliptic.

The oscillating motion of the pole, arising partly from the change of the obliquity and partly from the periodical term in ψ, is called the *Nutation of the Earth's Axis*.

465. It will be seen that the Precession and Nutation of the Earth's axis arise from the attraction of the Sun and Moon upon the protuberant parts of the Earth, i. e. upon the portion by which it exceeds a sphere touching it internally. For if the form of the Earth were spherical, then $C = A$ and the variable terms in θ and ψ would vanish.

We proceed to calculate the effect of the Moon upon the position of the Earth's axis.

PROP. *To find the motion of the Earth's axis with respect to the plane of the Moon's orbit caused by the action of the Moon.*

466. Let θ' and ψ' be the same quantities as θ and ψ in Art. 463. with this difference that the plane of the Moon's orbit is used instead of the ecliptic; i the inclination of the Moon's orbit to the ecliptic; this does never much exceed 5^0, and is therefore so small that we may neglect its square: we shall also consider i constant, since its variations are very small, as is shewn by observation: I' the inclination of the equator

to the Moon's orbit: M the mass of the Moon: a the radius of the Moon's orbit.

Now for $\frac{S}{r^3}$ in Art. 463. we must put $\frac{M}{a^3}$.

Let n'' be the mean motion of the Moon about the Earth:

$$\text{then } n'' = \sqrt{\frac{M+E}{a^3}}, \quad \therefore \frac{M}{a^3} = \frac{Mn''^2}{M+E} = \frac{n''^2}{1+\nu},$$

where E is the mass of the Earth, and ν the ratio of this mass to that of the Moon.

Hence $\frac{S}{r^3}$ in Art. 463. must be replaced by $\frac{n''^2}{1+\nu}$;

$$\therefore \frac{d\theta'}{dt} = -\frac{3n''^2}{2(1+\nu)} \frac{C-A}{nC} \sin I' \sin 2n''t$$

$$\frac{d\psi'}{dt} = \frac{3n''^2}{2(1+\nu)} \frac{C-A}{nC} \cos I' (1 - \cos 2n''t).$$

The periodical quantities $\sin 2n''t$ and $\cos 2n''t$ go through their changes in half a month; in consequence of the shortness of their period and the smallness of their coefficients they never accumulate so much as to produce a sensible effect: and are therefore omitted. Hence the inclination of the Earth's axis to the Moon's orbit suffers no sensible change from the Moon's attraction; but the line of intersection of the equator and the plane of the Moon's orbit does change its position, which is determined by the equation

$$\psi' = \frac{3n''}{2(1+\nu)} \frac{C-A}{nC} \cos I' . (n''t + \text{const.})$$

In order to calculate the Lunar Precession and Nutation, we must refer the angle ψ' to the ecliptic. Since the oscillations of the plane of the Moon's orbit are insensible no Lunar Nutation can arise from them; but the Moon's line of Nodes continually regresses (Art. 342.) performing a revolution in 18 years and 7 months: and this is the cause of Lunar Nutation. We proceed to calculate this and Lunar Precession.

Prop. *To calculate Lunar Precession.*

467. Let K, K', P be respectively the poles of the ecliptic, Moon's orbit, and the Earth's equator (fig. 107).

Now P revolves about K' with an angular velocity $\frac{d\psi'}{dt}$:

hence the linear vel. of P about $K' = \frac{d\psi'}{dt}\sin\theta'$, rad. of sphere = 1

the resolved part of this about $K = \frac{d\psi'}{dt}\sin\theta'\cos KPK'$: and therefore

P revolves about K with an angular velocity $= \frac{d\psi'}{dt}\frac{\sin\theta'}{\sin\theta}\cos KPK'$

and .... ♈ .... $= \frac{d\psi'}{dt}\sin\theta'\cos \Upsilon PK'$;

$$\therefore \frac{d\psi}{dt} = \frac{d\psi'}{dt}\frac{\sin\theta'}{\sin\theta}\cos KPK' = \frac{3n''^2}{2(1+\nu)}\frac{C-A}{nC}\frac{\cos\theta'\sin\theta'}{\sin\theta}\cos KPK'$$

$$= \frac{3n''^2}{2(1+\nu)}\frac{C-A}{nC}\frac{\cos\theta'}{\sin\theta}\frac{\cos i - \cos\theta\cos\theta'}{\sin\theta}$$

$$= \frac{3n''^2}{2(1+\nu)}\frac{C-A}{nC}\frac{1}{\sin^2\theta}(\cos\theta\cos i + \sin\theta\sin i\cos\Omega)$$

$$\times(\cos i - \cos^2\theta\cos i - \cos\theta\sin\theta\sin i\cos\Omega)$$

where Ω is the longitude of the Moon's node measured in a retrograde direction

$$= \frac{3n''^2}{2(1+\nu)}\frac{C-A}{nC}(\cos\theta\cos i + \sin\theta\sin i\cos\Omega)(\cos i - \cot\theta\sin i\cos\Omega)$$

$$= \frac{3n''^2}{2(1+\nu)}\frac{C-A}{nC}\left(\cos\theta\cos^2 i - \frac{\cos 2\theta}{2\sin\theta}\sin 2i\cos\Omega - \cos\theta\sin^2 i\cos^2\Omega\right)$$

$$= \frac{3n''^2}{2(1+\nu)}\frac{C-A}{nC}\left\{\cos I\,(\cos^2 i - \tfrac{1}{2}\sin^2 i)\right.$$

$$\left. - \frac{\cos 2I\sin 2i}{2\sin I}\cos\left(\frac{2\pi t}{\tau} + \Omega_0\right) - \tfrac{1}{2}\cos I\sin^2 i\cos\left(\frac{4\pi t}{\tau} + 2\,\Omega_0\right)\right\},$$

in which τ = the periodic time of the Moon's nodes: and Ω_0 is the longitude of the ascending node when $t = 0$;

$$\therefore \psi = \frac{3n''^2}{2(1+\nu)} \frac{C-A}{nC} \left\{ \cos I (\cos^2 i - \tfrac{1}{2}\sin^2 i)\, t - \frac{\tau}{4\pi} \frac{\cos 2I \sin 2i}{\sin I} \sin\left(\frac{2\pi t}{\tau} + \Omega_0\right) - \frac{\tau}{8\pi} \cos I \sin^2 i \sin\left(\frac{4\pi t}{\tau} + 2\Omega_0\right) \right\} + \text{const.}$$

Hence the Lunar Precession

$$= \frac{3n''}{2(1+\nu)} \frac{C-A}{nC} \cos I (\cos^2 i - \tfrac{1}{2}\sin^2 i)\, n'' t.$$

The second term of ψ is periodical as well as the third; but the third is so small, in consequence of its coefficient $\sin^2 i$, that it may be neglected. They are both parts of Lunar Nutation.

PROP. *To find the effect of the Moon on the obliquity of the ecliptic.*

468. We have from the last Proposition

$$\frac{d\theta}{dt} = \frac{d\psi'}{dt} \sin\theta' \cos \Upsilon PK' = \frac{3n''^2}{2(1+\nu)} \frac{C-A}{nC} \cos\theta' \sin\theta' \cos \Upsilon PK'$$

$$= \frac{3n''^2}{2(1+\nu)} \frac{C-A}{nC} (\cos\theta \cos i + \sin\theta \sin i \cos\Omega) \sin i \sin\Omega,$$

$$= \frac{3n''^2}{2(1+\nu)} \frac{C-A}{nC} \left\{ \tfrac{1}{2} \cos I \sin 2i \sin\left(\frac{2\pi t}{\tau} + \Omega_0\right) + \tfrac{1}{2} \sin I \sin^2 i \sin\left(\frac{4\pi t}{\tau} + 2\Omega_0\right) \right\};$$

$$\therefore \theta = I - \frac{3n''^2}{2(1+\nu)} \frac{C-A}{nC} \left\{ \frac{\tau}{4\pi} \cos I \sin 2i \cos\left(\frac{2\pi t}{\tau} + \Omega_0\right) + \frac{\tau}{8\pi} \sin I \sin^2 i \cos\left(\frac{4\pi t}{\tau} + 2\Omega_0\right) \right\}$$

the variable terms in this are periodical, and the last is so small as to be insensible. These terms and the periodical terms in the value of ψ make up the whole of Lunar Nutation.

469. Let x and y be the parts of Lunar Nutation which have been determined in Arts. 467, 468;

$$\therefore x^2\left\{\frac{8\pi n(1+\nu)C}{3\tau n''^2(C-A)\cos 2I\sin 2i}\right\}^2+y^2\left\{\frac{8\pi n(1+\nu)C}{3\tau n''^2(C-A)\cos I\sin 2i}\right\}^2=1.$$

This is the equation to an ellipse of which the axes are in the ratio $\cos 2I : \cos I$. This explains the construction mentioned in works on Plane Astronomy. Woodhouse's *Plane Astronomy*, p. 857. Maddy's *Plane Astronomy*, 2[nd] Edition.

The whole Precession, both Solar and Lunar, equals

$$\frac{3}{2n}\,\frac{C-A}{C}\cos I\left\{n'^2+\frac{n''^2}{1+\nu}(\cos^2 i-\tfrac{1}{2}\sin^2 i)\right\}t,$$

(see *Mécanique Céleste*, Liv. v. Chap. I. §. 14.) and the Nutation is given by the equations of Arts. 467, 468.

470. Annual Precession

$$=\frac{C-A}{C}\,\frac{3n'}{n}\cos I\left\{1+\frac{n''^2}{n'^2}\,\frac{1-\frac{3}{2}\sin^2 i}{1+\nu}\right\}180^0$$

$$I=23^0\,28'\,18'',\quad i=5^0\,8'\,50'',\quad \frac{n}{n'}=365.26,\quad \frac{n''}{n'}=\frac{36526}{2732}.$$

$$\log_{10}\sin i=\bar{2}.9528656$$

$$\log_{10}(\tfrac{3}{2}\sin^2 i)=\bar{2}.0818225=\log_{10}.0120732;$$

$$\therefore \log_{10}(1-\tfrac{3}{2}\sin^2 i)=\log_{10}.9879268=\bar{1}.9947248$$

$$\log_{10}\frac{n''^2}{n'^2} \qquad =2.2522428$$

$$\therefore \log_{10}\left\{\frac{n''^2}{n'^2}(1-\tfrac{3}{2}\sin^2 i)\right\}=2.2469676=\log_{10}176.5906;$$

$$\frac{n''^2}{n'^2}\left(1-\tfrac{3}{2}\sin^2 i\right)=176.5906$$

$$\log_{10}\left\{\frac{3n'}{n}\times 180\times 60\times 60\times \cos I\right\}=2.3856065$$

$$3.9030900$$

$$\bar{1}.9625076 - 2.5626021$$

$$6.2512041$$

$$2.5626021$$

$$3.6886020=\log_{10}4882.05\,;$$

$$\therefore \text{ Annual Precession}=\frac{C-A}{C}\left(1+\frac{176.5906}{1+\nu}\right)4882''.05.$$

471. We have supposed in these calculations that the Earth is wholly solid. Laplace shews, however, that the variation of the motion of the terrestrial nucleus, covered by a fluid, are the same as if the sea formed a solid mass with it: *Mécanique Céleste*, Liv. v. §. 10—12.

CHAPTER XIII.

MOTION OF A RIGID BODY ACTED ON BY IMPULSIVE FORCES.

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472. In the preceding Chapters we have obtained differential equations for calculating the motion of a body acted on by any forces of finite intensity. Since these differential equations are of the second order their first integrals will be of the first order, and will therefore be functions of the velocities and co-ordinates of position of the various parts of the body. The values of the arbitrary constants introduced by the process of integration are determined by knowing the velocity and position of the parts of the body at any given instant of the motion: the instant generally chosen is the epoch from which the time is measured. In calculating the motion of a heavenly body the values of the arbitrary constants are found by observations made on the position and velocity of the body at any given time; since we are altogether unacquainted with the initial circumstances of the motion. But we may wish to calculate the motion of a rigid body when the original circumstances of projection are known. Now instantaneous velocities are generated by forces of the nature we have termed impulsive (Art. 201). It becomes necessary, then, to calculate the motion of a body which results from the action of impulsive forces.

By *finite forces* we mean such as require a finite time to produce motion, or a change of motion, in a body: such as the moving force produced by the attraction of the Earth. They are measured, when uniform, by the momentum generated in a unit of time; when variable, by the momentum that would be generated if they were to remain uniform after the epoch at which they are to be estimated. But by *Impulsive Forces* we mean such as act only for an indefinitely short time, and yet produce a finite velocity in a body; such as the force of explosion of a cannon; the force of impact of one body against another. These forces are measured by the momentum generated in the body on which they act.

In Art. 226. we have enunciated a Principle by means of which we can make the calculation of the motion depend upon the conditions of equilibrium of forces acting on a rigid body.

PROP. *To obtain the equations of motion of a rigid body acted on by impulsive forces.*

473. Let V be the velocity impressed on a particle m of the system, the co-ordinates of the particle being xyz: then are $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ the effective velocities parallel to the axes of co-ordinates, that is, the actual velocities which the particle acquires in consequence of the action of the impulsive forces: a, b, c the angles which the direction of the velocity V makes with the axes.

Then according to the Principle explained in Art 226. the impulsive forces

$$m\left(V\cos a - \frac{dx}{dt}\right),\quad m\left(V\cos b - \frac{dy}{dt}\right),\quad m\left(V\cos c - \frac{dz}{dt}\right)$$

acting on m parallel to the axes of co-ordinates, together with similar forces acting on the other particles of the system ought to be in equilibrium.

Hence we have from Art. 65. the equations

$$\Sigma.m\left(V\cos a - \frac{dx}{dt}\right) = 0,\quad \Sigma.m\left(V\cos b - \frac{dy}{dt}\right) = 0,$$

$$\Sigma.m\left(V\cos c - \frac{dz}{dt}\right) = 0,$$

$$\Sigma . m \left\{ y \left(V \cos c - \frac{dz}{dt} \right) - z \left(V \cos b - \frac{dy}{dt} \right) \right\} = 0,$$

$$\Sigma . m \left\{ z \left(V \cos a - \frac{dx}{dt} \right) - x \left(V \cos c - \frac{dz}{dt} \right) \right\} = 0,$$

$$\Sigma . m \left\{ x \left(V \cos b - \frac{dy}{dt} \right) - y \left(V \cos a - \frac{dx}{dt} \right) \right\} = 0.$$

By means of these six equations we shall be able to calculate the motion of a rigid body acted on by any impulsive forces.

They lead immediately to two fundamental principles, analogous to those of Arts. 428, 429. for finite forces.

PROP. *The centre of gravity of the body will move in the same manner as if the forces which act upon the various particles of the body were all transferred to that point, their directions being parallel to their former directions.*

474. For the first three of the equations of last Article give

$$\Sigma . m \frac{dx}{dt} = \Sigma . m V \cos a, \quad \Sigma . m \frac{dy}{dt} = \Sigma . m V \cos b,$$

$$\Sigma . m \frac{dz}{dt} = \Sigma . m V \cos c.$$

Now let $\bar{x}\bar{y}\bar{z}$ be the co-ordinates of the centre of gravity at the instant the impulsive forces act: then

$$\bar{x} \; \Sigma . m = \Sigma . mx, \quad \bar{y} \; \Sigma . m = \Sigma . my, \quad \bar{z} \; \Sigma . m = \Sigma . mz.$$

Differentiate these with *respect to* t, and we have by the equations of motion above

$$\frac{d\bar{x}}{dt} \; \Sigma . m = \Sigma . m V \cos a, \quad \frac{d\bar{y}}{dt} \; \Sigma . m = \Sigma . m V \cos b,$$

$$\frac{d\bar{z}}{dt} \; \Sigma . m = \Sigma . m V \cos c.$$

But these are the equations we should have obtained by supposing the forces transferred to the centre of gravity, their directions being preserved.

Hence the Proposition, as enunciated, is true.

PROP. *The motion of rotation of the body will be the same as if the centre of gravity were fixed.*

475. Let $x_{\prime} y_{\prime} z_{\prime}$ be the co-ordinates of m measured from the centre of gravity parallel to the original axes: then

$$x = \bar{x} + x_{\prime},\ y = \bar{y} + y_{\prime},\ z = \bar{z} + z_{\prime};$$

and by substituting these in the last three equations of motion in Art. 473. the first of these becomes

$$\Sigma . m \left\{ (\bar{y} + y_{\prime}) \left(\frac{d\bar{z}}{dt} + \frac{dz_{\prime}}{dt} \right) - (\bar{z} + z_{\prime}) \left(\frac{d\bar{y}}{dt} + \frac{dy_{\prime}}{dt} \right) \right\},$$

$$= \Sigma . m \left\{ (\bar{y} + y_{\prime}) V \cos c - (\bar{z} + z_{\prime}) V \cos b \right\}.$$

But $\Sigma . mx_{\prime} = 0$, $\Sigma . my_{\prime} = 0$, $\Sigma . mz_{\prime} = 0$, by Art. 413;

$$\therefore\ \Sigma . m \left(y_{\prime} \frac{dz_{\prime}}{dt} - z_{\prime} \frac{dy_{\prime}}{dt} \right) = \Sigma . m (y_{\prime} V \cos c - z_{\prime} V \cos b),$$

and the other equations by a similar process become

$$\Sigma . m \left(z_{\prime} \frac{dx_{\prime}}{dt} - x_{\prime} \frac{dz_{\prime}}{dt} \right) = \Sigma . m (z_{\prime} V \cos a - x_{\prime} V \cos c),$$

$$\Sigma . m \left(x_{\prime} \frac{dy_{\prime}}{dt} - y_{\prime} \frac{dx_{\prime}}{dt} \right) = \Sigma . m (x_{\prime} V \cos b - y_{\prime} V \cos a).$$

But these are exactly the equations we should arrive at by supposing the centre of gravity fixed.

