The mathematical principles of mechanical philosophy, and their application to the theory of universal gravitation

Pratt, John Henry

Cambridge, M.DCCC.XXXVI. [1836]

ETH-Bibliothek Zürich

Shelf Mark: Rar 32687

Persistent Link: https://doi.org/10.3931/e-rara-72776

Chapter III.

CHAPTER III.

MOTION OF TWO MATERIAL PARTICLES ATTRACTING EACH OTHER.

PROP. *Two material particles attract each other with forces varying inversely as the square of their distance and directly as the mass of the attracting body: required to determine the motion of their centre of gravity.*

264. Let M and m be the masses of the two particles: r their distance at the time t: then, if the unit of attraction be the attraction of a unit of mass at a unit of distance, the accelerating force produced in M by the attraction of $m = \frac{m}{r^2}$;

and that produced in m by M's attraction $= \frac{M}{r^2}$.

Let xyz be co-ordinates to M at time t,

$x'y'z'$ .... m ....

Then resolving the attractions parallel to the axes, and attending to the directions in which the resolved parts act, the equations of motion of M are

$$\frac{d^2x}{dt^2} = -\frac{m(x-x')}{r^3}, \quad \frac{d^2y}{dt^2} = -\frac{m(y-y')}{r^3}, \quad \frac{d^2z}{dt^2} = -\frac{m(z-z')}{r^3}$$

and those of m are

$$\frac{d^2x'}{dt^2} = \frac{M(x-x')}{r^3}, \quad \frac{d^2y'}{dt^2} = \frac{M(y-y')}{r^3}, \quad \frac{d^2z'}{dt^2} = \frac{M(z-z')}{r^3}.$$

Multiply the first three equations by M and the last three by m, and add the first, second, and third of the first set to the first, second, and third of the second set respectively;

$$\left.\begin{array}{l} \therefore\ M\dfrac{d^2x}{dt^2}+m\dfrac{d^2x'}{dt^2}=0,\quad M\dfrac{d^2y}{dt^2}+m\dfrac{d^2y'}{dt^2}=0,\\ \qquad\qquad M\dfrac{d^2z}{dt^2}+m\dfrac{d^2z'}{dt^2}=0.\end{array}\right\}\ldots\ldots(1).$$

Let $\bar{x}\,\bar{y}\,\bar{z}$ be the co-ordinates to the centre of gravity of the two bodies at the time t: then

$$(M+m)\,\bar{x}=Mx+mx',\quad (M+m)\,\bar{y}=My+my',$$

$$(M+m)\,\bar{z}=Mz+mz'.$$

Differentiating these twice with respect to t and making use of equations (1), we obtain

$$\frac{d^2\bar{x}}{dt^2}=0,\quad \frac{d^2\bar{y}}{dt^2}=0,\quad \frac{d^2\bar{z}}{dt^2}=0\ldots\ldots\ldots(2);$$

$$\therefore\ \frac{d\bar{x}}{dt}=a,\quad \frac{d\bar{y}}{dt}=b,\quad \frac{d\bar{z}}{dt}=c,$$

a, b, c being constants to be determined by the initial circumstances of the motion of the bodies.

Hence the velocity of the centre of gravity $=\sqrt{a^2+b^2+c^2}$, (Art. 210. Cor.) and is therefore uniform.

$$\text{Also } \frac{d\bar{x}}{d\bar{z}}=\frac{a}{c},\quad \frac{d\bar{y}}{d\bar{z}}=\frac{b}{c};$$

$$\therefore\ \bar{x}=\frac{a}{c}\bar{z}+a',\quad \bar{y}=\frac{b}{c}\bar{z}+b',$$

a', b' being constants to be determined as before.

These are the equations to the path of the centre of gravity; and, since they are the equations to a straight line in space, they prove that that point will move in a straight line.

If a, b, c each $=0$, then the expression for the velocity of the centre of gravity vanishes: and the general conclusion is, That *the centre of gravity of the two bodies will either remain at rest during the motion of the bodies, or move* *uniformly in a straight line.* Which of these will be the case is determined by the initial circumstances of the motion of the bodies.

Prop. *To determine the orbits the bodies describe about each other, and about their centre of gravity.*

265. Let us subtract the equations of motion for m from those of M respectively, and we obtain

$$\frac{d^2(x-x')}{dt^2} = -\frac{(M+m)(x-x')}{r^3}, \quad \frac{d^2(y-y')}{dt^2} = -\frac{(M+m)(y-y')}{r^3},$$

$$\frac{d^2(z-z')}{dt^2} = -\frac{(M+m)(z-z')}{r^3}.$$

These are the equations we should obtain by supposing either of the bodies at rest, and the force acting on the other to be the *sum* of the masses divided by the square of the distance.

Hence (Art. 252) each will describe relatively to the other a conic section, the nature of the path being determined by the circumstances of projection of the bodies.

266. To determine their paths about their centre of gravity, let $r_{,}$ and r' be the distances of M and m from that point at the time t: then

$$r_{,} = \frac{m}{M+m} r, \quad r' = \frac{M}{M+m} r.$$

Also, if P and Q be the two particles (fig. 80), G their centre of gravity,

$$\frac{PN}{PQ} = \frac{PN'}{PG}, \quad \therefore \frac{x-x'}{r} = \frac{x-\bar{x}}{r_{,}}:$$

and in the same way $\dfrac{y-y'}{r} = \dfrac{y-\bar{y}}{r_{,}}$ and $\dfrac{z-z'}{r} = \dfrac{z-\bar{z}}{r_{,}}$:

Now subtract equations (2) of Art. 264. from the equations of motion of m in that Article respectively:

$$\therefore \frac{d^2(x-\bar{x})}{dt^2} = -\frac{m(x-x')}{r^3} = -\frac{m^3}{(M+m)^2}\frac{x-\bar{x}}{r_{,}^3}$$

$$\frac{d^2(y-\bar{y})}{dt^2} = -\frac{m^3}{(M+m)^2}\frac{y-\bar{y}}{r_{,}^3} \text{ and } \frac{d^2(z-\bar{z})}{dt^2} = -\frac{m^3}{(M+m)^2}\frac{z-\bar{z}}{r_{,}^3}.$$

These are the equations of motion of M relatively to the centre of gravity of M and m, which as we have seen is at rest, or is moving uniformly in a straight line. They prove that the path about the centre of gravity is such as would be described about a force $\frac{m^3}{(M+m)^2}\cdot\frac{1}{r_{,}^2}$ residing in that point.

Hence the orbits of M and m relatively to the centre of gravity are conic sections, their nature and magnitude being determined by the circumstances of projection *relatively* to the centre of gravity of M and m.

PROP. *To compare the relative orbits of* M *and* m *about their centre of gravity.*

267. Let v, v' be the absolute velocities of projection of M and m: $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ the angles the directions of these velocities make with the axes.

V and V' the relative vels. of project. about centre of gravity,

R and R' the initial distances from the centre of gravity,

δ and δ' the relative angles of projection,

a and a' the semi-axes major of the orbits,

e and e' the eccentricities of the orbits,

μ and μ' the absolute forces.

Then by equations (1) (2) of Art. 252,

$$\frac{1-e^2}{1-e'^2} = \frac{2\mu - V^2R}{2\mu' - V'^2R'}\,\frac{RV^2\sin^2\delta}{R'V'^2\sin^2\delta'}\,\frac{\mu'^2}{\mu^2},$$

$$\text{and } \frac{a(1\sim e^2)}{a'(1\sim e'^2)} = \frac{V^2R^2\sin^2\delta}{V'^2R'^2\sin^2\delta'}\,\frac{\mu'}{\mu}.$$

$$\text{Also } \frac{R}{R'} = \frac{m}{M}, \text{ and } \frac{\mu'}{\mu} = \frac{M^3}{m^3} \text{ by Art. 266.}$$

To find V, V', δ, δ' we proceed as follows:

The velocities of the centre of gravity parallel to the axes are at first and therefore during the motion respectively (Art. 264.)

$$\frac{Mv\cos\alpha + mv'\cos\alpha'}{M+m}, \quad \frac{Mv\cos\beta + mv'\cos\beta'}{M+m}, \quad \frac{Mv\cos\gamma + mv'\cos\gamma'}{M+m}.$$

Also the *absolute* velocities of projection of M parallel to the axes are $v\cos\alpha$, $v\cos\beta$, $v\cos\gamma$: and therefore the *relative* velocities of projection of M about the centre of gravity parallel to the axes are

$$\frac{m(v\cos\alpha - v'\cos\alpha')}{M+m}, \quad \frac{m(v\cos\beta - v'\cos\beta')}{M+m}, \quad \frac{m(v\cos\gamma - v'\cos\gamma')}{M+m}.$$

Adding the squares of these, (Art. 210. Cor.) the square of the relative velocity of M about the centre of gravity (V^2) =

$$\frac{m^2}{(M+m)^2}\{(v\cos\alpha - v'\cos\alpha')^2 + (v\cos\beta - v'\cos\beta')^2 + (v\cos\gamma - v'\cos\gamma')^2\}$$

$$= \frac{m^2}{(M+m)^2}(v^2 + v'^2 - 2vv'\cos A),$$

where A is the angle between the directions of projection of M and m: and therefore determined by the equation

$$\cos A = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'.$$

Similarly $V'^2 = \dfrac{M^2}{(M+m)^2}(v^2 + v'^2 - 2vv'\cos A)$.