Hence the Proposition, as enunciated, is true.

476. The Principles proved in the last two Propositions reduce the calculation of the motion of a rigid body moving freely and acted on by impulsive forces to the calculation of the motion of a single particle, and of a rigid body moving about a fixed point. We shall now determine more con-

venient equations for calculating the rotatory motion of a body about its centre of gravity when acted on by impulsive forces.

Prop. *To calculate the rotatory motion of a body moving about a fixed axis and acted on by impulsive forces.*

477. The equation for determining the rotatory motion is (Art. 473.)

$$\Sigma . m \left\{ x \left(V \cos b - \frac{dy}{dt} \right) - y \left(V \cos a - \frac{dx}{dt} \right) \right\} = 0,$$

the axis of z being the axis of revolution; let r be the distance of any particle m from this axis: θ the angle which r makes with the plane zx;

$$\therefore \ x = r \cos \theta, \quad y = r \sin \theta;$$

$$\therefore \ \frac{dx}{dt} = -r \sin \theta \frac{d\theta}{dt} = -y \frac{d\theta}{dt}, \quad \frac{dy}{dt} = r \cos \theta \frac{d\theta}{dt} = x \frac{d\theta}{dt};$$

$$\therefore \ \Sigma . m \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) = \Sigma . m \, (x^2 + y^2) \frac{d\theta}{dt} = \frac{d\theta}{dt} \Sigma . mr^2;$$

$$\therefore \ \frac{d\theta}{dt} = \frac{\Sigma . m V \, (x \cos b - y \cos a)}{\Sigma . mr^2},$$

$$= \frac{\text{moment of the impressed forces about the axis}}{\text{moment of inertia about the axis}}.$$

Prop. *A body in which one point is fixed is acted on by impulsive forces: required to determine the motion.*

478. Let the fixed point be the origin: and let $r_1 r_2 r_3$ be the distances of the particle m from the axes of x, y, z: and let $\theta_1 \theta_2 \theta_3$ be the angles which $r_1 r_2 r_3$ make respectively with the planes xy, zy, zx,

$$y = r_1 \cos \theta_1, \quad z = r_2 \cos \theta_2, \quad x = r_3 \cos \theta_3,$$

$$z = r_1 \sin \theta_1, \quad x = r_2 \sin \theta_2, \quad z = r_3 \sin \theta_3;$$

$$\therefore\ y\frac{dz}{dt} - z\frac{dy}{dt} = r_1^2\frac{d\theta_1}{dt}, \quad z\frac{dx}{dt} - x\frac{dz}{dt} = r_2^2\frac{d\theta_2}{dt},$$

$$x\frac{dy}{dt} - y\frac{dx}{dt} = r_3^2\frac{d\theta_3}{dt}.$$

Hence the last three equations of Art. 473. become

$$\frac{d\theta_1}{dt}\ \Sigma . mr_1^2 = \Sigma . mV(y\cos c - z\cos b),$$

$$\frac{d\theta_2}{dt}\ \Sigma . mr_2^2 = \Sigma . mV(z\cos a - x\cos c),$$

$$\frac{d\theta_3}{dt}\ \Sigma . mr_3^2 = \Sigma . mV(x\cos b - y\cos a),$$

or

$$\frac{d\theta_1}{dt} = \frac{\text{sum of moments of the impressed forces about axis of } x}{\text{moment of inertia about axis of } x},$$

$$\frac{d\theta_2}{dt} = \frac{\text{sum of moments of the impressed forces about axis of } y}{\text{moment of inertia about axis of } y},$$

$$\frac{d\theta_3}{dt} = \frac{\text{sum of moments of the impressed forces about axis of } z}{\text{moment of inertia about axis of } z}.$$

When these are integrated we shall know the position of the body at the time t with respect to the fixed point.

We shall apply the principles proved in the last Articles to the solution of a few questions.

When a body at rest is acted on by any forces there is a line about which it *begins* to revolve. This line is called the *Axis of Spontaneous Rotation*.

PROP. *To find the position of the axis of spontaneous rotation of a body when it is acted on by an impulsive force.*

479. Let P be the momentum which measures the impulsive force: M the mass of the body.

Then by Art. 474. the centre of gravity moves with the velocity $\frac{P}{M}$ in a direction parallel to that of the impulse.

Let the line of the impulse be taken for the axis of x: and the plane through this and the centre of gravity for the plane of xy: h and r the distances of the centre of gravity from the line of impulse and from the projection on the plane xy of any particle m: θ the angle which r makes with the axis of x. Hence by Arts. 475. and 477.

$$\frac{d\theta}{dt} = \frac{\text{moment of } P}{\text{moment of inertia}} = \frac{Ph}{Mk^2},$$

k being the radius of gyration about the centre of gravity.

Let xy be the co-ordinates of the projection of m: then by compounding the velocities of translation and rotation, we have (fig. 108).

$$\text{vel. of } m \text{ parallel to } x = \frac{P}{M} - \frac{d\theta}{dt} r \sin\theta = \frac{P}{M} - \frac{Phr\sin\theta}{Mk^2},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; y = \frac{d\theta}{dt} r\cos\theta = \frac{Phr\cos\theta}{Mk^2}.$$

To find the point in the plane xy which is at rest at the begining of the motion we must equate these two velocities to zero;

$$\therefore\ k^2 - hr\sin\theta = 0, \quad r\cos\theta = 0;$$

$$\therefore\ \theta = 90^0, \quad \text{and } r = \frac{k^2}{h} = GO \text{ in the figure},$$

and therefore the axis of spontaneous rotation is at right angles to the direction of the impulse; and also cuts at right angles the perpendicular from the centre of gravity upon the direction of the impulse at a distance $\frac{k^2}{h}$, the centre of gravity lying *between* the axis of spontaneous rotation and the impulse.

The point O coincides with the centre of oscillation, if H be the projection of the axis of suspension: see Art. 432.

PROP. *A body revolves about a fixed axis and impinges upon a fixed point, so that the direction of the impulse is perpendicular to the plane passing through the axis and the centre of gravity: required to find the position of the fixed point so that the pressure on the fixed axis at the instant of impact may be wholly in the plane perpendicular to the direction of the impulse. The fixed point so found is called the* Centre of Percussion.

480. Let the fixed axis be the axis of z: and the plane passing through the centre of gravity at the instant of the impulse the plane yz. P the momentum which measures the impulse on the fixed point: $y_, z_,$ the co-ordinates to the point in which the direction of the impulse cuts the plane yz: r the distance of any particle m from the axis of rotation: θ the angle r makes with the plane xz at the instant of impact: then $x = r\cos\theta$, $y = r\sin\theta$: and the velocities of (m) parallel to x and y at the instant of impact are $\left(\frac{dx}{dt} =\right) -y\frac{d\theta}{dt}$ and $\left(\frac{dy}{dt} =\right) x\frac{d\theta}{dt}$: and therefore the momenta parallel to the axes are at that instant $-my\frac{d\theta}{dt}$ and $mx\frac{d\theta}{dt}$.

We shall suppose the axis to be fixed at two points (since if fixed at more they can always be reduced to two) of which the distances from the origin are a and a'; let $R\cos\alpha$, $R\cos\beta$, $R\cos\gamma$, $R'\cos\alpha'$, $R'\cos\beta'$, $R'\cos\gamma'$ be the momenta which measure the impulsive pressures parallel to the axes of x and y on these points at the instant of impact: h the distance of the centre of gravity from the axis of z.

The forces, then, which act upon the body at the instant of impact are

$R\cos\alpha$, $R'\cos\alpha'$, $-my\frac{d\theta}{dt}$ on each particle m, and P parallel to x,

$R\cos\beta$, $R'\cos\beta'$, $mx\frac{d\theta}{dt}$ on each particle m, ... y,

$R\cos\gamma$, $R'\cos\gamma'$, ... z.

But the body is reduced to rest, by hypothesis; and consequently by the six equations of Art. 65. we have

$$R \cos \alpha + R' \cos \alpha' - \frac{d\theta}{dt} \Sigma . my + P = 0,$$

$$R \cos \beta + R' \cos \beta' + \frac{d\theta}{dt} \Sigma . mx = 0, \quad R \cos \gamma + R' \cos \gamma' = 0,$$

$$- R \cos \beta . a - R' \cos \beta' . a' - \frac{d\theta}{dt} \Sigma . mxz = 0,$$

$$R \cos \alpha . a + R' \cos \alpha' . a' - \frac{d\theta}{dt} \Sigma . myz + Pz_{,} = 0,$$

$$\frac{d\theta}{dt} \Sigma . m (x^2 + y^2) - Py_{,} = 0,$$

in these $\Sigma . mx = 0$, and $\Sigma . my = Mh$, as the axes have been chosen. From these equations the pressures may be found.

Now for the centre of Percussion we must have $R \cos \alpha = 0$ and $R' \cos \alpha' = 0$: hence

$$- Mh \frac{d\theta}{dt} + P = 0, \quad - \frac{d\theta}{dt} \Sigma . myz + Pz_{,} = 0,$$

$$\frac{d\theta}{dt} \Sigma . m (x^2 + y^2) - Py_{,} = 0;$$

therefore $P = Mh \dfrac{d\theta}{dt}$ is known, since the motion previous to the impact, and consequently $\dfrac{d\theta}{dt}$, may be calculated by the principles of Chapter XI: and the co-ordinates to the centre of percussion are

$$y_{,} = \frac{\Sigma . m (x^2 + y^2)}{Mh} = \frac{k^2 + h^2}{h}; \quad z_{,} = \frac{\Sigma . myz}{Mh}.$$

If the body be symmetrical about a plane through the centre of gravity and perpendicular to the axis of z, then, if $\bar{z}$ be the distance of the centre of gravity from the plane xy, and if $z = \bar{z} + z'$ we have

$$\Sigma . myz = \bar{z} \Sigma . my + \Sigma . myz' = \bar{z} \Sigma . my = M\bar{z}h.$$

In this case $y_{,} = \dfrac{k^2 + h^2}{h}$, $z_{,} = \bar{z}$: and the centre of percussion will then coincide with the centre of oscillation: see Art. 432.

In the Chapter of Problems we shall give some examples of the impact of bodies.

CHAPTER XIV.

THE MOTION OF A FLEXIBLE BODY.

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481. In the present Chapter we shall calculate and explain some of the simpler cases of the motion of vibrating strings: for more information on this subject and on the motion of elastic springs we refer the reader to M. Poisson's *Traité de Mécanique*, Tom. II. Seconde Edition; to the *Journal de l'Ecole Polytechnique*, Cahier XVIII, p. 442; and lastly to M. Poisson's Memoir on the equilibrium and motion of elastic bodies in the *Mémoires de l'Académie des Sciences*, Tom. VIII.

Prop. *To determine equations for calculating the motion of a perfectly flexible cord, very slightly extensible, of the same thickness and density throughout, fixed at its two extremities, and very little disturbed from its position of rest.*

482. Let A and B be the fixed extremities of the cord (fig. 109), we shall suppose that the cord is straight when in equilibrium: Let P be the position of a particle of the cord in motion at the time t which is at Q when the cord is at rest: $AQ = x$, $AM = x + u$, $MN = y$, $NP = z$, $AB = l$: M the mass of the cord, T the tension at the point P: the resolved parts of T parallel to the axes are

$$T\frac{d(x+u)}{ds}, \quad T\frac{dy}{ds}, \quad T\frac{dz}{ds}; \text{ where } ds = PP';$$

the excesses of the corresponding tensions at P' over those at P are therefore, by Taylor's Theorem,

$$\frac{d}{dx}\left(T\frac{d(x+u)}{ds}\right)dx, \quad \frac{d}{dx}\left(T\frac{dy}{ds}\right)dx, \quad \frac{d}{dx}\left(T\frac{dz}{ds}\right)dx:$$

these are the *impressed* forces acting on PP': the mass of $PP' = M\dfrac{ds}{l}$, hence the *effective* forces acting on PP' are

$$M\frac{ds}{l}\,\frac{d^2u}{dt^2}, \quad M\frac{ds}{l}\,\frac{d^2y}{dt^2}, \quad M\frac{ds}{l}\,\frac{d^2z}{dt^2}:$$

hence by the first Principle of Art. 226 we have

$$\frac{d}{dx}\left\{T\frac{d(x+u)}{ds}\right\} = \frac{M}{l}\,\frac{d^2u}{dt^2}, \quad \frac{d}{dx}\left(T\frac{dy}{ds}\right) = \frac{M}{l}\,\frac{d^2y}{dt^2},$$

$$\frac{d}{dx}\left(T\frac{dz}{ds}\right) = \frac{M}{l}\,\frac{d^2z}{dt^2}.$$

Let W be the tension of the cord when at rest; then it is found by experiment that the change in extension (as $ds - dx$,) of a piece of cord dx varies as the change in tension $T - W$: suppose that $T - W = Q\dfrac{ds - dx}{dx}$.

$$\text{Now } \frac{ds^2}{dx^2} = \left(1 + \frac{du}{dx}\right)^2 + \frac{dy^2}{dx^2} + \frac{dz^2}{dx^2} = \left(1 + \frac{du}{dx}\right)^2,$$

neglecting small quantities of the second order. Hence $T - W = Q\dfrac{du}{dx}$ and our equations of motion become

$$\frac{d^2u}{dt^2} = b^2\frac{d^2u}{dx^2}, \quad \frac{d^2y}{dt^2} = a^2\frac{d^2y}{dx^2}, \quad \frac{d^2z}{dt^2} = a^2\frac{d^2z}{dx^2},$$

if we neglect small quantities of the second order, and put $Ql = Mb^2$, and $Wl = Ma^2$: hence a^2 and b^2 are in the ratio $W : Q$.

The variables u, y, z are separated in these equations; from which we conclude that the vibrations of the cord parallel to the axes of x, y, z are independent of each other, and co-exist without any interference. The transversal vibrations are the same in the directions of y and z. We shall calculate the motion parallel to y.

PROP. *To integrate the equations of motion, and to interpret the integrals.*

483. For the transversal vibrations,

$$\frac{d^2y}{dt^2} = a^2 \frac{d^2y}{dx^2}.$$

To integrate this add to each side $a \dfrac{d^2y}{dx\,dt}$;

$$\therefore \frac{d}{dt}\left\{\frac{dy}{dt} + a\frac{dy}{dx}\right\} = a\frac{d}{dx}\left\{\frac{dy}{dt} + a\frac{dy}{dx}\right\},$$

$$\text{or } \frac{dv}{dt} = a\frac{dv}{dx}, \text{ if we put } \frac{dy}{dt} + a\frac{dy}{dx} = v:$$

$$\therefore dv = \frac{dv}{dt}dt + \frac{dv}{dx}dx = \frac{dv}{dx}d(x + at),$$

$$\therefore v, \text{ or } \frac{dy}{dt} + a\frac{dy}{dx}, = \phi'(x + at);$$

ϕ' representing any arbitrary function of $x + at$.

In like manner by subtracting $a\dfrac{d^2y}{dx\,dt}$ from each side we have

$$\frac{dy}{dt} - a\frac{dy}{dx} = \psi'(x - at)$$

ψ' representing another arbitrary function.

$$\text{Hence } \frac{dy}{dt} = \tfrac{1}{2}\phi'(x + at) + \tfrac{1}{2}\psi'(x - at),$$

$$\frac{dy}{dx} = \frac{1}{2a}\phi'(x + at) - \frac{1}{2a}\psi'(x - at);$$

$$\therefore \; dy = \frac{dy}{dt}dt + \frac{dy}{dx}dx$$

$$= \frac{1}{2a}\phi'(x+at)\,d(x+at) - \frac{1}{2a}\psi'(x-at)\,d(x-at);$$

$$\therefore \; y = F(x+at) + f(x-at),$$

F and f being arbitrary functions depending on ϕ' and ψ'; but we shall cease to use ϕ' and ψ'.

We proceed to explain how to determine the values of these functions. We are supposed to know the initial circumstances of the motion, namely the values of y and $\frac{dy}{dt}$ for all values of x between $x = 0$ and $x = l$ when $t = 0$: hence $F(x) + f(x)$ is known for all values of x between 0 and l: and also $\frac{dF(x)}{dx} - \frac{df(x)}{dx}$ is known and consequently $F(x)$ and $f(x)$ are known between these limits; but the values of these quantities for values of x greater than l and less than 0 are not given, nor are they necessarily known since the functions $F(x)$ and $f(x)$ may be discontinuous; that is, the original form of the curve need not be such as can be expressed in analysis, but may be a series of pieces of curve so long as they have the same tangent at their points of junction.

The condition that the extremities of the cord are always stationary enables us to determine the values of $F(x+at)$ and $f(x-at)$ for all values of x and t. For by this condition $y = 0$ and $\frac{dy}{dt} = 0$ when $x = 0$ and l whatever t be: hence by putting $at = v$ we have

$$F(v) + f(-v) = 0 \; \ldots \; (1), \qquad F(l+v) + f(l-v) = 0 \; \ldots \; (2)$$

for all positive values of v.