Let the line joining M and m at the commencement of the motion be the axis of x: then the cosine of the angle which the direction of V makes with the distance of projection (which coincides with the axis of x), or $\cos\delta$, equals the relative velocity parallel to the axis of x divided by the whole relative velocity (V) =

$$\frac{m}{M+m} \cdot \frac{v\cos\alpha - v'\cos\alpha'}{V}.$$

Similarly $\cos \delta' = \dfrac{M}{M+m} \cdot \dfrac{v' \cos a' - v \cos a}{V'}$.

Substituting these in the expression given above for $\dfrac{1-e^2}{1-e'^2}$ we find

$$\frac{1-e^2}{1-e'^2} = 1; \quad \therefore e = e',$$

or the orbits are similar to each other.

$$\text{Also} \quad \frac{a}{a'} = \frac{m^4}{M^4} \cdot \frac{M^3}{m^3} = \frac{m}{M},$$

or the linear dimensions of the orbits of M and m are in the ratio of m to M.

268. Cor. 1. It follows from this that the perturbation of the Sun by any planet is very small, because his mass is so much the greater of the two masses.

In the same way it will be shewn that the combined effect of the heavenly bodies in moving the Sun is very slight; and therefore the error in Kepler's Laws, anticipated in Art. 261, owing to the supposed immobility of the Sun, is not very great. Thus far, then, we are confirmed in our hypothesis of Universal Gravitation.

269. Cor. 2. We have seen (Art. 257.) that if μ be the absolute force of a centre of which the law is that of the inverse square, and a the semi-axis major of the orbit described, the periodic time

$$(T) = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}} = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{M+m}}. \quad \text{(Art. 265.)}$$

M and m being the masses of the Sun and a planet.

Let m' be the mass of another planet: and a' the semi-axis major of its orbit, T' its period;

$$\therefore T' = \frac{2\pi a'^{\frac{3}{2}}}{\sqrt{M+m'}}, \quad \text{and} \therefore \frac{T^2}{T'^2} = \frac{a^3}{a'^3} \frac{M+m'}{M+m}.$$

This shews that Kepler's Third Law would not be true even if we suppose that the planets do not attract each other, unless their masses were equal to each other. The deviation, however, from the truth is extremely small.

270. The investigations in Arts. 252, 265, shew us that if our law of gravitation be true, the only orbits which a heavenly body will describe, supposed to be acted on only by the Sun, are an ellipse, a parabola, or a hyperbola with the Sun's centre in the focus.

The manner in which the magnitude and position of the orbit of a heavenly body is determined by actual observation will be found in Works on Plane Astronomy. We shall here briefly explain the process. There are six quantities which determine the position and magnitude of an elliptic or hyperbolic orbit, and the place of the body in its orbit: these are called the *elements* of the body's orbit, and are (1) the inclination of the orbit to the ecliptic, and (2) the longitude of the ascending node, these determine the *position of the plane of the orbit* in space: next (3) the longitude of the perihelion, (or point of the orbit nearest the Sun), which determines the *position of the orbit itself:* then (4) the mean distance, and (5) eccentricity, which determine the *magnitude* of the orbit, and lastly (6) the epoch, or the time of the planet's being in the perihelion, this determines the *position of the body itself* in its orbit.

The elements of a parabolic orbit are five in number, being the same as the above, if we replace the mean distance and eccentricity by the perihelion distance.

The elements of a circular orbit are only four in number, the eccentricity and longitude of the perihelion not being required.

In order to determine the numerical values of the elements of any heavenly body (supposed to move in a conic section with the Sun in the focus) two Trigonometrical equations* are deduced connecting the elements with the right ascension and

* For a parabolic and circular orbit see Maddy's Plane Astronomy, Chap. XIV. Woodhouse's Plane Astronomy, Chap. XXIV.

But for other orbits the reader may consult the Work of Lalande; Gauss's Theoria Motus Corporum Cœlestium; the Mécanique Céleste, Vol. I.; Lagrange's Mec. Analytique; Pontécoulant's Théorie Anal. du Système du Monde, and Mr Lubbock's Mathematical Tracts and various Papers in the Transactions of the Philosophical and Astronomical Societies.

declination of the body and the distance of the Earth from the Sun.

Since there are five or six quantities to be determined three independent observations must be made on the declination and right ascension of the body: when these are substituted successively in the two equations mentioned above we shall have six equations involving the elements: by means of which we shall be able to calculate the magnitude and position of the orbit.

271. By methods of this nature Kepler discovered his three planetary Laws.

Also Astronomers have in this way proved, that comets move in orbits most of which are parabolic, some elliptic, and others probably hyperbolic. In consequence of the vast distances to which comets penetrate into space, they are invisible except when near the Sun. During their appearance numerous observations are made, in order that the elements may be determined with the greatest possible accuracy. The calculations for parabolic motion are less laborious than for elliptic or hyperbolic motion. The elements are therefore first calculated on the supposition that the orbit is a parabola. If the elements thus calculated shew that the comet has passed so near any of the planets as to have experienced a sensible perturbation the elements must be corrected in a manner to be explained hereafter.

If a parabola will not coincide with the orbit calculations must be made for an ellipse or hyperbola. It is thus found that "three or four comets describe very long ellipses: and nearly all the others that have been observed are found to move in curves which cannot be distinguished from parabolas. There is reason to think that two or three comets move in hyperbolas." (Airy's *Gravitation*, page 15.)

272. Our calculations have been hitherto respecting the nature of the orbits described. We now proceed to deduce formulæ for determining the time that the body occupies in moving through a given angle; and conversely the angle described in a given time: by the former we know the time of the body being at a given place, and by the latter we know the place of the body at a given time.

PROP. *To find the time of motion of a planet or comet through any portion of an elliptic orbit, the Sun's centre being in the focus.*

273. Let θ and ϖ be the longitudes of the body and the perihelion, that is, the point of the orbit nearest the Sun: a the semi-axis major of the orbit: e the eccentricity: μ the sum of the masses of the Sun and the body (Art. 265): then the equation to the orbit is

$$\frac{1}{r} = \frac{1 + e\cos(\theta - \varpi)}{a(1 - e^2)}. \quad \text{Also } \frac{dt}{d\theta} = \frac{r^2}{h}, \text{ Art. 242.}$$

Now h must be determined in terms of the quantities above given, since the orbit to be described is known and not the original circumstances of projection. The following method, which we here apply to the ellipse, will answer our purpose in every case. By Art. 243, $h = vp$ at every point of a central orbit; v being the velocity and p the perpendicular from the centre of force on the tangent at that point: also by Art. 245, the velocity is that due to one-fourth the chord of curvature through the centre of force;

$$\therefore v^2 = \frac{2\mu}{r^2} \cdot \tfrac{1}{2} p \frac{dr}{dp}: \text{ but } p^2 = \frac{b^2 r}{2a - r} \text{ from the focus;}$$

$$\therefore h = vp = \sqrt{\frac{\mu}{r^2} p^3 \frac{dr}{dp}} = \sqrt{\frac{\mu b^2}{a}} = \sqrt{\mu a(1 - e^2)}.$$

Then the time of moving from the perihelion through the angle $\theta - \varpi =$

$$t = \int_{\varpi}^{\theta} \frac{r^2 d\theta}{h} = \frac{a^{\frac{3}{2}}(1 - e^2)^{\frac{3}{2}}}{\sqrt{\mu}} \int_{\varpi}^{\theta} \frac{d\theta}{\{1 + e\cos(\theta - \varpi)\}^2}$$

$$= \frac{a^{\frac{3}{2}}(1 - e^2)^{\frac{3}{2}}}{\sqrt{\mu}} \int_{\varpi}^{\theta} \frac{d\theta}{\{(1 + e)\cos^2\frac{1}{2}(\theta - \varpi) + (1 - e)\sin^2\frac{1}{2}(\theta - \varpi)\}^2}$$

$$= \frac{2a^{\frac{3}{2}}(1 - e^2)^{\frac{3}{2}}}{\sqrt{\mu}} \int_{\varpi}^{\theta} \frac{\sec^2\frac{1}{2}(\theta - \varpi)\dfrac{d\tan\frac{1}{2}(\theta - \varpi)}{d\theta}\, d\theta}{\{(1 + e) + (1 - e)\tan^2\frac{1}{2}(\theta - \varpi)\}^2}.$$