Put $l + v$ for v in (2), then by (2) and (1)

$$F(2l+v) = F(v) \; \ldots\ldots\ldots\ldots \; (3).$$

The initial circumstances make known $F(v)$ and $f(v)$ from $v = 0$ to $v = l$: then (2) gives $F(v)$ from $v = l$ to $v = 2l$, and then (3) gives $F(v)$ from $v = 2l$ to $4l$, then to $6l$ and so on to ∞: hence $F(x + at)$ is known for all positive values of t. Again, by (1) $f(-v)$ is known from $v = 0$ to $v = \infty$: but also $f(v)$ is known from $v = 0$ to $v = l$ by the initial circumstances, hence $f(x - at)$ is known for every point of the cord during the whole motion. Hence the value of y, and therefore the form of the string, is known at every instant. There is nothing to make known $F(v)$ for negative values of v or $f(v)$ for values of v between l and ∞.

In (1) put $v + 2l$ for v, then

$$f(-2l - v) = -F(2l + v) = -F(v) \text{ by (3)}$$
$$= f(-v) \text{ by (1) } \ldots\ldots\ldots (4).$$

Hence
$$y = F(x + at) + f(x - at)$$
$$= F(x + at + 2l) + f(x - at - 2l)$$
$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$= F(x + at + 2nl) + f(x - at - 2nl) \text{ by (3) (4)}$$

n a positive integer: hence the cord repeatedly assumes the same form relatively to the plane xz, performing a vibration in the time $\dfrac{2l}{a}$; substituting for a its value,

$$\text{time of vibration} = 2\sqrt{\frac{Ml}{W}}:$$

the same is true of the motion parallel to z: and also parallel to x the oscillations take place in the time $2\sqrt{\dfrac{Ml}{Q}}$.

Prop. *A portion only of the cord is set in motion at first, as a piano-forte wire by the sudden blow of the hammer of a key: required to determine the motion.*

484. To simplify the calculation we shall at first suppose that one end of the string is at an indefinitely great distance. Let the original displacement extend over a small space 2α; and let the origin of x be at the mid-point of this space: h the

distance of the nearest extremity. Then when $t=0$ we have $y=0$ and $\frac{dy}{dt}=0$ from $x=-\infty$ to $x=-a$ and from $x=a$ to $x=h$;

$$\therefore\ F(v)=0 \text{ and } f(v)=0 \dots\dots\dots\dots (1),$$

from $v=-\infty$ to $v=-a$, and from $v=a$ to $v=h$.

Because of the fixed extremity

$$F(h+v)+f(h-v)=0,$$

for positive values of v. Hence when v is greater than h,

$$F(v)=-f(2h-v) \dots\dots\dots\dots\dots (2).$$

Since $x-at$ is always negative for negative values of x, it follows by (1) that beyond the limits of disturbance, that is when x is $< -a$, we have $f(x-at)=0$: also for negative values of x we have $F(x+at)=0$ unless t lie between $\frac{-a-x}{a}$ and $\frac{a-x}{a}$. Hence the initial disturbance is propagated to the left and each particle of the cord begins to move after a time $\frac{-a-x}{a}$, vibrates for a time $\frac{2a}{a}$ and then returns to rest.

In the same way the motion is propagated to the right: but in consequence of the fixed extremity this will require a little further examination.

Let us consider the motion of a particle at a distance x;

$y=F(x+at)+f(x-at)$ is its general displacement.

When $t=0$ this particle is at rest, for $y=0$ by (1): and it remains so till $t=\frac{x-a}{a}$, for, then $y=f(a)$, and the particle moves till $t=\frac{x+a}{a}$, and after this is at rest again for a time. Ever after this $x-at$ is negative and $< -a$, and therefore $f(x-at)=0$ by (1). But when t becomes

$$\frac{x-a}{a} + \frac{2h-2x}{a} \text{ or } \frac{2h-x-a}{a};$$

$$y = F(2h-a) = -f(a) \text{ by (2)},$$

and as t increases and becomes $\frac{x+a}{a} + \frac{2h-2x}{a} = \frac{2h-x+a}{a}$;

$$y = F(2h+a) = -f(-a) \text{ by (2)}.$$

Hence at a time $\frac{2h-2x}{a}$ after the particle began to move, it again begins to move: and ceases to move at the same time after it ceased before. Likewise the displacements of the particle are exactly the same that they were before, but on the *opposite side* of the line of rest.

When t is $> \frac{2h-x+a}{a}$, $x + at$ is $> 2h + a$;

$$\therefore F(x+at) = -f\{2h-(x+at)\} = 0 \text{ always}.$$

Hence the particle oscillates for a period $\frac{2a}{a}$ commencing at the time $\frac{x-a}{a}$: it then rests: and after a time $\frac{2(h-x)}{a}$, it oscillates for the same period in a manner precisely similar to the former; except on the opposite side of the line of rest: after this the particle remains permanently at rest. This is true whatever x be: and it is to be remarked that $\frac{2(h-x)}{a}$ = the time the bent portion of the cord, or the *pulse*, would take to move from the particle to the fixed point and *back again*. Hence, since this second motion arises solely in consequence of the fixity of one extremity of the string, it follows that when the right hand pulse reaches the fixed point *it is reflected*, but to the opposite side of the string.

Hence the original disturbance divides itself into two pulses, one moving continually to the left; the other to the right till it reaches the fixed point, after which it moves back, towards the left, on the other side of the line and with the same velocity as before.

Let the string be of definite length. Then the pulse after reflexion will be reflected again at the other extremity and move on the upper part of the string to the right.

Let C, D (fig. 110) be the fixed extremities of the string and A the origin of disturbance. The initial disturbance divides into two a and b: b is reflected at D and moves along to b' at the same time that a moves to a' having been reflected at C: a' and b' meet at B and confirm each other forming a disturbance exactly similar to the original disturbance.

Evidently B is such a point that

$$AD + DB = AC + CB, \quad \text{or } AD = CB.$$

Now the interval between two maximum disturbances at A = time of describing $AC + CD + DA = \frac{2l}{a}$. Hence the velocity of the pulses $= a$.

CHAPTER XV.

GENERAL DYNAMICAL PRINCIPLES.

485. In the present Chapter we shall prove some general principles of the motion of a material system, which are consequences of the laws of motion.

Prop. *When a material system is in motion under the action of forces, whether finite in intensity or impulsive, but none of them being extraneous to the system, then the centre of gravity moves uniformly in a straight line or remains at rest.*

486. The internal forces which act upon the particles of the system may be of the nature we have denominated impulsive, or they may be of finite intensity, or they may be some impulsive and some finite. But impulsive forces are such as act only for an indefinitely short time; and consequently during the period of their action the finite forces can produce no effect. We may therefore consider the action of these two systems of forces on the system separately.

I. *Suppose the forces are of finite intensity.*

Let XYZ be the internal accelerating forces acting on a particle m of the system parallel to the axes, not including the molecular action: xyz the co-ordinates to m. Then the first three equations of Art. 427, give

$$\Sigma . m\left(X-\frac{d^2x}{dt^2}\right)=0,\quad \Sigma . m\left(Y-\frac{d^2y}{dt^2}\right)=0,\quad \Sigma . m\left(Z-\frac{d^2z}{dt^2}\right)=0.$$

Now since, by hypothesis, the forces acting on the system are all internal, or, in other words, arise from the mutual action of the particles of the system, it follows that if $m'X'$, $m'Y'$, $m'Z'$ be forces acting on the particle m' there must be in the system forces equal to $-m'X'$, $-m'Y'$, $-m'Z'$ acting on some other particle: hence these forces will disappear in the expressions $\Sigma . mX$, $\Sigma . mY$, $\Sigma . mZ$; and this will be the case with all the forces;

$$\therefore \ \Sigma . mX = 0, \quad \Sigma . mY = 0, \quad \Sigma . mZ = 0,$$

and consequently we obtain

$$\Sigma . m \frac{d^2 x}{dt^2} = 0, \quad \Sigma . m \frac{d^2 y}{dt^2} = 0, \quad \Sigma . m \frac{d^2 z}{dt^2} = 0.$$

Let $\bar{x}\,\bar{y}\,\bar{z}$ be the co-ordinates to the centre of gravity of the system at the time t; $x_{,} y_{,} z_{,}$ co-ordinates to m measured from the centre of gravity: then $x = \bar{x} + x_{,}$, $y = \bar{y} + y_{,}$, $z = \bar{z} + z_{,}$; and since $\Sigma . mx_{,} = 0$, $\Sigma . my_{,} = 0$ and $\Sigma . mz_{,} = 0$, the last three equations give

$$\frac{d^2 \bar{x}}{dt^2} = 0, \quad \frac{d^2 \bar{y}}{dt^2} = 0, \quad \frac{d^2 \bar{z}}{dt^2} = 0;$$

$$\therefore \ \frac{d\bar{x}}{dt} = V_0 \cos \alpha, \quad \frac{d\bar{y}}{dt} = V_0 \cos \beta, \quad \frac{d\bar{z}}{dt} = V_0 \cos \gamma,$$

V_0 being the velocity of the centre of gravity when $t = 0$ and $\alpha \beta \gamma$ the angles its direction makes with the axes;

$$\therefore \text{ vel. of centre of gravity at time } t = \sqrt{\frac{d\bar{x}^2}{dt^2} + \frac{d\bar{y}^2}{dt^2} + \frac{d\bar{z}^2}{dt^2}} = V_0,$$

and consequently is constant.

If $V_0 = 0$ then the centre of gravity remains at rest.

Again, by integration we have

$$\bar{x} = V_0 t \cos \alpha, \quad \bar{y} = V_0 t \cos \beta, \quad \bar{z} = V_0 t \cos \gamma,$$

the origin of co-ordinates being chosen at the point where the centre of gravity lies when $t = 0$;

$$\therefore \ \bar{x} = \frac{\cos \alpha}{\cos \gamma} \bar{z}, \quad \bar{y} = \frac{\cos \beta}{\cos \gamma} \bar{z}.$$

These equations shew that the centre of gravity, if it be not at rest, moves in a straight line.

II. *Suppose the forces are impulsive.*

Let V be the velocity that any particle m has when the impulsive forces begin to act; a, b, c the angles its direction makes with the axes: V' the velocity which measures the internal impulsive force acting on m: $a' b' c'$ the angles of its direction.

Then the first three equations of Art. 473, give

$$\Sigma . m \left(V \cos a + V' \cos a' - \frac{dx}{dt} \right) = 0,$$

$$\Sigma . m \left(V \cos b + V' \cos b' - \frac{dy}{dt} \right) = 0,$$

$$\Sigma . m \left(V \cos c + V' \cos c' - \frac{dz}{dt} \right) = 0.$$

Now since the impulsive forces are all internal we shall have, as may be shewn in the same manner as before, that

$$\Sigma . m V' \cos a' = 0, \quad \Sigma . m V' \cos b' = 0, \quad \Sigma . m V' \cos c' = 0.$$

Hence the equations become

$$\Sigma . m \left(V \cos a - \frac{dx}{dt} \right) = 0, \quad \Sigma . m \left(V \cos b - \frac{dy}{dt} \right) = 0,$$

$$\Sigma . m \left(V \cos c - \frac{dz}{dt} \right) = 0.$$

Let V_0 be the velocity of the centre of gravity at the commencement of the action of the impulsive forces: $\alpha \beta \gamma$ the angles its direction makes with the axes: then (Art. 413.)

$$\Sigma . m V \cos a = V_0 \cos \alpha \Sigma . m, \quad \Sigma . m V \cos b = V_0 \cos \beta \Sigma . m,$$

$$\Sigma . m V \cos c = V_0 \cos \gamma \Sigma . m.$$

$$\text{Also } \Sigma . m \frac{dx}{dt} = \frac{d\bar{x}}{dt} \Sigma . m, \quad \Sigma . m \frac{dy}{dt} = \frac{d\bar{y}}{dt} \Sigma . m,$$

$$\Sigma . m \frac{dz}{dt} = \frac{d\bar{z}}{dt} \Sigma . m ;$$

$$\therefore \frac{d\bar{x}}{dt} = V_0 \cos \alpha, \quad \frac{d\bar{y}}{dt} = V_0 \cos \beta, \quad \frac{d\bar{z}}{dt} = V_0 \cos \gamma,$$

consequently the velocity of the centre of gravity is V_0, and therefore is not altered by the action of the impulsive forces.

Also, if the centre of gravity be in motion, the indefinitely small space it describes during the action of the impulsive forces is a continuation of the straight line in which it was moving before their action.

For $\bar{x} = \frac{\cos \alpha}{\cos \gamma} \bar{z}$, $\bar{y} = \frac{\cos \beta}{\cos \gamma} \bar{z}$; the origin of co-ordinates being taken at the point where the centre of gravity is situated when the forces begin to act.

Wherefore the Proposition, as enunciated, is true. This Principle is called *The Principle of the Conservation of the Motion of the Centre of Gravity.*

PROP. *When a material system is in motion under the action of forces, whether of finite intensity or impulsive, but none of which are extraneous to the system; then the sum of the products of each particle multiplied by the projection on any plane of the area swept out by the radius vector of this particle measured from any fixed point varies as the time of motion. This is called the* Principle of the Conservation of Areas.

487. I. *Suppose the forces are of finite intensity.*

Using the same rotation as in the last Proposition the last three equations of Art. 427 are

$$\Sigma . m \left\{ y \frac{d^2 z}{dt^2} - z \frac{d^2 y}{dt^2} \right\} = \Sigma . m \, (yZ - zY),$$

$$\Sigma . m \left\{ z \frac{d^2 x}{dt^2} - x \frac{d^2 z}{dt^2} \right\} = \Sigma . m (zX - xZ),$$

$$\Sigma . m \left\{ x \frac{d^2 y}{dt^2} - y \frac{d^2 x}{dt^2} \right\} = \Sigma . m (xY - yX).$$

We shall now shew that the second sides of these equations vanish when all the forces are internal.

Let $m'X'$, $m'Y'$, $m'Z'$ and $m''X''$, $m''Y''$, $m''Z''$ be the resolved parts parallel to the axes of the mutual actions P' and P'' of two particles m' and m''; then $P' = -P''$ and also

$$m'X' = -m''X'', \quad m'Y' = -m''Y'', \quad m'Z' = -m''Z''.$$

If the particles be in contact let $x'y'z'$ be their co-ordinates; then these mutual actions will enter the expression $\Sigma . m (yZ - zY)$ in the form

$$m'(y'Z' - z'Y') + m''(y'Z'' - z'Y'')$$

but this vanishes: and in the same manner it may be shewn that the mutual actions of particles in contact will disappear from all three of the equations of motion.

Again; suppose the particles are not in contact, their co-ordinates being $x'y'z'$ and $x''y''z''$: let r be their distance; then $\frac{x'-x''}{r}$, $\frac{y'-y''}{r}$, $\frac{z'-z''}{r}$ are the cosines of the angles which r makes with the axes; and we have

$$m'X' = P'\frac{x'-x''}{r}, \quad m'Y' = P'\frac{y'-y''}{r}, \quad m'Z' = P'\frac{z'-z''}{r},$$

and consequently the mutual actions of m' and m'' enter the expression $\Sigma . m (yZ - zY)$ under the form

$$\frac{P'}{r}\{y'(z'-z'') - z'(y'-y'')\} + \frac{P''}{r}\{y''(z'-z'') - z''(y'-y'')\}$$

and this vanishes since $P' = -P''$: and consequently the mutual actions in this case also disappear from the equations of motion.

Since then all the forces are supposed to be internal the equations of motion become

$$\Sigma . m \left(y \frac{d^2 z}{dt^2} - z \frac{d^2 y}{dt^2} \right) = 0, \ \Sigma . m \left(z \frac{d^2 x}{dt^2} - x \frac{d^2 z}{dt^2} \right) = 0,$$

$$\Sigma . m \left(x \frac{d^2 y}{dt^2} - y \frac{d^2 x}{dt^2} \right) = 0,$$

and therefore by integration

$$\Sigma . m \left(y \frac{dz}{dt} - z \frac{dy}{dt} \right) = h, \ \Sigma . m \left(z \frac{dx}{dt} - x \frac{dz}{dt} \right) = h',$$

$$\Sigma . m \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) = h'',$$

h, h', h'' being constants.

Let $A_x A_y A_z$ be the areas swept out by the projections of the radius vector of the particle m on the co-ordinate planes respectively perpendicular to the axes of xyz during the time t: then by the last three equations

$$\Sigma . m \frac{dA_x}{dt} = h, \ \Sigma . m \frac{dA_y}{dt} = h', \ \Sigma . m \frac{dA_z}{dt} = h'';$$

$$\therefore \Sigma . m A_x = ht, \ \Sigma . m A_y = h't, \ \Sigma . m A_z = h''t,$$

since the areas are measured from the epoch when $t = 0$.

Wherefore the Principle, as enunciated, is true for the three co-ordinate planes arbitrarily chosen; and consequently true for any plane.

II. *Suppose the forces are impulsive.*

Using the same notation as in Art. 486, the last three equations of motion of Art. 473, are

$$\Sigma . m \left(y \frac{dz}{dt} - z \frac{dy}{dt} \right)$$

$$= \Sigma . m \left\{ V (y \cos c - z \cos b) + V' (y \cos c' - z \cos b') \right\},$$

$$\Sigma . m \left(z \frac{dx}{dt} - x \frac{dz}{dt} \right)$$

$$= \Sigma . m \left\{ V (z \cos a - x \cos c) + V' (z \cos a' - x \cos c') \right\},$$

$$\Sigma . m \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right)$$

$$= \Sigma . m \left\{ V (x \cos b - y \cos a) + V' (x \cos b' - y \cos a') \right\}.$$

As in the previous part of this Proposition, we might prove that the mutual forces will disappear from these equations. Hence

$$\Sigma . m \left(y \frac{dz}{dt} - z \frac{dy}{dt} \right) = \Sigma . m V (y \cos c - z \cos b) = h,$$

$$\Sigma . m \left(z \frac{dx}{dt} - x \frac{dz}{dt} \right) = \Sigma . m V (z \cos a - x \cos c) = h',$$

$$\Sigma . m \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) = \Sigma . m V (x \cos b - y \cos a) = h'',$$

and therefore the principle of the Conservation of Areas is true whatever the internal forces be.

488. Cor. This Principle is also true when extraneous forces act on the system, provided their directions all pass through the same point and the areas are estimated about that point.

For, this point being the origin of co-ordinates, suppose that P' is one of the extraneous forces acting at the point $(x' y' z')$ at the distance r' from the origin: then $\frac{x'}{r'} P'$, $\frac{y'}{r'} P'$, and $\frac{z'}{r'} P'$ are the resolved parts of P': and this force consequently vanishes from the expressions $\Sigma . m (yZ - zY)$ and $\Sigma . m V' (y \cos c' - z \cos b')$ and from all the analogous expressions.

Prop. *To prove that when a material system is acted on by forces, whether of finite intensity or impulsive, but* *none of which are extraneous to the system, there is a plane invariable in position during the motion, with respect to which the motion may be estimated: and to find its position.*

489. We have seen in Art. 487. that

$$\Sigma . m A_x = ht, \ \Sigma . m A_y = h't, \ \Sigma . m A_z = h''t,$$

$h h' h''$ being constant quantities which depend upon the configuration of the system: and which may be found by observing the motions of the bodies of the system and then calculating $\Sigma . m A_x$, $\Sigma . m A_y$, $\Sigma . m A_z$.

Hence a plane drawn at the time t perpendicular to the straight line which makes with the axes the angles of which the cosines are

$$\frac{\Sigma . m A_x}{\sqrt{(\Sigma . m A_x)^2 + (\Sigma . m A_y)^2 + (\Sigma . m A_z)^2}}, \ \frac{\Sigma . m A_y}{\sqrt{(\Sigma . m A_x)^2 + (\Sigma . m A_y)^2 + (\Sigma . m A_z)^2}};$$

$$\text{and } \frac{\Sigma . m A_z}{\sqrt{(\Sigma . m A_x)^2 + (\Sigma . m A_y)^2 + (\Sigma . m A_z)^2}}$$

remains invariable in position during the motion.

For this reason it is termed the *Invariable Plane.*

PROP. *To prove that the Invariable Plane is that with respect to which the sum of the moments of the momenta of the different particles of the system is a maximum.*

490. Let $\lambda \mu \nu$ be the angles which the Invariable Plane makes with the co-ordinate planes respectively perpendicular to the axes of xyz.

The momentum of the particle m parallel to the plane yz is $m \sqrt{\frac{dy^2}{dt^2} + \frac{dz^2}{dt^2}}$: the perpendicular from the origin upon the tangent to the projection of the curve in which m is moving on the plane $yz = \frac{y \frac{dz}{dt} - z \frac{dy}{dt}}{\sqrt{\frac{dy^2}{dt^2} + \frac{dz^2}{dt^2}}}$. Hence the moment of the momentum of m about the axis of $x = m\left(y\dfrac{dz}{dt} - z\dfrac{dy}{dt}\right)$ and the sum of the moments $= \Sigma . m\left(y\dfrac{dz}{dt} - z\dfrac{dy}{dt}\right) = h$ and the sums of the moments about the axes of y and z are h' and h''. Hence the sum of the moments about the line perpendicular to the plane which makes the angles $\lambda\mu\nu$ with the co-ordinate planes equals

$$h\cos\lambda + h'\cos\mu + h''\cos\nu;$$

and when this is a maximum

$$h\sin\lambda + h''\sin\nu\frac{d\nu}{d\lambda} = 0 \text{ and } h'\sin\mu + h''\sin\nu\frac{d\nu}{d\mu} = 0,$$

$$\text{but } \cos^2\lambda + \cos^2\mu + \cos^2\nu = 1;$$

$$\therefore \frac{h}{h''} = -\frac{\sin\nu}{\sin\lambda}\frac{d\nu}{d\lambda} = \frac{\cos\lambda}{\cos\nu} \text{ and } \frac{h'}{h''} = -\frac{\sin\nu}{\sin\mu}\frac{d\nu}{d\mu} = \frac{\cos\mu}{\cos\nu};$$

$$\therefore \frac{h^2 + h'^2 + h''^2}{h''^2} = \frac{1}{\cos^2\nu}, \quad \therefore \cos\nu = \frac{h''}{\sqrt{h^2 + h'^2 + h''^2}},$$

and in like manner

$$\cos\mu = \frac{h'}{\sqrt{h^2 + h'^2 + h''^2}} \text{ and } \cos\lambda = \frac{h}{\sqrt{h^2 + h'^2 + h''^2}}.$$

But these determine also the position of the Invariable Plane: (see Art. 489.) Hence the Invariable Plane possesses the property that the sum of the moments is greater on that than on any other plane. Also the maximum sum of moments is always the same, since it equals

$$h\cos\lambda + h'\cos\mu + h''\cos\nu = \sqrt{h^2 + h'^2 + h''^2}.$$

491. If the position of this plane be calculated upon the supposition that the heavenly bodies are intense particles without rotatory motion it is found that h, h', h'' are constant even in carrying the approximation to the squares and products of the masses, whatever changes the secular variations may induce in the course of ages: hence it follows that the invariable plane retains its position notwithstanding the secular variations in the

elliptical elements of the planetary system. The determination of the position of the invariable plane requires a knowledge of the masses of all the bodies in the system and of the elements of their orbits. Now we know the masses of the planets only approximately; but of the masses of the comets we are in total ignorance. But from the agreement of theory and observation, mentioned above, we learn, that, hitherto at least, the action of the comets on the planetary system is insensible. Laplace has shewn that the comet of 1770 passed through the system of Jupiter and his satellites without producing the smallest effect, though its own motion was much perturbed.

If the position of the ecliptic in the beginning of 1750 be taken as the fixed plane of xy and the longitudes measured from the line of the equinoxes, it is found that at the epoch 1750 the longitude of the ascending node of the invariable plane was $102^\circ 57' 30''$, and its inclination on the ecliptic was $1^\circ 35' 31''$: and if these be calculated for 1950 they are $102^\circ 57' 15''$ and $1^\circ 35' 31''$; these differ but very little from the former, and therefore shew that the motion of the ecliptic in space is exceedingly slow.

492. It is important to remark that the terms in the equations of motion of Art. 487. which depend upon the mutual action of the parts of the system would disappear even when the intensity of the forces varies with the time, independently of the distance: i. e. when the expression for the force is an explicit function of the time. For in this case the invariability of the principal moment and of the direction of its axis is preserved.

493. This shews that the loss of heat sustained by the particles of the system by radiation though it diminishes the intensity of their mutual action, yet has no effect on the position of the invariable plane or on the principal moment. So that if we leave out of consideration the action of the Sun Moon and planets on the Earth, and suppose that our planet were at one time in a gaseous state, and become solid by refrigeration without losing any position of its ponderable matter, we may feel assured that the principal moment of the system has not altered in magnitude nor has its axis changed its position during the change of condition of the globe.

If M be the whole mass: k the radius of gyration about the axis of principal moments through the centre of gravity at the time t, ω the angular velocity about this axis; then

$$Mk^2\omega = \text{principal moment} = \text{constant}.$$

This shews that if the Earth radiate its heat into space so as to diminish its radius by contracting its dimensions, then, since k varies as the radius, w will be increasing and the length of the day shortening.

Now it has been proved in Art. 461, by calculations of eclipses, that within the last 2556 years the length of the day is not become shorter by even a ten millionth part: and therefore, since ω varies inversely as k^2, or the length of the day varies as the square of the mean radius, the mass remaining the same, the mean radius of the Earth has not changed within the last five and twenty centuries by even a twenty-millionth part.

494. The appearance of fossil remains of tropical plants and animals in these higher latitudes has induced geologists to adopt the hypothesis that the temperature of the Earth was in ages gone by far higher than at present. The results of the last Article shew that no objection can be urged against this theory, at least upon mechanical principles. If this hypothesis be true we learn that the radiation goes on now very slowly, whatever its rapidity may have been at more ancient epochs.

495. The Principle of the Conservation of Areas, or rather the Principle of the Conservation of the Principal Moments which springs from it, proves that earthquakes, volcanic explosions, the action of winds upon the surface of the Earth, the friction and pressure of the Ocean upon the solid nucleus of the terrestrial spheroid, produce no variation in the principal moment on the direction of its axis: since these forces all arise from the mutual action of the parts of the system. And since the displacements produced by these causes in any portions composing the Earth's mass are too inconsiderable sensibly to alter the value of k, it follows that their effect on the angular velocity (ω) and upon the length of the day will be inappreciable.

Prop. *When a material system is in motion under the action of forces not impulsive, and none of which are extraneous to the system; then the change of the Vis Viva of the system during a given time depends only on the co-ordinates of the particles of the system at the beginning and end of the given time, and not at all on the curves which the particles describe.*

496. This is called the *Principle of Vis Viva.*

Let XYZ be the impressed accelerating forces which act upon the particle m resolved parallel to the axes of co-ordinates: xyz the co-ordinates to m at the time t: then the forces

$$m\left(X-\frac{d^2x}{dt^2}\right),\quad m\left(Y-\frac{d^2y}{dt^2}\right),\quad m\left(Z-\frac{d^2z}{dt^2}\right),$$

acting on m, and similar forces acting on the other particles of the system will satisfy the conditions of equilibrium; (Art. 226.) Wherefore by the Principle of Virtual Velocities (Art. 72.) we have

$$\Sigma\,.\,m\left\{\left(X-\frac{d^2x}{dt^2}\right)\delta x+\left(Y-\frac{d^2y}{dt^2}\right)\delta y+\left(Z-\frac{d^2z}{dt^2}\right)\delta z\right\}=0,$$

δx, δy, δz, being any small spaces geometrically described by m parallel to the axes, in a manner consistent with the connexion of the parts of the system one with another at the time t.

Now the spaces actually described by the particle m during the instant after the time t parallel to the axes are consistent with the connexion of the parts of the system one with another: hence we may take

$$\delta x=\frac{dx}{dt}\delta t,\quad \delta y=\frac{dy}{dt}\delta t,\quad \delta z=\frac{dz}{dt}\delta t,$$

and the above equation becomes

$$\Sigma\,.\,m\left\{\frac{dx}{dt}\frac{d^2x}{dt^2}+\frac{dy}{dt}\frac{d^2y}{dt^2}+\frac{dz}{dt}\frac{d^2z}{dt^2}\right\}=\Sigma\,.\,m\left(X\frac{dx}{dt}+Y\frac{dy}{dt}+Z\frac{dz}{dt}\right)$$

$$\therefore\ \Sigma . m \left\{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2}\right\}$$

$$= 2\,\Sigma . m \int \left(X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt}\right) dt + C.$$

But by the Differential Calculus

$$\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2} = \frac{ds^2}{dt^2} = (\text{velocity})^2 = v^2,$$

$$\therefore\ \Sigma . mv^2 = 2\,\Sigma . m \int \left(X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt}\right) dt + C.$$

Now let P be the mutual pressure of two particles m and m' in contact at the point xyz : $\alpha\beta\gamma$ the angles its direction makes with the axes.

Then the expression $m \int \left(X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt}\right) dt,$

for the particle m becomes

$$\int P \left(\cos\alpha \frac{dx}{dt} + \cos\beta \frac{dy}{dt} + \cos\gamma \frac{dz}{dt}\right) dt,$$

and for the particle m' it becomes

$$-\int P \left(\cos\alpha \frac{dx}{dt} + \cos\beta \frac{dy}{dt} + \cos\gamma \frac{dz}{dt}\right) dt,$$

and the sum of these = 0, and therefore P will not appear in our final equation.

Again, let xyz, $x'y'z'$ be the co-ordinates of two particles m and m' not in contact; r their distance; P their mutual action, supposed to be a function of r: then the cosines of the angles which the direction of P makes with the axes are $\frac{x-x'}{r}$, $\frac{y-y'}{r}$, $\frac{z-z'}{r}$: and the expression

$m \int \left(X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt}\right) dt$ becomes, for the particle m

$$\int P\left\{\frac{x-x'}{r}\frac{dx}{dt}+\frac{y-y'}{r}\frac{dy}{dt}+\frac{z-z'}{r}\frac{dz}{dt}\right\}dt$$

and for the particle m'

$$-\int P\left\{\frac{x-x'}{r}\frac{dx'}{dt}+\frac{y-y'}{r}\frac{dy'}{dt}+\frac{z-z'}{r}\frac{dz'}{dt}\right\}dt,$$

Hence P will appear in $2\Sigma.m\int\left(X\frac{dx}{dt}+Y\frac{dy}{dt}+Z\frac{dz}{dt}\right)dt$ under the form

$$2\int\frac{P}{r}\left\{(x-x')\frac{d(x-x')}{dt}+(y-y')\frac{d(y-y')}{dt}+(z-z')\frac{d(z-z')}{dt}\right\}dt,$$

or $2\int P\,dr$; since $r^2=(x-x')^2+(y-y')^2+(z-z')^2$.

Wherefore we have finally

$$\Sigma.mv^2=2\Sigma.\int P\,dr+C,$$

and since P is a function of r the second side of this equation, when taken between limits, will be a function solely of the initial and final co-ordinates of the particles of the system.

Hence the Principle is true.

497. Cor. 1. The expression for the vis viva of a system acted on by any forces (not impulsive) is given by the equation

$$\Sigma.mv^2=2\Sigma.m\int\left(X\frac{dx}{dt}+Y\frac{dy}{dt}+Z\frac{dz}{dt}\right)dt.$$

498. Cor. 2. Any force which acts upon a fixed point of the system will not appear in the equation of vis viva, since the velocities of the point are nothing. In this way the mutual pressures of any parts of the system against immoveable obstacles will not appear. Neither will the force of friction which acts upon a body rolling (*not* partly rolling and partly sliding) upon a fixed obstacle appear in the vis viva; since the point of contact is for an instant at rest.

499. Cor. 3. If forces act upon none of the particles of the system except such as remain invariably connected during the motion, then the vis viva remains the same throughout the motion. For in this case $\frac{dr}{dt} = 0$; and therefore $\Sigma . mv^2 = C$.

This is called the *Principle of the Conservation of Vis Viva.*

Prop. *The vis viva of a material system in motion is equal to the vis viva arising from the motion of translation of the centre of gravity in space added to the vis viva arising from the motion about the centre of gravity.*

500. Let xyz be co-ordinates to m at time t,

$\overline{x}\,\overline{y}\,\overline{z}$ be co-ordinates to centre of gravity of the system,

and let $x = \overline{x} + x_{,}$, $y = \overline{y} + y_{,}$, $z = \overline{z} + z_{,}$;

$$\therefore v^2 = (\text{vel. of } m)^2 = \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2} = \frac{d\overline{x}^2}{dt^2} + \frac{d\overline{y}^2}{dt^2} + \frac{d\overline{z}^2}{dt^2}$$

$$+ 2\left\{\frac{d\overline{x}}{dt}\frac{dx_{,}}{dt} + \frac{d\overline{y}}{dt}\frac{dy_{,}}{dt} + \frac{d\overline{z}}{dt}\frac{dz_{,}}{dt}\right\} + \frac{dx_{,}^2}{dt^2} + \frac{dy_{,}^2}{dt^2} + \frac{dz_{,}^2}{dt^2},$$

and observing that $\Sigma . m\frac{dx_{,}}{dt} = 0$, $\Sigma . m\frac{dy_{,}}{dt} = 0$, $\Sigma . m\frac{dz_{,}}{dt} = 0$

we have

$$\Sigma . mv^2 = \overline{V}^2 \Sigma . m + \Sigma . mv_{,}^2,$$

$\overline{V}$ being the velocity of the centre of gravity of the system and $v_{,}$ the velocity of m relative to the centre of gravity.

501. By Art. 496. we have

$$2\,\Sigma . mPdr = \frac{d(\Sigma . mv^2)}{dt}\,dt$$

therefore whenever during the motion the particles of the system assume such a relative position that the vis viva is a maximum or minimum $\Sigma . mPdr = 0$, and therefore (Art. 81.) the system is at that instant in a position in which the forces are in equilibrium.

When the vis viva is a *maximum* the position which the system assumes would be a position of *stable* equilibrium, if all velocity be destroyed: and when the vis viva is a *minimum* the position would be one of *unstable* equilibrium. This readily follows from considerations analogous to those mentioned in Art. 79. Also since a function passes through its maximum and minimum values *alternately* as the variable increases continuously, the system when in motion will pass through the positions of stable and unstable equilibrium alternately.

PROP. *When a material system in motion is acted on by impulsive forces, none of which are supposed external to the system, vis viva is lost or gained according as the impulse is of the nature of collision or explosion. When the system is perfectly elastic the vis viva is the same before and after the impulse.*

502. Let $V\cos a$, $V\cos b$, $V\cos c$ be the velocities of any particle of the system m parallel to the axes at the commencement of the impulse: P the resultant of the internal forces acting on m, and $\alpha\beta\gamma$ the angles its direction makes with the axes. Then the forces

$$mV\cos a + P\cos\alpha - m\frac{dx}{dt},\quad mV\cos b + P\cos\beta - m\frac{dy}{dt},$$

$$mV\cos c + P\cos\gamma - m\frac{dz}{dt}$$

acting on m parallel to the axes and similar forces acting on all the other particles of the system will satisfy the conditions of equilibrium (Art. 226).