To simplify this let

$$\tan \tfrac{1}{2}(\theta - \varpi) = \sqrt{\frac{1+e}{1-e}} \tan \frac{u}{2} \ldots\ldots\ldots (1);$$

$$\therefore\ t = \frac{2a^{\frac{3}{2}}(1-e^2)^{\frac{3}{2}}}{\sqrt{\mu}} \int_0^u \frac{\left(1 + \frac{1+e}{1-e}\tan^2\frac{u}{2}\right)}{(1+e)^2 \sec^4 \frac{u}{2}} \frac{d}{du}\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2}\right) du$$

$$= \frac{a^{\frac{3}{2}}}{\sqrt{\mu}} \int_0^u \left\{(1-e)\cos^2\frac{u}{2} + (1+e)\sin^2\frac{u}{2}\right\} du$$

$$= \frac{a^{\frac{3}{2}}}{\sqrt{\mu}} \int_0^u (1 - e\cos u)\, du$$

$$= \frac{a^{\frac{3}{2}}}{\sqrt{\mu}}(u - e\sin u), \text{ let } \frac{a^{\frac{3}{2}}}{\sqrt{\mu}} = \frac{1}{n};$$

$$\therefore\ nt = u - e\sin u \ldots\ldots\ldots (2).$$

When θ is given we calculate u by (1), and substituting in (2) we know t.

The angle $\theta - \varpi$, or the excess of the longitude of the body over the longitude of the perihelion, is called the *true anomaly:* and nt is called the *mean anomaly*, since it varies uniformly with the time and coincides with the true anomaly at the end of each revolution, as the formulæ (1) (2) shew. Also the angle u is called the *eccentric anomaly*, since it equals the angle QCA (fig. 81), as may easily be proved: P is the body, APa the ellipse, S the focus, AQa a circle on Aa.

Cor. 1. If t be not measured from the epoch of passing the perihelion, but from the time when $u = u_{,}$, then

$$t = \frac{a^{\frac{3}{2}}}{\sqrt{\mu}}\left\{(u - u_{,}) - e(\sin u - \sin u_{,})\right\}.$$

Cor. 2. Whenever u increases by 2π, θ increases by 2π, and t by $\dfrac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}}$. This, then, is the periodic time of the planet: it is remarkable that it is independent of the eccentricity of the orbit.

To solve the converse of this Proposition, that is, to find the position of a heavenly body in its elliptic orbit at any time in terms of the time and the elements of the orbit, we must effect several expansions.

PROP. *To expand the true anomaly in terms of the eccentric anomaly.*

274. By last Article $\tan \dfrac{\theta - \varpi}{2} = \sqrt{\dfrac{1+e}{1-e}} \tan \dfrac{u}{2}$.

Substituting the exponential expressions for the tangents,

$$\frac{\varepsilon^{(\theta-\varpi)\sqrt{-1}} - 1}{\varepsilon^{(\theta-\varpi)\sqrt{-1}} + 1} = m \frac{\varepsilon^{u\sqrt{-1}} - 1}{\varepsilon^{u\sqrt{-1}} + 1}, \quad \sqrt{\frac{1+e}{1-e}} = m,$$

in which ε is the base of Naperian logarithms.

$$\therefore \varepsilon^{(\theta-\varpi)\sqrt{-1}} = \frac{(m+1)\,\varepsilon^{u\sqrt{-1}} - (m-1)}{(m+1) - (m-1)\,\varepsilon^{u\sqrt{-1}}} = \varepsilon^{u\sqrt{-1}} \frac{1 - \lambda \varepsilon^{-u\sqrt{-1}}}{1 - \lambda \varepsilon^{u\sqrt{-1}}}, \quad \lambda = \frac{m-1}{m+1};$$

$$\therefore (\theta - \varpi)\sqrt{-1} = u\sqrt{-1} + \log_\varepsilon (1 - \lambda \varepsilon^{-u\sqrt{-1}}) - \log_\varepsilon (1 - \lambda \varepsilon^{u\sqrt{-1}})$$

$$= u\sqrt{-1} + \lambda (\varepsilon^{u\sqrt{-1}} - \varepsilon^{-u\sqrt{-1}}) + \frac{\lambda^2}{2} (\varepsilon^{2u\sqrt{-1}} - \varepsilon^{-2u\sqrt{-1}}) + \dots$$

$$\therefore \theta - \varpi = u + 2\lambda \sin u + \frac{2\lambda^2}{2} \sin 2u + \frac{2\lambda^3}{3} \sin 3u + \dots$$

$$\text{in which } \lambda = \frac{\sqrt{1+e} - \sqrt{1-e}}{\sqrt{1+e} + \sqrt{1-e}} = \frac{1 - \sqrt{1-e^2}}{e}.$$