Hence by the Principle of Virtual Velocities (Art. 72.)

$$\Sigma . m\left\{\left(V\cos a + \frac{P}{m}\cos\alpha - \frac{dx}{dt}\right)\delta x\right.$$

$$\left. + \left(V\cos b + \frac{P}{m}\cos\beta - \frac{dy}{dt}\right)\delta y + \left(V\cos c + \frac{P}{m}\cos\gamma - \frac{dz}{dt}\right)\delta z\right\} = 0,$$

δx, δy, δz being any small spaces geometrically described by m parallel to the axes in a manner consistent with the connexion of the parts of the system one with another at the time t.

We shall first observe, that P will disappear from the equation above. For if P be the action between two bodies of the system which touch each other in the point xyz, then δx, δy, δz will be the virtual velocities of the point of contact with respect to P acting on one, and $-\delta x$, $-\delta y$, $-\delta z$ those with respect to the other body; and consequently in the above expression when a term of the form $P\cos a\delta x$ occurs we find also $-P\cos a\delta x$; and therefore P disappears, and the equation becomes

$$\Sigma . m\left\{\left(V\cos a - \frac{dx}{dt}\right)\delta x + \left(V\cos b - \frac{dy}{dt}\right)\delta y + \left(V\cos c - \frac{dz}{dt}\right)\delta z\right\} = 0 \ldots (1).$$

In applying this equation to calculate the motion of a system suddenly acted on by impulsive forces we must make a few important remarks. When a body yields or expands the centres of its particles approach or recede from each other; but, during the action of the impulsive forces, the spaces through which they yield or recede are so extremely small, that we wholly neglect them; but this is not the case with their velocities, for although the change of distance of the centres of the particles during the impulse is indefinitely small, yet this change divided by the time elapsed during the impulse will give a difference of velocities which is not necessary insensible. In consequence of this, when two bodies come into collision the particles in contact do not move with the same velocity at the first instant of the contact, but after all compression ceases and the restitution of figure has not begun to take place, at this instant and at this instant alone, do the particles in contact move with the same velocity. Again, when two bodies are acted upon by impulsive forces of the nature of internal explosion, the particles in contact move with the same velocity at the first instant of the action of the forces, but at every other instant of the action they move with different velocities.

Now δx, δy, δz may be any small spaces provided they be consistent with the connexion of the parts of the system one with another at the time of the impulse; this connexion remains the same during the impulse, because all small spaces described in that time are insensible. Wherefore we must not give to these quantities such arbitrary values as will imply, that the particles in contact at the point (xyz) separate, or penetrate each other, or (in other words) move with opposite or unequal velocities.

If, then, the impulsive forces be of the nature of *collision* and xyz be co-ordinates of the point of contact, the initial velocities of the particles in contact will not be the same, but after the collision ceases they will have the same effective velocities $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$. Hence, in this case we may put

$$\delta x = \frac{dx}{dt}\delta t, \quad \delta y = \frac{dy}{dt}\delta t, \quad \delta z = \frac{dz}{dt}\delta t,$$

since these virtual velocities are consistent with the connexion of the parts of the system one with another, and they imply that the particles in contact remain in contact when the principle of virtual velocities is applied to the system in its imaginary state of equilibrium.

If the impulsive forces be of the nature of internal *explosion*, then it will easily be seen, after what has been said, that we may put

$$\delta x = V\cos a\,\delta t, \quad \delta y = V\cos b\,\delta t, \quad \delta z = V\cos c\,\delta t,$$

but we must not put the other values for δx, δy, δz.

I. Suppose the impulse is of the nature of collision, the bodies being inelastic.

Then substituting for δx, δy, δz in equation (1), and putting v for the resulting velocity of m

$$\Sigma \,.\, mv^2 = \Sigma \,.\, mV\left\{\frac{dx}{dt}\cos a + \frac{dy}{dt}\cos b + \frac{dz}{dt}\cos c\right\};$$

$$\therefore \ \Sigma . mv^2 = \Sigma . m V^2$$

$$- \Sigma . m \left\{ \left(V \cos a - \frac{dx}{dt} \right)^2 + \left(V \cos b - \frac{dy}{dt} \right)^2 + \left(V \cos c - \frac{dz}{dt} \right)^2 \right\},$$

and, since the last term of this is essentially negative, we see that vis viva is lost during the collision.

II. Suppose the impulse is of the nature of internal explosion.

By substituting in (1) the values of δx, δy, δz above specified we have

$$\Sigma . m V^2 = \Sigma . m V \left\{ \frac{dx}{dt} \cos a + \frac{dy}{dt} \cos b + \frac{dz}{dt} \cos c \right\};$$

$$\therefore \ \Sigma . mv^2 = \Sigma . m V^2$$

$$+ \Sigma . m \left\{ \left(V \cos a - \frac{dx}{dt} \right)^2 + \left(V \cos b - \frac{dy}{dt} \right)^2 + \left(V \cos c - \frac{dz}{dt} \right)^2 \right\}$$

and consequently vis viva is gained during the separation.

III. Suppose that the impulse is in part of the nature of collision, and in part of the nature of explosion.

In this case we must combine the cases already mentioned. When, for instance, the bodies are perfectly elastic the impulsive forces which act during the collision are the same exactly as those which act during the separation of the bodies; it follows, by examining the above expressions, that the vis viva lost during the collision is exactly regained during the separation, and that the state of the system is consequently unaffected by the whole impulse.

503. The degradation of rocks and the consequent action of collision which is incessantly taking place in large portions of matter on the surface of the Earth, the unceasing action of waves on the sea shore and the collision of the waters of the ocean upon the solid nucleus of the Earth, and other like causes are continually causing a loss of vis viva in the Earth's mass, and if allowed to act without any compensating pheno-

mena would in the course of time produce a sensible effect in the length of the day: but on the other hand the explosions of volcanoes are compensating causes. Also the downward motion of rivers, the descent of vapour and cloud in the form of rain, the descent of boulders and avalanches, and various other causes, all tend to remove large portions of matter nearer to the Earth's centre and would in the course of time produce a sensible increase in the length of the day, since we have seen (Art. 499.) that the vis viva of the Earth is constant, if we neglect the attraction of the Sun, Moon, and planets and consider only the action of finite forces. But the ascent of vapour by evaporation, and the effect of earthquakes and volcanoes in removing masses of matter to a greater distance from the centre have an opposite effect. On the whole all these causes balance each other, since observations have shewn that the length of the day has been invariable for many ages, Art. 461.

PROP. *To prove that the variation of* $\Sigma . \mathrm{m} \int \mathrm{v\,ds}$ *taken between given limits equals zero, where* v *is the velocity and* ds *is the element of the space described in the short time* dt *by the particle* m *of a material system in free motion: if any particle move on a surface it is supposed to continue on the surface in taking the variation.*

504. This is called the *Principle of Least Action*; because, in general, $\Sigma . m \int v\,ds$ is a minimum.

Let δ be the symbol of variation in the Calculus of Variations: then

$$\delta\,(\Sigma . m \int v\,ds) = \Sigma . m \int \delta\,(v\,ds) = \Sigma . m \int (v\,\delta . ds + ds\,\delta v)$$

$$= \Sigma . m \int (v\,\delta . ds + \tfrac{1}{2}\,dt\,\delta . v^2).$$

Suppose the particle m rests on a curve surface, and that R is the normal pressure, $\alpha\beta\gamma$ the angles of its direction; X, Y, Z the accelerating forces acting on m, then (as in Art. 407)

$$\frac{d^2x}{dt^2} = X + \frac{R}{m}\cos\alpha, \quad \frac{d^2y}{dt^2} = Y + \frac{R}{m}\cos\beta, \quad \frac{d^2z}{dt^2} = Z + \frac{R}{m}\cos\gamma.$$

Let $L = 0$ be the equation to the surface; then

$$\cos\alpha = V\frac{dL}{dx}, \quad \cos\beta = V\frac{dL}{dy}, \quad \cos\gamma = V\frac{dL}{dz};$$

$$\text{where } \frac{1}{V^2} = \frac{dL^2}{dx^2} + \frac{dL^2}{dy^2} + \frac{dL^2}{dz^2}.$$

Hence $v^2 = 2\int(Xdx + Ydy + Zdz) = \phi(x, y, z) + \text{const.}$

$$\therefore\ \tfrac{1}{2}\delta . v^2 = X\delta x + Y\delta y + Z\delta z$$

$$= \frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z - \frac{R}{m}V\delta L = \frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z,$$

for if the particle do not rest on a surface $R = 0$, and if it do still $\delta L = 0$, because we suppose the motion to be such that particles on curve surfaces remain on the surfaces.

Again, $ds^2 = dx^2 + dy^2 + dz^2$,

$$\therefore\ ds\delta . ds = dx\delta . dx + dy\delta . dy + dz\delta . dz;$$

$$\therefore\ v\delta . ds = \frac{dx}{dt}\delta . dx + \frac{dy}{dt}\delta . dy + \frac{dz}{dt}\delta . dz.$$

$$\text{Hence } \int(v\delta . ds + \tfrac{1}{2}dt\delta . v^2) = \frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z + \text{const.}$$

and at the limits $\delta x = 0$, $\delta y = 0$, $\delta z = 0$ because the first and last positions are given,

$$\therefore\ \int(v\delta . ds + \tfrac{1}{2}dt\delta . v^2) = 0,$$

$$\therefore\ \delta(\Sigma . m\int v ds) = 0,$$

and $\Sigma . m\int v ds$ is a maximum or minimum. It is evidently a minimum, because a path of an indefinite length can always be found for any particle of the system.

Cor. 1. Since $ds = vdt$ we learn that $\Sigma . m\int v^2 dt$ is a minimum, or the quantity of vis viva generated or expended during any given time is a minimum.

Cor. 2. If the system consist of only one particle moving on a surface and no forces but the normal pressure act, then $\int v ds$ is a minimum: but v is a constant (Art. 407), therefore $\int ds$ is a minimum, or the particle will describe the shortest curve line that can be drawn on the surface between its positions at the beginning and end of the time t.

505. If we compare the principle of least action with the principles of the conservation of the motion of the centre of gravity, of the conservation of areas, and of vis viva we see that this principle only serves to determine the equations of motion, and is therefore comparatively useless since these are found by much simpler means; but the other principles, which develop important properties, have the advantage of furnishing three general integrals of the equations of motion, which are in most problems the only integrals that can be found.

Prop. *To shew that the calculation of the motion of a material system may be made to depend upon the integration of a single function.*

506. We shall shew this by proving a new dynamical principle discovered by Professor Hamilton and published in the *Philosophical Transactions*, 1834.

We have seen, Art. 497, that the Principle of Virtual Velocities leads us to the dynamical equation

$$\Sigma . m \left\{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2}\right\} = 2\, \Sigma . m \int \left\{X \frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt}\right\} dt.$$

Now it is easily shewn that

$$\Sigma . m \left(X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt}\right)$$

is a perfect differential coefficient with respect to t for all the forces which exist in nature; viz. forces tending to the centres of the particles of the material universe, whether fixed or moveable. Let therefore the second side $= 2(U + H)$, H being independent of t: and let $2T$ be the vis viva of the system at the time t; T_0, H_0 the values of T and H when $t = 0$;

$$\therefore\ T = U + H, \text{ and } T_0 = U_0 + H.$$

Now if the initial circumstances of the motion be varied, then H will vary, and so also will T and U: let δ be the symbol of these variations;

$$\therefore\ \delta T = \delta U + \delta H$$

$$\text{or } \Sigma\,.\,m\left\{\frac{dx}{dt}\delta\frac{dx}{dt} + \frac{dy}{dt}\delta\frac{dy}{dt} + \frac{dz}{dt}\delta\frac{dz}{dt}\right\}$$

$$= \Sigma\,.\,m\left\{\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z\right\} + \delta H;$$

$$\text{and therefore } 2\,\Sigma\,.\,m\left\{\frac{dx}{dt}\delta\frac{dx}{dt} + \frac{dy}{dt}\delta\frac{dy}{dt} + \frac{dz}{dt}\delta\frac{dz}{dt}\right\}$$

$$= \Sigma\,.\,m\frac{d}{dt}\left\{\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z\right\} + \delta H.$$

Now let the accumulation of the vis viva from the commencement to the termination of the time t be V;

$$\therefore\ \mathrm{V} = \int_0^t \Sigma\,.\,m\left\{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2}\right\}dt.$$

Then V is a function of the initial and final co-ordinates of the material particles, and

$$\delta\mathrm{V} = \Sigma\,.\left\{\frac{\delta\mathrm{V}}{\delta x}\delta x + \frac{\delta\mathrm{V}}{\delta y}\delta y + \frac{\delta\mathrm{V}}{\delta z}\delta z + \frac{\delta\mathrm{V}}{\delta a}\delta a + \frac{\delta\mathrm{V}}{\delta b}\delta b + \frac{\delta\mathrm{V}}{\delta c}\delta c\right\}$$

$$= 2\int_0^t \Sigma\,.\,m\left\{\frac{dx}{dt}\delta\frac{dx}{dt} + \frac{dy}{dt}\delta\frac{dy}{dt} + \frac{dz}{dt}\delta\frac{dz}{dt}\right\}dt$$

$$+ \Sigma\,.\left\{\frac{\delta\mathrm{V}}{\delta a}\delta a + \frac{\delta\mathrm{V}}{\delta b}\delta b + \frac{\delta\mathrm{V}}{\delta c}\delta c\right\}$$

$$= \Sigma\,.\,m\left\{\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z\right\} + t\delta H + H_{,},$$

$H_{,}$ being a function of the initial co-ordinates a, b, c.

But when $t = 0$, $\delta V = 0$, hence

$$\delta V = \Sigma . m \left\{\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z\right\}$$

$$- \Sigma . m \left\{\frac{da}{dt}\delta a + \frac{db}{dt}\delta b + \frac{dc}{dt}\delta c\right\} + t\delta H.$$

From this equation we obtain the following groups of equations; $x_1 y_1 z_1$ being co-ordinates to m_1......

$$\left.\begin{array}{l} \frac{\delta V}{\delta x_1} = m_1 \frac{dx_1}{dt}; \quad \frac{\delta V}{\delta x_2} = m_2 \frac{dx_2}{dt}; \ \ldots\ldots \\ \frac{\delta V}{\delta y_1} = m_1 \frac{dy_1}{dt}; \quad \frac{\delta V}{\delta y_2} = m_2 \frac{dy_2}{dt}; \ \ldots\ldots \\ \frac{\delta V}{\delta z_1} = m_1 \frac{dz_1}{dt}; \quad \frac{\delta V}{\delta z_2} = m_2 \frac{dz_2}{dt}; \ \ldots\ldots \end{array}\right\} \ldots\ldots\ldots (A).$$

Second group,

$$\left.\begin{array}{l} \frac{\delta V}{\delta a_1} = - m_1 \frac{da_1}{dt}; \quad \frac{\delta V}{\delta a_2} = - m_2 \frac{da_2}{dt}; \ \ldots \\ \frac{\delta V}{\delta b_1} = - m_1 \frac{db_1}{dt}; \quad \frac{\delta V}{\delta b_2} = - m_2 \frac{db_2}{dt}; \ \ldots \\ \frac{\delta V}{\delta c_1} = - m_1 \frac{dc_1}{dt}; \quad \frac{\delta V}{\delta c_2} = - m_2 \frac{dc_2}{dt}; \ \ldots \end{array}\right\} \ldots\ldots\ldots (B).$$

Lastly,

$$\frac{\delta V}{\delta H} = t \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (C).$$

The problem is therefore reduced to finding the function V, which Professor Hamilton denominates the *characteristic function* of the motion of a system. When V is calculated, then, by eliminating H from the equations (A) (C), we shall have the $3n$ integrals of the first order of the equations of motion by simply differentiating V. And by eliminating H from the equations (B) (C) we have the $3n$ final integrals by simple differentiation.

It may be observed that V must satisfy the two following partial differential equations

$$\tfrac{1}{2}\Sigma . \frac{1}{m}\left\{\frac{\delta V^2}{\delta x^2} + \frac{\delta V^2}{\delta y^2} + \frac{\delta V^2}{\delta z^2}\right\} = U + H,$$

$$\text{and } \tfrac{1}{2}\Sigma . \frac{1}{m}\left\{\frac{\delta V^2}{\delta a^2} + \frac{\delta V^2}{\delta b^2} + \frac{\delta V^2}{\delta c^2}\right\} = U_0 + H.$$

These equations furnish the principal means of discovering the form of the function V and are of essential importance in Professor Hamilton's Theory.

The equation

$$\delta V = \Sigma . m\left\{\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z\right\}$$

$$- \Sigma . m\left\{\frac{da}{dt}\delta a + \frac{db}{dt}\delta b + \frac{dc}{dt}\delta c\right\} + t . \delta H$$

is denominated the *law of varying action*.

507. "It has been shewn by Lagrange and others, in treating of the motion of a system, that the variation δV vanishes when the extreme co-ordinates and constant H are given (Art. 504): and they appear to have deduced from this result only the principle which is called the law of *least action*: namely, that if the particles of a system be imagined to move from a given set of initial to a given set of final positions, not as they do, nor even as they could move consistently with the general dynamical laws, or differential equations of motion, but so as not to violate any supposed geometrical connexions, nor that one dynamical relation between velocities and configuration which constitutes the law of vis viva: and if, moreover, this geometrically imaginable, but dynamically impossible motion, be made to differ infinitely little from the actual manner of motion of the system, between the given extreme positions, then the varied value of the definite integral called *action*, or the accumulated vis viva of the system in the motion thus imagined, will differ infinitely less from the actual value of that integral.