PROP. *To expand the eccentric anomaly in terms of the mean anomaly.*

275. By Art. 273, $u = nt + e \sin u$.

Hence by Lagrange's Theorem, putting $nt = z$,

$$u = z + e \sin z + \frac{e^2}{1.2} \frac{d \sin^2 z}{dz} + \frac{e^3}{1.2.3} \frac{d^2 \sin^3 z}{dz^2} + \dots.$$

$$= z + e \sin z + \tfrac{1}{2} e^2 \sin 2z + \tfrac{1}{2} e^3 (2 \sin z - 3 \sin^3 z) + \dots$$

$$= nt + e \sin nt + \tfrac{1}{2} e^2 \sin 2nt + \tfrac{1}{8} e^3 (3 \sin 3nt - \sin nt) + \dots$$

PROP. *To expand sin* u, *sin* 2u,... *in terms of the mean anomaly.*

276. By Lagrange's Theorem,

$$\sin u = \sin z + e \sin z \frac{d \sin z}{dz} + \frac{e^2}{1.2} \frac{d}{dz} \left\{ \sin^2 z \frac{d \sin z}{dz} \right\} + \dots$$

$$= \sin z + e \sin z \cos z + \tfrac{1}{2} e^2 (2 \cos^2 z \sin z - \sin^3 z) + \dots$$

$$= \sin z + \tfrac{1}{2} e \sin 2z + \tfrac{1}{8} e^2 (2 \sin 3z - \sin z) + \dots$$

$$= \sin nt + \tfrac{1}{2} e \sin 2nt + \tfrac{1}{8} e^2 (3 \sin 3nt - \sin nt) + \dots$$

Again, $\sin 2u = \sin 2z + e \sin z \dfrac{d \sin 2z}{dz} + \dots$

$$= \sin 2nt + 2e \sin nt \cos 2nt + \dots$$

$$= \sin 2nt + e (\sin 3nt - \sin nt) + \dots$$

$$\sin 3u = \sin 3nt + \dots$$

and so on.

PROP. *To expand the true anomaly in terms of the mean anomaly.*

277. By Art. 274 we have

$$\theta - \varpi = u + 2\lambda \sin u + \frac{2\lambda^2}{2} \sin 2u + \frac{2\lambda^3}{3} \sin 3u + \dots$$

$$\text{where } \lambda = \frac{1 - \sqrt{1 - e^2}}{e} = \frac{e}{2} + \frac{e^3}{8} + \dots$$

Then substituting for u, $\sin u$, $\sin 2u$ ... the values obtained in the last two Articles, and retaining powers of e as far as the cube,

$$\theta - \varpi = nt + e\sin nt + \tfrac{1}{2}e^2 \sin 2nt + \tfrac{1}{8}e^3(3\sin 3nt - \sin nt) + \dots$$
$$+ 2\lambda\left\{\sin nt + \tfrac{1}{2}e\sin 2nt + \tfrac{1}{8}e^2(3\sin 3nt - \sin nt) + \dots\right\}$$
$$+ \lambda^2\left\{\sin 2nt + e(\sin 3nt - \sin nt) + \dots\dots\dots\dots\right\}$$
$$+ \tfrac{2}{3}\lambda^3 \sin 3nt + \dots\dots\dots\dots$$
$$= nt + \left(2e + \frac{e^3}{4}\right)\sin nt + \frac{5e^2}{4}\sin 2nt + \frac{13e^3}{12}\sin 3nt + \dots$$

which is true as far as terms involving e^3.

278. Cor. If the time t be not measured from the time of perihelion passage, suppose ϵ is the mean longitude of the body when $t = 0$; then the mean longitude at the time t is $nt + \epsilon$; and the mean anomaly is $nt + \epsilon - \varpi$: in this case, then,

$$\theta - \varpi = nt + \epsilon - \varpi + \left(2e + \frac{e^3}{4}\right)\sin(nt + \epsilon - \varpi)$$
$$+ \frac{5e^2}{4}\sin 2(nt + \epsilon - \varpi) + \dots$$

ϵ is called the *epoch*.

Prop. *To expand the radius vector* r *in terms of the mean anomaly.*

279. The radius vector

$$r = \frac{a(1-e^2)}{1+e\cos(\theta-\varpi)} = \frac{a(1-e^2)}{(1+e)\cos^2\frac{1}{2}(\theta-\varpi) + (1-e)\sin^2\frac{1}{2}(\theta-\varpi)}$$

$$= \frac{a(1-e^2)\sec^2\frac{1}{2}(\theta-\varpi)}{1+e+(1-e)\tan^2\frac{1}{2}(\theta-\varpi)} = \frac{a(1-e^2)\left\{1 + \dfrac{1+e}{1-e}\tan^2\dfrac{u}{2}\right\}}{(1+e)\sec^2\dfrac{u}{2}}$$

$$= a\left\{(1-e)\cos^2\frac{u}{2} + (1+e)\sin^2\frac{u}{2}\right\} = a(1 - e\cos u).$$

But $u = nt + e\sin u$; putting $nt = z$,

$$\cos u = \cos z + e\sin z\frac{d\cos z}{dz} + \frac{e^2}{1.2}\frac{d}{dz}\left\{\sin^2 z\frac{d\cos z}{dz}\right\} + \dots$$
$$= \cos z - \tfrac{1}{2}e(1 - \cos 2z) - \tfrac{1}{8}e^2(3\cos z - 3\cos 3z) + \dots$$
$$\therefore\ \frac{r}{a} = 1 + \frac{e^2}{2} - e\cos nt - \frac{e^2}{2}\cos 2nt - \frac{3e^3}{8}(\cos 3nt - \cos nt) + \dots$$

280. Cor. If t be measured as in Art. 278, then

$$r=a\left\{1+\frac{1}{2}e^2-e\cos(nt+\varepsilon-\varpi)-\frac{1}{2}e^2\cos 2(nt+\varepsilon-\varpi)-\ldots\right\}.$$

The time of describing a given portion of an elliptic hyperbolic or parabolic orbit may be found in terms of the radius vectors at the extremities of the arc and the chord of the arc. These expressions are useful in determining the elements of a heavenly body. They will be found in Maddy's *Plane Astronomy*, Chapter XIII. New Edition: and in the *Système du Monde* of M. Pontécoulant, Tom. I. Liv. II. Chap. V.

Prop. *To find the time of describing a given portion of a parabolic orbit about the Sun in the focus.*

281. We have $r^2\dfrac{d\theta}{dt}=h$: $h=\sqrt{2\mu D}$, and $r=\dfrac{D}{\cos^2\frac{1}{2}(\theta-\varpi)}$ is the equation to the parabola, θ and ϖ being the longitude of the comet and of its perihelion measured from the Sun, and D the perihelion distance;

$$\therefore\ t=\frac{D^{\frac{3}{2}}}{\sqrt{2\mu}}\int_{\varpi}^{\theta}\frac{d\theta}{\cos^4\dfrac{\theta-\varpi}{2}}$$

$$=\frac{2D^{\frac{3}{2}}}{\sqrt{2\mu}}\int_{\varpi}^{\theta}\frac{d\tan\frac{1}{2}(\theta-\varpi)}{d\theta}\left\{1+\tan^2\tfrac{1}{2}(\theta-\varpi)\right\}d\theta$$

$$=\sqrt{\frac{2}{\mu}}\,D^{\frac{3}{2}}\left\{\tan\tfrac{1}{2}(\theta-\varpi)+\tfrac{1}{3}\tan^3\tfrac{1}{2}(\theta-\varpi)\right\},$$

t being measured from the time of the perihelion passage.

By this equation it is easy to calculate the time of describing a given angle.

Prop. *To find the position of the comet in a parabolic orbit at a given time.*

282. This would require the solution of the cubic equation in the last Article. This is, however, obviated in the following manner.

$$\text{Let }\sqrt{\frac{\mu}{2D^3}}=n;$$

$$\therefore\ nt=\tan\tfrac{1}{2}(\theta-\varpi)+\tfrac{1}{3}\tan^3\tfrac{1}{2}(\theta-\varpi).$$

A Table is formed consisting of two columns: one with values of t and the other with the corresponding values of $\theta - \varpi$ calculated from this formula for an orbit in which $n = 1$. Suppose, then, that we wish to find the position of a comet in a given parabolic orbit (the mean motion in which is n) at a given time t. We must multiply t by n and look for the value of $\theta - \varpi$ opposite the value of nt in the first column. This gives the position of the comet.

PROP. *To find the place of a comet at a given time in a very eccentric elliptic orbit.*

283. By Art. 273. $\dfrac{dt}{d\theta} = \dfrac{a^{\frac{3}{2}}(1-e^2)^{\frac{3}{2}}}{\sqrt{\mu}} \dfrac{1}{\{1 + e\cos(\theta - \varpi)\}^2}$.