But when this principle of least action, or," as Professor Hamilton proposes to call it, "of *stationary action*, is applied to the determination of the actual motion of a system, it serves only to form, by the rules of the Calculus of Variations, the differential equations of motion of the second order, which can always be otherwise found."

In this, then, appears the excellence of this new principle called the *law of varying action*, that we pass from an actual motion to another motion dynamically possible, by varying the extreme positions of the system and (in general) the quantity H: but more especially that it serves to express, by means of a single function, not the mere differential equations of motion, but their intermediate and their final integrals.

We hope that the slight sketch we have given of this new principle will tempt our readers to consult the original Memoirs in the Transactions of the Royal Society of London for the years 1834, 1835, from which this notice has been gathered.

PROP. *To prove the general laws of the very small oscillations of a vibrating system of particles.*

508. If the oscillations of the particles be extremely small we may always reduce the equations of motion to linear equations and obtain approximately the co-ordinates in terms of the time. Very many and various phenomena depend upon the principles of small oscillations.

Let i be the number of particles, and m the number of equations $L = 0$, $L' = 0$, ... connecting their co-ordinates: let $3i - m = n$, then these equations determine m of the variable co-ordinates in terms of the other n, or, more generally, all the co-ordinates may be determined by means of these equations in functions of n independent variables. Let α, β .... be the initial values of these variables, and $\alpha + u$, $\beta + v$ .... their values at the time t; in which we suppose that u, v .... are very small during the whole motion: hence the co-ordinates x, y, z, x', ... can be expanded in very converging series of u, v ....

$$\text{Let } x = p + a\,u + b\,v + \tfrac{1}{2}c\,u^2 + \tfrac{1}{2}e\,v^2 + f\,uv + \ldots\ldots$$

$$y = p_1 + a_1 u + b_1 v + \tfrac{1}{2}c_1 u^2 + \tfrac{1}{2}e_1 v^2 + f_1 uv + \ldots\ldots$$

$$z = p_2 + a_2 u + b_2 v + \tfrac{1}{2} c_2 u^2 + \tfrac{1}{2} e_2 v^2 + f_2 uv + \ldots\ldots$$

$$x' = p' + a' u + b' v + \tfrac{1}{2} c' u^2 + \tfrac{1}{2} e' v^2 + f' uv + \ldots\ldots$$

....

Also since the forces $X, Y, Z, X', \ldots\ldots$ are supposed to be functions of the co-ordinates, these may be expanded in converging series: let

$$X = P + Au + Bv + \ldots, \qquad Y = P_1 + A_1 u + B_1 v + \ldots,$$
$$Z = P_2 + A_2 u + B_2 v + \ldots, \ \&c.$$

$P, A, B \ldots$ being functions of $p, a, b, c \ldots$

Now by Art. 496, we have

$$\Sigma . m \left\{ \left(\frac{d^2 x}{dt^2} - X \right) \delta x + \left(\frac{d^2 y}{dt^2} - Y \right) \delta y + \left(\frac{d^2 z}{dt^2} - Z \right) \delta z \right\} = 0,$$

and $\delta x = (a + cu + fv + \ldots) \delta u + (b + ev + fu + \ldots) \delta v + \ldots$

....

If we substitute these and put the coefficients of the n arbitrary quantities δu, δv ...... equal to zero, we have

$$\Sigma . m \left\{ \left(\frac{d^2 x}{dt^2} - X \right) (a + cu + fv + \ldots) + \left(\frac{d^2 y}{dt^2} - Y \right) (a_1 + c_1 u + f_1 v + \ldots) \right.$$
$$\left. + \left(\frac{d^2 z}{dt^2} - Z \right) (a_2 + c_2 u + f_2 v + \ldots) \right\} = 0.$$

....

It remains to substitute for $X, Y, Z \ldots x, y, z \ldots$, this substitution being made we shall neglect the squares and products of $u, v \ldots$ and of their second differential coefficients with respect to t: we shall thus have n linear equations of the form

$$\left. \begin{array}{l} D \dfrac{d^2 u}{dt^2} + E \dfrac{d^2 v}{dt^2} + \ldots\ldots + Fu + Gv + \ldots\ldots = Q \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array} \right\} \ldots (1),$$

where D, E, F, G, Q ... are given functions of the constants which enter the formulæ for x, y, z ... X, Y, Z ... We may suppose $Q = 0$, since we can always add to u, v ... such constant values as to strike Q out: this amounts to supposing that α, β, γ ... are the values of the n independent variables which correspond to a state of equilibrium of the system; since when $u = 0$, $v = 0$ ... the accelerating forces vanish.

We may satisfy the equations (1), putting $Q = 0$, by $u = RN \sin(t\sqrt{\rho} - r)$, $v = RN' \sin(t\sqrt{\rho} - r)$ ...... R and r being arbitrary constants of which the second may be considered positive and less than π, and ρ, N, N', ... are constants to be determined. By putting these values in (1), we have n equations,

$$\left.\begin{array}{l}(DN + EN' + ...)\,\rho = FN + GN' + ... \\ ....\end{array}\right\} \quad .... (2).$$

In eliminating from these $n - 1$ of the quantities N, N' .... the n^{th} equation will be of the n^{th} degree in ρ and will be free from all the quantities N, N' .... in consequence of the form of equations (2). And the values of $n - 1$ of N, N', ... *viz.* N' .... suppose, obtained from (2) will be rational fractions of the n^{th} degree with respect to ρ, having a common denominator, and being each multiplied by N, which remains indeterminate; we may therefore choose N equal to the common denominator, and N, N' .... will be expressed in terms of symmetrical functions of ρ of the n^{th} degree.

In consequence of the linear form of equations (1) they are satisfied not only by the values of u, v .... corresponding to each of the n values of ρ, but also by taking for u, v .... the sums of these particular values, in which we may change the values of R and r as ρ changes.

If then, we call $\rho\,\rho_1\,\rho_2\,\rho_3$ .... the values of ρ and use corresponding subscript figures for the other letters, we have the following general solutions of equations (1),

$$\left.\begin{array}{l}u = RN\sin(t\sqrt{\rho} - r) + R_1N_1\sin(t\sqrt{\rho_1} - r_1) + ... \\ v = RN'\sin(t\sqrt{\rho} - r) + R_1N'_1\sin(t\sqrt{\rho_1} - r_1) + ... \\ ....\end{array}\right\} \quad .... (3),$$

R, R_1 .... r, r_1 .... being the $2n$ arbitrary constants in these complete integrals. The constants must be determined in terms of the initial values of u, v .... and their differential coefficients: they are small because the original displacements are small. If the values $\rho\,\rho_1\,\rho_2$ .... be all real, then the motions of the particles will be periodical and will always be very small. If, however, one or more of $\rho\,\rho_1\,\rho_2$ .... be imaginary, we must replace the circular functions by exponentials, and therefore as the time increases u, v .... will increase indefinitely and the above formulæ will cease to be true. In the first case the state of equilibrium of the system is stable; in the second unstable.

509. Suppose, for example, that all of R, R_1 .... except the first vanish: then

$$\left.\begin{aligned} x &= p + (a\,N + b\,N' + \ldots\ldots)\,R\sin(t\sqrt{\rho} - r), \\ y &= p_1 + (a_1 N + b_1 N' + \ldots\ldots)\,R\sin(t\sqrt{\rho} - r), \\ z &= p_2 + (a_2 N + b_2 N' + \ldots\ldots)\,R\sin(t\sqrt{\rho} - r), \\ x' &= p' + (a' N + b' N' + \ldots\ldots)\,R\sin(t\sqrt{\rho} - r), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \ldots\ldots(4).$$

Hence the particles all perform their oscillations in the same period, *viz.* $\dfrac{2\pi}{\sqrt{\rho}}$: and all the particles return to their places of equilibrium at the same instant.

510. A system of material particles, in which the relations connecting the co-ordinates are of such a number as to leave n of them independent variables, will, when slightly disturbed from the position of rest, assume a number (n) of oscillatory motions, each analogous to that described in the last Article, corresponding to the n values $\rho\,\rho_1\,\rho_2$ .... And in virtue of equations (3) and the corresponding values of x, y, z .... all the oscillations, or only some of them may exist at the same time in the system: and conversely, whatever be the initial derangement we may always resolve the motion of each particle parallel to each co-ordinate axis into n or less than n simple

oscillations analogous to that represented by equations (4), the periods being $\frac{2\pi}{\sqrt{\rho}}$, $\frac{2\pi}{\sqrt{\rho_1}}$ ....: when these are commensurable the whole system will return to the same state in a period equal to the least common multiple of these periods: this is the case in vibrating cords, and vibrating surfaces. The principle proved in this Article is called the *Principle of the Co-existence of Small Vibrations.*

511. Suppose that U, V .... are values of u, v .... when the system is in vibration under the action of one set of forces, the initial values of u, v .... $\frac{du}{dt}, \frac{dv}{dt}$ .... being u_0, v_0 .... u_1, v_1 ..... Again suppose that U', V' .... are the values of u, v .... when the system is under the action of a second set of forces and u_0', v_0' .... u_1', v_1' .... the initial values of u, v .... $\frac{du}{dt}, \frac{dv}{dt}$ .... and so on: then, if the initial values of u, v .... $\frac{du}{dt}, \frac{dv}{dt}$..... be $u_0 + u_0' + ...$, $v_0 + v_0' + ...$, ... $u_1 + u_1' + ...$, $v_1 + v_1' + ...$ the general values of u, v .... are

$$u = U + U' + ..., \quad v = V + V' + ...$$

This principle, the truth of which arises from equations (1) being linear, is called the *Principle of the Superposition of Small Motions*: see Art. 288.

CHAPTER XVI.

PROBLEMS ON THE MOTION OF RIGID BODIES, AND ANY MATERIAL SYSTEM.

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512. We shall commence this Chapter with some observations upon the best methods of solving dynamical problems, and the application of the general principles proved in the last Chapter in facilitating their solution.

To determine the motion of a rigid body in space we have six differential equations of the second order: these contain the three co-ordinates to the centre of gravity and the three angles of position of the principal axes of the body; see Arts. 428, 446 and 447. These are the only relations that can exist among the *mechanical quantities* (Art. 144).

If all the forces and other quantities involved in these equations be known, then we have sufficient equations for solving the problem, and determining the position of the body at every instant.

If, however, the equations involve unknown forces, or unknown geometrical quantities (as angular and linear measures), or both, then there must exist as many more equations as there are of these unknown quantities; and, moreover, these relations must be among the geometrical quantities, since the six equations of motion, as we have mentioned, are the only mechanical relations that can exist.

Suppose that from the nature of the problem we have, involved in the six mechanical equations, one unknown force, and n unknown geometrical quantities besides those necessarily contained in the six equations: then we must have $n + 1$ additional equations among the $n + 6$ geometrical quantities:

when we have obtained these we have enough equations for the solution of the problem. To determine then, these $n + 6$ geometrical quantities, and therefore to determine the position of the body, we have already $n + 1$ equations free from unknown mechanical quantities, and must therefore obtain five more such equations; these are found by eliminating the unknown force from the six equations of motion. In the same way we should proceed if there were two, three, or more unknown forces. The equations which we obtain among the unknown geometrical quantities must be integrated, that we may have these quantities in terms of the time.

The same remarks will apply when the system is acted on by impulsive forces.

Now the principles of the conservation of motion of the centre of gravity, and the conservation of areas, and the principle of vis viva demonstrated in the last Chapter are the first integrals of the equations of motion under peculiar suppositions as to the nature of the forces which act upon the system. If, then, in any proposed problem one or more of these principles apply we may write them down as the integrals of our equations, and so diminish the labour of elimination and integration. If the integrals involved in these principles cannot be obtained in consequence of their involving unknown forces, the principles, though they may be true in these cases, will nevertheless not answer our purpose.

To find the unknown forces we must obtain their values from the equations of motion in terms of the geometrical quantities and their differential coefficients; and since these are supposed to be found the forces will be known also; see Problem 16.

If we find, after all the equations are written down, that there are more unknown quantities than equations, then the general solution of the problem is indeterminate; though it does not necessarily follow that all the unknown quantities are indeterminate (as in Art. 438). If we find more equations than unknown quantities it follows, that the general solution of the problem is impossible unless certain relations among the known quantities are fulfilled, the number of these relations being equal to the number by which the equations exceed the unknown quantities. Nevertheless, as in the last, some of the unknown quantities may be independent of these conditions.

We shall illustrate the remarks which we have made upon the solution of problems by referring to Art. 436. Here we have a case of motion in parallel planes, and therefore only three equations of motion: but these contain the unknown forces F and R beside the three necessary geometrical measures of position x, y, θ: hence two more equations must exist, and these among x, y, θ; these are equations (4) (5) in that Article; and we require only one more relation connecting x, y, θ; this we have by elimination from the equations of motion. But since the point of application of the forces F and R has no velocity, (for the body at each instant is revolving about that point as an instantaneous centre of rotation), F and R will not appear in the equation of vis viva of Art. 497. Hence this equation gives the integral we require; and we have (by Art. 500),

$$M\left\{\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}\right\}+Mk^2\frac{d\theta^2}{dt^2}=2\Sigma\,.\,m\int g\,dy',$$

y' being the vertical ordinate of m,

$$=2g\Sigma\,.\,m\,(y'+\text{constant})=2Mg\,(y+\text{constant}),\ \text{(Art. 413)}.$$

This is the equation obtained in Art 436, by elimination.

We shall now give some Problems; we shall solve a few, or give hints to guide to their solution.

PROB. 1. A sphere rolls down an inclined plane; required to determine the motion: (fig. 111.)

Since the motion of the centre of gravity is evidently parallel to the fixed inclined plane we shall measure its distance (z) from the point C which it occupies at the commencement of the motion, E the point which was then in contact at B with the plane, $\angle\, EOD=\theta$, P the pressure of the plane, F the friction acting upwards, a the radius of the sphere. Then for the motion of the centre of gravity (Art. 428.) and the motion of rotation about the centre of gravity (Art. 429, 431),

$$\frac{d^2z}{dt^2}=g-\frac{F}{M}\ \ldots\ldots\ (1),\qquad \frac{d^2\theta}{dt^2}=\frac{Fa}{Mk^2}\ \ldots\ldots\ (2);$$

three unknown quantities; we want another equation, this is

$$z = a\theta \;\ldots\ldots\; (3).$$

Since the object is to determine the position of the body at a given time we must obtain an equation between z and θ in addition to (3); this is obtained by eliminating F from (1) and (2): we thus have

$$\frac{d^2z}{dt^2} = g - \frac{k^2}{a}\frac{d^2\theta}{dt^2} = g - \frac{k^2}{a^2}\frac{d^2z}{dt^2} \text{ by } (3);$$

$$\therefore \frac{d^2z}{dt^2} = \frac{a^2g}{a^2+k^2}, \quad \frac{dz}{dt} = \frac{a^2gt}{a^2+k^2}, \text{ constant} = 0,$$

$$z = \frac{a^2g}{a^2+k^2}\frac{t^2}{2}, \quad \theta = \frac{ag}{a^2+k^2}\frac{t^2}{2}.$$

We might have used the principle of vis viva to obtain the second equation between z and θ, since F does not occur in the equation of vis viva, because the velocity of its point of application equals zero: but the elimination was so simple that we preferred that method.

Cor. If the body partly roll and partly slide, then F is constant, and must be determined by experiment. Hence equation (3) does not hold, and, in short, (1) (2) are sufficient for determining the motion in this case.

Prob. 2. Suppose the inclined plane, or wedge, on which the cylinder rests is capable of moving on a smooth horizontal plane: to determine the motion of the sphere and wedge: (fig. 111.)

The quantities as before, except that x and y are the horizontal and vertical co-ordinates of O measured from A in the horizontal plane: x' the horizontal co-ordinate to the point K of the wedge, M and M' the masses of the sphere and wedge. Then for the sphere we have the three equations (Art. 428, 429, 431.)

$$\frac{d^2x}{dt^2} = \frac{F\cos\alpha - P\sin\alpha}{M} \ldots (1), \quad \frac{d^2y}{dt^2} = \frac{F\sin\alpha + P\cos\alpha}{M} \ldots (2),$$

$$\frac{d^2\theta}{dt^2} = \frac{Fa}{Mk^2} \;\ldots\ldots\; (3).$$

For the wedge $\frac{d^2x'}{dt^2} = \frac{P \sin \alpha - F \cos \alpha}{M'}$ ...... (4).

Here are six unknown quantities, there must therefore be two relations connecting x, y, θ, x': these are

$x' - x - a \sin \alpha = a\theta \cos \alpha$ ...... (5), $y = h - a\theta \sin \alpha$ ...... (6),

h the initial value of y.

We must obtain two relations connecting x, y, θ, x' from (1) (2) (3) (4). But since there are no forces acting externally to the system of the sphere and wedge parallel to the horizon, there is a conservation of the horizontal motion of the centre of gravity (Art. 486): hence

$$M\frac{dx}{dt} + M'\frac{dx'}{dt} = \text{constant} = 0,$$

in our case, since there is supposed to be no initial velocity;

$$\therefore Mx + M'x' = \text{constant} = 0 \text{ ...... (7)},$$

if we properly choose the origin A.

Again, the principle of vis viva gives us an integral; for although the point of application of P and F does move in this case, yet the velocity of this point will have exactly opposite signs relatively to P and F acting on the sphere, and P and F acting on the wedge, and therefore P and F will not occur in the equation of vis viva: in short, they are *internal* forces;

$$\therefore M\left\{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + k^2\frac{d\theta^2}{dt^2}\right\} + M'\frac{dx'^2}{dt^2} = 2\Sigma . m\int -g\,dy'$$

$$= 2gM(h - y) \text{ ...... (8)}.$$

The equations (5) (6) (7) (8) will determine the position.

By (5) (7) (6) $\frac{dx'}{dt} = \frac{Ma \cos \alpha}{M + M'}\frac{d\theta}{dt} = -\frac{M}{M'}\frac{dx}{dt}$, $\frac{dy}{dt} = -a \sin \alpha \frac{d\theta}{dt}$;

$$\therefore \text{by (8) } \frac{d\theta^2}{dt^2}\left\{a^2 + k^2 - \frac{M}{M + M'}a^2 \cos^2 \alpha\right\} = 2ag\,\theta \sin \alpha;$$

$$\therefore \theta = \left\{a^2 + k^2 - \frac{M}{M + M'}a^2 \cos^2 \alpha\right\}^{-1} . \tfrac{1}{2} agt^2,$$

this coincides with the result of Prob. 1. if we put $M' = \infty$.

The equation to the path of O is, by (5) (6) (7)

$$y = h + \frac{M + M'}{M'} x \tan \alpha ;$$

therefore the path of the centre of the sphere is a straight line.

PROB. 3. A groove in the form of a cycloid with its vertex downwards and base horizontal is cut in a solid vertical board: determine the motion of a ball moving along it while the board itself is capable of moving freely along a smooth horizontal plane, and the curve which the ball describes in space.

Let x, y be the horizontal and vertical co-ordinates to the ball at time t: x' the co-ordinate to the vertex of the cycloid supposed to be in the horizontal plane: s the distance of the ball from the vertex measured along the groove, R the mutual pressure of the ball and groove, M and m the masses of the board and ball: then the equations of the problem are

$$\frac{d^2x}{dt^2} = -\frac{R}{m}\frac{dy}{ds} \;\ldots\ldots (1), \qquad \frac{d^2y}{dt^2} = -g + \frac{R}{m}\frac{d(x - x')}{ds} \;\ldots\ldots (2),$$

$$\frac{d^2x'}{dt^2} = \frac{R}{M}\frac{dy}{ds} \;\ldots\ldots (3),$$

$$x - x' = a \operatorname{vers}^{-1}\frac{y}{a} + \sqrt{2ay - y^2} \;\ldots\ldots\ldots\ldots\ldots (4).$$

The principle of conservation of the horizontal motion of the centre of gravity and the principle of vis viva both apply: they give

$$Mx' + mx = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

by choosing the origin under the initial position of the centre of gravity, and also

$$M\frac{dx'^2}{dt^2} + m\left\{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right\} = 2mg(h - y) \;\ldots\ldots (6),$$

h the initial value of y.

By (4) (6) the equation to the path of the ball in space is

$$\frac{M+m}{M}x = a \operatorname{vers}^{-1}\frac{y}{a} + \sqrt{2ay - y^2} \dots\dots\dots (7).$$

$$\text{By (7) (5) } \frac{dx}{dt} = \frac{M}{M+m}\sqrt{\frac{2a-y}{y}}\frac{dy}{dt} = -\frac{M}{m}\frac{dx'}{dt};$$

$$\therefore \text{ by (6) } m\frac{dy^2}{dt^2}\left\{\frac{M}{M+m}\,\frac{2a-y}{y} + 1\right\} = 2mg\,(h-y),$$

from which the motion must be calculated.

PROB. 4. Two equal balls are fixed to the end of a rod without weight; the rod is connected at its middle point with a fixed vertical axis, so as to allow the rod to move in a vertical plane passing through the axis, and to revolve with the axis in a horizontal direction: required the motion of the balls.

Let M be the mass of each ball, $2a$ the length of the rod, θ the angle the rod makes with the vertical axis at the time t, ϕ the angle the plane in which θ is measured makes with the initial position of that plane. We shall not write down the equations of motion in this case, but resort immediately to the principles of the conservation of areas (which applies since the resultant of the weights of the balls always passes through the fixed point,) and the conservation of vis viva. The principle of areas gives for a horizontal plane

$$2Ma^2\frac{d\phi}{dt} = \text{const.} \quad \therefore \frac{d\phi}{dt} = \alpha.$$

The conservation of vis viva gives

$$2M\left\{a^2\frac{d\theta^2}{dt^2} + a^2\frac{d\phi^2}{dt^2}\right\} = \text{const.} \quad \therefore \frac{d\theta}{dt} = \beta.$$

α and β being the initial angular velocities: then $\phi = \alpha t$, $\theta = \theta_0 + \beta t$: and if xyz be co-ordinates to one of the balls from the centre of the rod, the vertical axis being the axis of z, we have

$$x = a\sin(\theta_0+\beta t)\cos\alpha t,\; y = a\sin(\theta_0+\beta t)\sin\alpha t,\; z = a\cos(\theta_0+\beta t).$$

Prob. 5. A rod acts by one extremity with a uniform force in the direction of its length on the fly-wheel-crank of a steam engine, the other extremity moving in a straight line passing through the centre of the fly-wheel, and a uniform resistance is to be overcome by the fly-wheel. Find the velocity of the wheel at any time: and find the relation between the forces when they are so adjusted that after half a revolution the velocity may be unaltered.

Prob. 6. A uniform lever ACB, of which the arms AC and BC are at right angles to each other, rests in equilibrium when AC is inclined at α^0 to the vertical: shew that if AC be raised to a horizontal position (C being fixed) it will fall through an angle θ, such that $\cos\theta = \cot(45^\circ + \alpha)$.

Prob. 7. Given the radii and masses of the wheels in Atwood's Machine (Art. 216.) and the constant friction on the fixed axles of the wheels A and B (fig. 75); shew that the accelerating force of P and Q when in motion is much less effected by the friction at A and B, than if the wheel C turned about a fixed axle.

Prob. 8. A horizontal wheel moves freely about a vertical axis through its centre; a string of definite length is wrapt round its circumference, and passing through a ring has fixed to it a weight which falls by gravity; determine the whole motion.

Prob. 9. A hemisphere rests on a horizontal plane with a string fastened to its edge, which, passing over a pully, supports a weight: when the string is cut find the motion of the hemisphere.

Prob. 10. A beam is drawn from a horizontal to a vertical position about one extremity, which is fixed, by means of a string which is attached to the other extremity of the beam and after passing over a pully placed above the fixed extremity at a height equal to the length of the beam is attached to a falling body; determine the motion.

Prob. 11. A beam is projected perpendicular upwards, and has a rotatory motion round its centre of gravity in a vertical plane; it is observed at a given altitude to be in one of its horizontal positions, and to be then ascending with a given velocity; after this it performs a given number of revolutions and strikes the ground at a given angle: find the angular velocity.

Prob. 12. An inflexible straight rod is set in motion round a vertical axis passing through one extremity, about which it is capable of revolving freely in an horizontal plane: determine the motion of a ring sliding freely along it: and prove that the whole vis viva of the system is constant.

Prob. 13. A body is placed on a smooth wedge which rests upon a smooth horizontal plane, and the wedge is acted on by a horizontal and constant force f in a vertical plane perpendicular to the inclined plane of the wedge: determine the motion: and find f when the body is at rest on the plane.

Prob. 14. A semi-cylinder rests with its plane surface on the ground, on which it is capable of moving freely; shew that a body sliding down its curved surface will describe an ellipse; and determine the time of descent.

Prob. 15. Determine the motion of two heavy particles connected by an inflexible rod without weight, one of which moves on a surface of revolution and the other is constrained to move in the axis of the surface, this axis being vertical. Find the velocity of the particle on the surface when the other continues stationary.

Prob. 16. A cylinder rolls down a fixed quadrant; find where the cylinder will leave the quadrant.

The pressure must be calculated; the body leaves at the instant that this is zero.

Prob. 17. A sphere revolves round an axis touching its surface, find the length of the simple isochronous pendulum.

Prob. 18. A sector of a circle revolves round an axis perpendicular to its plane, and passing through the centre of the circle; find the angle of the sector when the length of the isochronous simple pendulum equals one half the length of the arc.

Prob. 19. For what axes of suspension is the time of a small oscillation of a solid body an absolute minimum? Take the case of an ellipsoid.

Prob. 20. A rough vertical cylinder, capable of revolving about a concentric but smooth and smaller cylinder as an axis, rests upon a rough horizontal plane, on every point of which the pressure is the same: determine the force applied by a string wrapt round the cylinder which will just make it move. If the force be greater than this determine the motion.

PROB. 21. A cylinder is made to rotate about it axis, and is then suddenly placed in contact with a rough horizontal plane with its axis parallel to the plane; the force of friction is of finite intensity and is not sufficiently great to prevent the line of contact of the cylinder from *sliding* on the plane at the beginning of the motion: required to determine the motion, and to shew how long the cylinder will continue to combine a sliding motion with its rolling motion.

Let ω be the angular velocity communicated to the cylinder before the contact: the friction does not affect this velocity at the first instant of the contact because the force of friction by hypothesis is of finite intensity: θ the angle described in the time t by that radius of the cylinder that was in contact with the plane at first: a the radius: x the distance of the axis of the cylinder at time t from its initial position: F the friction. Then F is constant and has its greatest value so long as the cylinder slides as well as rolls; in which case the equations of motion are

$$\frac{d^2\theta}{dt^2} = -\frac{Fa}{Mk^2} \ldots\ldots (1), \quad \frac{d^2x}{dt^2} = \frac{F}{M} \ldots\ldots (2),$$

but when the sliding motion ceases, if F' be the friction, which is then not necessarily constant, we must put F' for F in (1) and (2), and add the equation

$$x = a\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

I. So long as the sliding motion continues we have, then,

$$\frac{d\theta}{dt} = \omega - \frac{Fat}{Mk^2}, \quad \frac{dx}{dt} = \frac{Ft}{M},$$

$$\theta = \omega t - \frac{Fat^2}{2Mk^2}, \quad x = \frac{Ft^2}{2M}.$$

The sliding motion ceases when the motion of translation and the motion of rotation give exactly equal and opposite motions to the point of contact, or when

$$\frac{dx}{dt} = a\frac{d\theta}{dt}, \quad \text{and } \therefore\ t = \frac{M}{F}\,\frac{k^2 a\omega}{a^2 + k^2}, \quad \text{and } \frac{d\theta}{dt} = \frac{k^2\omega}{a^2 + k^2}.$$

II. After the sliding motion ceases, the equations of motion are

$$\frac{d^2\theta}{dt^2} = -\frac{F'a}{Mk^2} \dots (1), \quad \frac{d^2x}{dt^2} = \frac{F'}{M} \dots (2), \quad x = a\theta \dots (3).$$

$$\text{By (1) (2) } \frac{k^2}{a}\frac{d^2\theta}{dt^2} + \frac{d^2x}{dt^2} = 0, \quad \therefore \text{ by (3) } \frac{d^2\theta}{dt^2} = 0;$$

$$\therefore\ F' = 0 \text{ and } \frac{d\theta}{dt} = \text{constant} = \frac{k^2\omega}{a^2 + k^2}.$$

From this we learn that the friction has gradually reduced the angular motion of the body till the velocity of the point of contact is zero, and after that the body proceeds to move uniformly and to rotate uniformly and no friction is called into play.

Prob. 22. A rough body lies upon a rough board, and this lies upon a smooth horizontal plane, the friction between the body and board is of finite intensity (as in the last Problem): the board is projected with a given velocity, determine the motion of the body and board.

Prob. 23. A sphere is fastened by an inflexible rod to a horizontal axis fixed at two points: when the sphere revolves about the axis required the pressure on the two fixed points.

If we use the notation of Art. 438, and put $\gamma = 90^0$, $\gamma' = 90^0$ and therefore $\beta = 90^0 - \alpha$, then, the axis of rotation being the axis of z and the plane in which the centre of the sphere moves the plane of xy and the axis of x drawn vertically downwards, the moving forces $m(g + yf + x\omega^2)$, $m(y\omega^2 - xf)$, 0 acting on m parallel to the axes of x, y, z and similar forces acting on all the other particles of the system, together with the pressures of the fixed points ought to be in equilibrium at the time t. Hence

$$P \cos\alpha + P' \cos\alpha' + \Sigma . m\,(g + yf + x\omega^2) = 0,$$

$$P \sin\alpha + P' \sin\alpha' + \Sigma . m\,(y\omega^2 - xf) = 0,$$

$$-P \sin\alpha . a - P' \sin\alpha' . a' - \Sigma . m\,(gz + yzf + xz\omega^2) = 0;$$

$$P \cos\alpha . a + P' \cos\alpha' . a' + \Sigma . m\,(yz\omega^2 - xzf) = 0,$$

$$\Sigma . m\,(xy\omega^2 - x^2 f - gy - y^2 f - xy\omega^2) = 0.$$

Let $\bar{x}\,\bar{y}\,0$ be the co-ordinates to the centre of gravity; then, since every axis through the centre of a sphere is a principal axis, we have

$$\Sigma . m\,(y - \bar{y})\,z = 0,\quad \Sigma . m\,(x - \bar{x})\,z = 0,\quad \Sigma . m\,(y - \bar{y})\,(x - \bar{x}) = 0:$$

$$\therefore\ \Sigma . myz = \bar{y}\,\Sigma . mz = 0,\quad \Sigma . mxz = 0,\quad \Sigma . myx = M\bar{x}\,\bar{y}.$$

Hence the equations become

$$P \cos\alpha + P' \cos\alpha' + M\,(g + f\bar{y} + \omega^2\bar{x}) = 0,$$

$$P \sin\alpha + P' \sin\alpha' + M\,(\omega^2\bar{y} - f\bar{x}) = 0;$$

$$Pa \sin\alpha + P'a' \sin\alpha' = 0,\quad Pa \cos\alpha + P'a' \cos\alpha' = 0,\quad Mfk_{\prime}^{2} + Mg\bar{y} = 0.$$

From which P, P', a, a' may be found.

Prob. 24. If a body revolve round an axis by the action of a constant force in a direction always perpendicular to the plane passing through the axis and the centre of gravity of the body, determine how the point of application of this force must vary with the *time*, so that there may be no pressure on the axis, except in the plane to which the direction of the force is perpendicular.

Prob. 25. A hemisphere oscillates about a horizontal axis, which coincides with a diameter of the base; shew that if the base be at first vertical, the ratio of the greatest pressure on the axis to the weight of the hemisphere = 109 ÷ 64.

Prob. 26. A sphere, when acted on separately by three forces, revolves round three diameters inclined at the same angle to each other and with the same angular velocity, determine the angular velocity and the new axis of rotation when the three forces are applied at the same instant.

Prob. 27. A sphere attracted to a given centre of force varying as the distance is projected with a given velocity along a plane passing through that centre, friction being such as to destroy all sliding: prove that the path will be an ellipse, and find the velocity that the ellipse may be a circle.

Prob. 28. A cone of given form, and supported at G its centre of gravity, has a motion communicated to it round an axis through G perpendicular to the line joining G with a point in the circumference of the base, and in a plane passing through this point and the axis of the cone: determine the position of the *invariable plane;* and explain the motion of the cone's vertex.

Prob. 29. Explain how the rotation of a hoop preserves it from falling.

Prob. 30. A solid of revolution moveable about its centre of gravity G, which is fixed and is the origin, and having its axis inclined to the axis of z at an angle ϕ, has an angular motion impressed upon it about a line between these two axes, and inclined to the former at an angle θ, such that $k^2 \tan\phi = k'^2 \tan\theta$, where k and k' are the radii of gyration about its axis and a line perpendicular to the axis through G: prove that the axis of the solid will constantly preserve the same inclination to the axis of z, and will revolve uniformly about it; and the solid will at the same time revolve uniformly about its own axis, which is in motion.

Prob. 31. If the Moon moved in the ecliptic, shew that the force of the Earth to produce rotation about her axis perpendicular to that plane would nearly $= \frac{3\mu \sin 2\theta}{2r^3} \frac{A-B}{C}$; μ, r being the Earth's mass and distance from the Moon, A, B, C the principal moments of inertia of the lunar spheroid, and θ the angular distance, at the Moon's centre, of the Earth from one of the principal axes which are in the ecliptic.

We shall now give some problems in which the action of impulsive forces is considered.

Prob. 32. Two inelastic balls impinge upon each other, their motions being in the same straight line: required their velocity after impact.

Let M, M' be the masses of the balls: V, V' their velocities at the instant the contact commences; v, v' their velocities after the impulse ceases: P the momentum which measures their mutual pressure during the collision. Then by Art. 474. for the motion of the centre of gravity of M

$$MV - P - Mv = 0 \ldots\ldots\ldots\ldots\ldots (1),$$

for the ball M'

$$M'V' + P - M'v' = 0 \ldots\ldots\ldots\ldots\ldots (2).$$

Here we have two equations with three unknown quantities P, v, v'. The third equation is the condition that the particles in contact move with the same velocity the instant the compression ceases.

$$\text{Hence } v - v' = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Eliminating P from (1) (2)

$$Mv + M'v' = MV + M'V';$$

$$\therefore \text{ by (3), } v = v' = \frac{MV + M'V'}{M + M'},$$

and the balls will therefore each move with a velocity $\frac{MV + M'V'}{M + M'}$ and remain in contact.

$$\text{Also } P = M(V - v) = \frac{MM'}{M + M'}(V - V') \text{ by (1)}.$$

Prob. 33. Suppose the two balls are imperfectly elastic.