Let D be the perihelion distance; $\therefore D = a(1-e)$;

$$\therefore \frac{dt}{d\theta} = \frac{D^{\frac{3}{2}}(1+e)^{\frac{3}{2}}}{\sqrt{\mu}} \frac{\sec^4\frac{1}{2}(\theta-\varpi)}{\{(1+e)+(1-e)\tan^2\frac{1}{2}(\theta-\varpi)\}^2}$$

$$= \frac{D^{\frac{3}{2}}}{\sqrt{\mu(1+e)}} \sec^4\tfrac{1}{2}(\theta-\varpi)\left\{1 + \frac{1-e}{1+e}\tan^2\tfrac{1}{2}(\theta-\varpi)\right\}^{-2}.$$

Expanding in powers of $1-e$, and neglecting powers of $1-e$ higher than the first, because $e = 1$ nearly;

$$\therefore nt = \tfrac{1}{2}\left(1 - \frac{1-e}{2}\right)^{-\frac{1}{2}} \int_{\varpi}^{\theta} \sec^4\tfrac{1}{2}(\theta-\varpi)\{1-(1-e)\tan^2\tfrac{1}{2}(\theta-\varpi)\}\,d\theta$$

$$= \int_{\varpi}^{\theta} \frac{d\tan\frac{1}{2}(\theta-\varpi)}{d\theta}\{1+\tan^2\tfrac{1}{2}(\theta-\varpi)$$

$$+ (1-e)\left[\tfrac{1}{4} - \tfrac{3}{4}\tan^2\tfrac{1}{2}(\theta-\varpi) - \tan^4\tfrac{1}{2}(\theta-\varpi)\right]\}\,d\theta;$$

$$\therefore nt = \tan\tfrac{1}{2}(\theta-\varpi) + \tfrac{1}{3}\tan^3\tfrac{1}{2}(\theta-\varpi)$$

$$+ (1-e)\left\{\tfrac{1}{4}\tan\tfrac{1}{2}(\theta-\varpi) - \tfrac{1}{4}\tan^3\tfrac{1}{2}(\theta-\varpi) - \tfrac{1}{5}\tan^5\tfrac{1}{2}(\theta-\varpi)\right\}.$$

The following is a convenient method for calculating the value of $\theta - \varpi$ for a given value of t.

Suppose $\theta' - \varpi$ is the true anomaly of a comet at the time t moving in a parabolic orbit of which D is the perihelion distance; then by Art. 282.

$$nt = \tan \tfrac{1}{2}(\theta' - \varpi) + \tfrac{1}{3}\tan^3 \tfrac{1}{2}(\theta' - \varpi).$$

Let $\theta - \varpi = \theta' - \varpi + x$: then putting this for $\theta - \varpi$ in the first expression for nt, and neglecting the squares and products of x and e, we have by Taylor's Theorem

$$nt = \tan \tfrac{1}{2}(\theta' - \varpi) + \tfrac{1}{3}\tan^3 \tfrac{1}{2}(\theta' - \varpi)$$

$$+ \frac{x}{2}\sec^4 \frac{\theta' - \varpi}{2} + \frac{1-e}{4}\tan \tfrac{1}{2}(\theta' - \varpi)\left\{1 - \tan^2 \tfrac{1}{2}(\theta' - \varpi)\right.$$

$$\left. - \tfrac{4}{5}\tan^4 \tfrac{1}{2}(\theta' - \varpi)\right\},$$

and eliminating nt from these last two equations

$$x = \tfrac{1}{10}(1-e)\tan \tfrac{1}{2}(\theta' - \varpi)\left\{4 - 3\cos^2 \tfrac{1}{2}(\theta' - \varpi) - 6\cos^4 \tfrac{1}{2}(\theta' - \varpi)\right\}.$$

A third column must now be added to the Table mentioned in Art. 282. consisting of values of $\dfrac{x}{1-e}$ for the corresponding values of t and $\theta - \varpi$. When this is constructed the manner of using it is as follows. Suppose $\sqrt{\dfrac{\mu}{2D^3}} = n$ in our orbit: then in the first column look for the time nt; and take the corresponding values of $\theta - \varpi$ and $\dfrac{x}{1-e}$: multiply the latter by $1 - e$, which will depend upon the form of the orbit, and then the true anomaly at the time t will be this quantity added to the value of $\theta - \varpi$ thus found.