In this case the Problem divides itself into two parts: first the motion during compression, and secondly the motion during the restitution of the figure of the bodies. Now bodies differ in their elasticity owing to their physical constitution; but the law to which we are led by experiment is this, that for the same material the momentum gained by the restitution bears a constant ratio to the momentum lost by the collision: this ratio we write e, and is called the *elasticity* of the material of which the bodies are made (Art. 220).

By the previous Problem the bodies are moving with a velocity $\dfrac{MV + M'V'}{M + M'}$ at the instant the restitution of figure commences: and the momentum which measures their mutual action $= P = \dfrac{MM'}{M + M'}(V - V')$: therefore the mutual pressure during restitution $= Pe = \dfrac{MM'e}{M + M'}(V - V')$:

Let u, u' be the velocities after the restitution ceases: then

$$M\frac{MV+M'V'}{M+M'} - P' - Mu = 0, \quad M'\frac{MV+M'V'}{M+M'} + P' - M'u' = 0;$$

$$\therefore\ u = \frac{MV + M'V'}{M + M'} - e\frac{M'(V - V')}{M + M'},$$

$$u' = \frac{MV + M'V'}{M + M'} + e\frac{M(V - V')}{M + M'}.$$

Cor. If the bodies are considered perfectly elastic, then $e = 1$.

$$u = V - \frac{2M'}{M + M'}(V - V'), \quad u' = V' + \frac{2M}{M + M'}(V - V').$$

Prob. 34. A smooth but imperfectly elastic ball moves in a horizontal plane and impinges on a hard vertical plane obliquely, its direction making an angle α with the normal to the plane: find the velocity and direction of the motion after impact.

Let V be the velocity before impact; v the velocity and θ the direction of motion at the instant compression ceases; u the velocity and ϕ the direction of motion after impact: P the mutual pressure during compression, Pe the pressure during restitution, M the mass of the ball: then when compression ceases

$$MV\cos\alpha - Mv\cos\theta = 0\ldots(1),\quad MV\sin\alpha - P - Mv\sin\theta = 0\ldots(2),$$

but we have three unknown quantities P, v, θ: a third equation is given by the condition that the plane is immoveable, and hence the velocity of the body perpendicular to the plane is zero at the instant compression ceases, therefore

$$v\sin\theta = 0\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

when restitution of figure ceases,

$$Mv\cos\theta - Mu\cos\phi = \ldots(4),\quad Pe - Mu\sin\phi = 0\ldots(5).$$

By (1) (3) (4) $u\cos\phi = V\cos\alpha$, by (2) (3) (5), $u\sin\phi = eV\sin\alpha$;

$$\therefore\ \tan\phi = e\tan\alpha,\ \text{and}\ u = V\sqrt{\cos^2\alpha + e^2\sin^2\alpha},\ = V\cos\alpha \div \cos\phi,$$

which determine the direction and the velocity after impact.

PROB. 35. Two imperfectly elastic and smooth balls impinge upon each other, the motion of their centre taking place in the same plane: required their velocities after impact.

Since the balls are perfectly smooth there will be no rotatory motion produced by the impulse. We must first consider the motion till the compression ceases. Let P be the mutual pressure acting in the common normal at the points in contact; V, V' the velocities at the commencement of the contact: $\alpha\alpha'$ the angles their directions make with the line passing through their centres when the contact takes place; v, v' the velocities of the balls at the instant the compression ceases: θ, θ' the angles their directions make with the axes.

Therefore

$$MV\cos\alpha - P - Mv\cos\theta = 0\ldots(1),\quad MV\sin\alpha - Mv\sin\theta = 0\ldots(2),$$

$$M'V'\cos\alpha + P - M'v'\cos\theta' = 0\ldots(3),\quad M'V'\sin\alpha' - M'v'\sin\theta' = 0\ldots(4);$$

and since the points in contact move with the same velocities in the direction of the normal at the instant the compression ceases, then

$$v\cos\theta - v'\cos\theta' = 0\ldots\ldots\ldots\ldots\ldots(5).$$

Again, during the restitution of figure the mutual pressure $= Pe$: and if u and u' be the velocities after the restitution of figure is complete, and ϕ and ϕ' the angles of the directions of motion, the equations of motion are

$$Mv\cos\theta - Pe - Mu\cos\phi = 0 \ldots (6), \quad Mv\sin\theta - Mu\sin\phi = 0 \ldots (7);$$

$$M'v'\cos\theta' + Pe - M'u'\cos\phi' = 0 \ldots (8), \quad M'v'\sin\theta' - M'u'\sin\phi' = 0 \ldots (9).$$

In these nine equations are involved nine unknown quantities $P, v, v', \theta, \theta', u, u', \phi, \phi'$: we have to determine u, u', ϕ, ϕ'.

By (1) (2) (5) $(M+M')\,v\cos\theta = MV\cos\alpha + M'V'\cos\alpha'$;

eliminating e by (1) (6) $(M+M')\,u\cos\phi$

$$= (M+M')(1+e)\,v\cos\theta - (M+M')\,V\cos\alpha$$

$$= M'V'\cos\alpha' - M'V\cos\alpha + e\,(MV\cos\alpha + M'V'\cos\alpha'),$$

by (2) (7) $u\sin\phi = V\sin\alpha$;

from which u and ϕ may easily be determined: in the same way u' and ϕ' may be determined.

Prob. 36. A rough ball A is placed on a rough horizontal table, and another rough ball B lying on the table is struck in a direction not passing through the centre of gravity, but so as to cause B to strike A: find the motion after impact, the bodies being inelastic.

Prob. 37. Supposing, in the last Problem, that the friction of the Table is so slight as not altogether to prevent sliding, find the conditions that B may move through its original place of rest.

The four following Problems are intended to illustrate the action of springs in removing the shock arising from the sudden collision of bodies.

Prob. 38. A ball A moves along a smooth horizontal plane with a velocity V, and sets in motion another ball B, equal to A and originally at rest, by impinging upon a spring CD (fig. 112), which is fastened to B at the point D: the inertia of the spring is neglected, and we suppose the force of the spring to vary as the space through which it is compressed: required to determine the motion of the balls.

Let O be the place of A, the centre of the first ball, at the time of first contact with the spring: $OA = x$, $CD = z$ ($= b$ when the spring is not compressed), $OB = x'$: then the force exerted by the spring on the balls at the time t varies as $b - z$; let it $= c^2(b-z)$. The equations of motion are

$$\frac{d^2x}{dt^2} = -c^2(b-z) \ldots\ldots (1), \quad \frac{d^2x'}{dt^2} = c^2(b-z) \ldots\ldots (2);$$

also $x' - x = 2a + z \ldots\ldots (3)$, a the radius of the balls, three equations and three unknown quantities x, x', z.

Differentiating (3) and subtracting (1) (2) we have

$$\frac{d^2z}{dt^2} = 2c^2(b-z);$$

$$\therefore \frac{dz^2}{dt^2} = \text{constant} - 2c^2(b-z)^2 = V^2 - 2c^2(b-z)^2,$$

since the point C of the spring (having no inertia) instantly acquires the velocity (V) of the body A at the first contact;

$$\therefore z = b - \frac{V}{c\sqrt{2}} \sin(c\sqrt{2}\,t + C') = b - \frac{V}{c\sqrt{2}} \sin c\sqrt{2}\,t \ldots (4).$$

This shews that the greatest compression of the spring is equal to $\frac{V}{c\sqrt{2}}$, and that the time of compression $= \frac{\pi}{2c\sqrt{2}}$; after an equal duration of time the spring is restored to its original form, since z equals b when $c\sqrt{2}\,t = \pi$.

$$\text{By (1) (4)} \quad \frac{d^2x}{dt^2} = -\frac{Vc}{\sqrt{2}} \sin c\sqrt{2}t;$$

$$\therefore \frac{dx}{dt} = \text{const.} + \frac{V}{2}\cos c\sqrt{2}t = \frac{V}{2}(1 + \cos c\sqrt{2}t);$$

$$\therefore x = \frac{V}{2}\left(t + \frac{1}{c\sqrt{2}} \sin c\sqrt{2}t\right);$$

$$\therefore \text{ by (3) } x' = 2a + b + \frac{V}{2}\left(t - \frac{1}{c\sqrt{2}} \sin c\sqrt{2}t\right);$$

$$\therefore \frac{dx'}{dt} = \frac{V}{2}(1 - \cos c\sqrt{2}t).$$

From these equations we readily gather the following results. The ball A stops when $\frac{dx}{dt} = 0$, or $t = \frac{\pi}{c\sqrt{2}}$; but at this instant (as we have shewn) the spring has returned to its natural form, consequently the contact between A and the spring at this instant ceases, and A remains permanently at rest: the space through which A has moved during the action of the spring $= \frac{\pi V}{2c\sqrt{2}}$. The velocity of B is zero when the spring begins to act and is V when its action ceases, and with this velocity B henceforth moves uniformly along the plane. Hence A *gradually* imparts all its velocity to B: and the duration of time which this communication of velocity occupies is $\frac{\pi}{c\sqrt{2}}$. If the elastic force of the spring be of very great intensity, as is the case with the forces put into play by the impact of hard balls of ivory, c is very great, and the duration of collision is exceedingly short.

PROB. 39. Suppose that A and B (in the last Problem) are of the same size, but of different masses M and M', and that they move with the velocities V and V' before they come in contact: required to determine the motion.

PROB. 40. Suppose, in the last Problem, that the force exerted by the spring during the restitution of its figure is less than the force exerted during the compression in the ratio $e : 1$, but that a complete restitution of figure takes place: required to determine the motion.

PROB. 41. A heavy carriage (represented in fig. 113.) rests upon a spring B, and is also held in its place by two springs pressing at C and C': the carriage moves uniformly along a horizontal plane with a velocity V, and its four wheels (two only of which are seen in the figure) which are all of the same size suddenly impinge at the same instant on four very small and equal pointed obstacles, and move over them; the force exerted by each spring is supposed to vary as the extent of displacement of its point of contact with the carriage, and the springs are supposed to be bent into such a form that for all small displacements of the body of the carriage the resultant of their pressures always passes through the centre of gravity of the body and so prevent rotatory motion: required the motion of the centre of gravity.

We shall merely give the results with a few of the steps of the calculation.

Let the dotted lines in the figure represent the state of things at the instant of the impact: and the dark lines the state of things at a time t after the impact: a the radius of each wheel. In consequence of the elasticity of the springs (which is supposed perfect) the body of the carriage is not rigidly connected with the wheels and axle-tree, and therefore the body can produce no instantaneous effect upon the velocity when the impulse takes place. By the impact of the wheels on the obstacles the parts of the springs which are connected with the axes of the wheels and the axle-tree have their motion suddenly changed, this causes the springs to assume new forms and in that way the forces are brought into action which gradually change the motion of the body.

Let $x'y'$ be the horizontal and vertical spaces described by the point B of the axle-tree in the time t; x and y the spaces described by the centre of gravity of the body: let c^2 and e^2 be constants which depend upon the elasticity of the springs at C and C' and that at B; we neglect the downward effect of the spring B on the axle-tree but consider only the dead weight to act at B: let θ be the angle which the spoke of each wheel, which passes through the obstacle, makes with the vertical at the time t; $\theta = \alpha$ when $t = 0$: M' = mass of each wheel: ω the angular velocity of each wheel after impact.

The equations of motion are, for each wheel,

$$\frac{d^2\theta}{dt^2} = \frac{(M' + \frac{1}{4}M)\,ga\sin\theta}{M'k^2} = \frac{\sin\theta}{n^2}, \text{ suppose } \ldots\ldots (1),$$

for the motion of G

$$\frac{d^2x}{dt^2} = -c^2(x - x') \dots\dots (2), \quad \frac{d^2y}{dt^2} = -g + e^2(y' - y) \dots\dots (3),$$

and x', y', θ are connected by the equations

$$x' = a(\sin\alpha - \sin\theta) \dots\dots (4), \quad y' = a(\cos\theta - \cos\alpha) \dots\dots (5).$$

These equations are sufficient to solve the problem: but they cannot be integrated unless α and θ (and therefore the obstacles) be supposed small: we shall neglect powers higher than the second. After reducing the equations their integrals will be found to be of the forms

$$x = A\sin(ct + B) + a\alpha + h\varepsilon^{-\frac{t}{n}} + k\varepsilon^{\frac{t}{n}},$$

$$y = C\sin(et + D) - l - m\varepsilon^{-\frac{2t}{n}} - p\varepsilon^{\frac{2t}{n}},$$

A, B, C, D being arbitrary constants to be determined by the initial circumstances, and h, k, l, m, p being written for known quantities. After determining these it will be found that when $t = 0$, $\frac{dy}{dx} = 0$ and therefore the original rectilinear path of G is a tangent at the first point to the curve described by G; also the values of the constants will shew that the velocity is V at first, and gradually decreases: hence there is no *jerk* in the body of the carriage.

We might in the same way obtain the circumstances after the wheels again come to the horizontal plane.

Prob. 42. A rectangular parallelopiped slides down a smooth inclined plane and meets a fixed obstacle: determine the impulse and the subsequent motion.

Prob. 43. A beam is projected in any manner along a smooth horizontal plane and impinges upon a fixed obstacle: determine the impulse.

Prob. 44. A beam is fixed at one extremity, what vertical force applied instantaneously at the other will throw it exactly vertical?

Prob. 45. A rectangular parallelopiped revolves about one of its edges, which rests in a horizontal groove, and impinges on a fixed line parallel to the groove and in the same horizontal plane with it: find the angle through which the parallelopiped must fall so that it may be just on the point of revolving about the fixed line as a new axis, all *sliding* being prevented by friction.

Prob. 46. A beam is placed with one end against a smooth vertical wall and the other on a smooth horizontal plane so as to move in a vertical plane when left to the action of gravity; the horizontal plane does not extend to the wall, but is terminated by a straight edge parallel to the wall: find the distance of this edge from the wall that the beam may just be prevented from revolving about the edge and finally falling beneath the horizontal plane.

Prob. 47. At what point must a given uniform circular body be struck by a force perpendicular to its plane, that in the first instant of the body's motion one extremity of a given diameter may remain at rest?

Prob. 48. A beam falls from a vertical position by revolving about one extremity which rests on a rough horizontal plane, and impinges on a vertical post: determine the magnitude and direction of the impulse on the post and on the horizontal plane at the immoveable extremity of the beam.

Prob. 49. In the last Problem determine the initial circumstances that the beam may just fall over the post.

Prob. 50. If a rough ball be projected against a rough beam on a smooth horizontal plane determine the centre of spontaneous rotation.

Prob. 51. An elastic beam falls upon a horizontal fixed line: determine the motion.

Prob. 52. A beam, moveable about a fixed horizontal axis at a given altitude above a horizontal plane, falls through a given angle: determine the point at which a given sphere should be opposed to its impact, that it may be projected to the greatest possible distance on the horizontal plane, the beam being in its vertical position at the instant of impact.

Prob. 53. A hoop rolling down an inclined plane suddenly comes in contact with a horizontal plane; find the change in angular velocity.

Prob. 54. In lowering a bale of goods from the higher story of a warehouse by means of a given crane, the whole weight of the bale is allowed to wind off the rope freely from the axle, and when the bale is half way down, the handle of the crane suddenly flies off; determine the motion.

Prob. 55. Explain the use of fly-wheels in machinery, and if a fly-wheel of given dimensions and weight move with a given angular velocity what force applied perpendicularly at a given point of one of the spokes of the wheel will instantaneously destroy the motion.

Prob. 56. A perfectly flexible chain has one end fixed to a peg, which is at the extremity and highest point of a quadrant of a circle of which the plane is vertical, and all the chain is collected at that point; it will just cover the quadrant, and being suffered to descend freely, it is required to find the stress upon the peg at the end of the motion.

Prob. 57. A groove is cut in a horizontal table in the form of a regular hexagon and an inelastic ball is projected with a given velocity along one of its sides, find the velocity with which it will successively describe each of the other sides of the figure.

Prob. 58. A perfectly elastic solid of revolution, turning about its axis at a given rate, impinges on a hard smooth plane: if before impact the centre of gravity move perpendicular to the plane with a velocity V, determine the motion of rotation after impact, and prove that the centre of gravity will move in the same direction with a velocity $\frac{p^2 - k^2}{p^2 + k^2} V$, where p is the perpendicular from the centre of gravity on the normal at the point of impact, and k is the radius of gyration round an axis through the centre of gravity perpendicular to the axis of the solid.

Prob. 59. A solid sphere is placed in a hollow sphere, which rests on a smooth horizontal plane; determine the small oscillations, when they are slightly disturbed from the state of rest.

Prob. 60. Prove, by means of the principle of least action, that the orbit a body describes about a centre of force varying inversely as the square of the distance is a conic section.

Prob. 61. Prove the laws of reflexion and refraction of light by the principle of least action, on the supposition that light consists of luminous particles moving uniformly in the same homogeneous medium, but with different velocities in different media.

Prob. 62. A bullet is fired into a thick board hanging from a fixed horizontal axis about which it is capable of revolving; the board has a sheet of iron on its back to prevent the bullet from passing through: a ribbon is fastened to the bottom of the board and runs through a ring touching the bottom of the board in its position of rest: shew how to compare the velocities of bullets by observing the lengths of ribbon drawn out by the motion of the board.

This is Robins' Ballistic Pendulum.

Prob. 63. The weights suspended from a wheel and axle are in motion, the wheel and axle move about a fixed axis very nearly fitting into the cylindrical aperture concentric with the axle, so as to suffer only one point to be in contact: determine the position of this point when friction is considered and when it is neglected.