The mathematical principles of mechanical philosophy, and their application to the theory of universal gravitation

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Chapter VII.

CHAPTER VII.

MOTION OF A PARTICLE ON CURVES AND SURFACES. SIMPLE PENDULUM.

PROP. *A material particle moves on a curve in a vertical plane, and acted upon by gravity: required to determine the motion.*

393. Let A be the lowest point of the curve (fig. 94.) Ax the axis of x drawn vertically upwards: P the position of the body on the curve AP at the time t: $AM=x$, $MP=y$: let R be the pressure of the curve against the body, this acts in the normal line PG: M the mass of the body: then $\frac{R}{M}$ is the accelerating force resulting from the action of R (Art. 225): g the force of gravity.

Now the forces acting vertically are g downwards and $\frac{R}{M}\cos PGM$ or $\frac{R}{M}\frac{dy}{ds}$ upwards, the only horizontal force is $\frac{R}{M}\frac{dx}{ds}$.

Hence, attending to the *directions* of the forces, we have the following equations of motion:

$$\frac{d^2x}{dt^2} = -g + \frac{R}{M}\frac{dy}{ds}\ldots\ldots(1),\quad \frac{d^2y}{dt^2} = -\frac{R}{M}\frac{dx}{ds}\ldots\ldots(2).$$

Multiply these respectively by $2\frac{dx}{dt}$, $2\frac{dy}{dt}$ and add, then

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} + 2\frac{dy}{dt}\frac{d^2y}{dt^2} = -2g\frac{dx}{dt} + \frac{2R}{M}\left(\frac{dx}{dt}\frac{dy}{ds} - \frac{dy}{dt}\frac{dx}{ds}\right)$$

$$= -2g\frac{dx}{dt};$$

$$\therefore \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} = \text{const.} - 2gx,$$

$$\text{or } \frac{ds^2}{dt^2} = \text{const.} - 2gx.$$

At the commencement of the motion let $x = h$;

$$\therefore 0 = \text{const.} - 2gh;$$

$$\therefore \frac{ds^2}{dt^2} = 2g(h - x).$$

This expression shews that the velocity at any time is independent of the form of the curve on which the body moves; and depends solely on the vertical space through which it passes.

Extracting the square root and inverting the two sides of the equation

$$-\frac{dt}{ds} = \frac{1}{\sqrt{2g}} \frac{1}{\sqrt{h - x}}$$

the negative sign being taken because s diminishes as t increases (Note in page 208).

$$\therefore t = -\frac{1}{\sqrt{2g}} \int \frac{dx}{\sqrt{h - x}} \frac{ds}{dx}.$$

We must determine $\frac{ds}{dx}$ from the equation to the curve by the formula $\frac{ds}{dx} = \sqrt{1 + \frac{dy^2}{dx^2}}$, then by integration we shall know t in terms of x and therefore x in terms of t. In this manner, then, we shall know the velocity and position of the body at every assigned instant.

Prop. *To find the pressure upon the curve.*

394. The equations of motion being

$$\frac{d^2x}{dt^2} = -g + \frac{R}{M} \frac{dy}{ds}, \quad \frac{d^2y}{dt^2} = -\frac{R}{M} \frac{dx}{ds}$$

we multiply them respectively by $\frac{dy}{dt}$, $\frac{dx}{dt}$ and subtract;

$$\therefore \frac{dy}{dt}\frac{d^2x}{dt^2} - \frac{dx}{dt}\frac{d^2y}{dt^2} = -g\frac{dy}{dt} + \frac{R}{M}\left(\frac{dy}{ds}\frac{dy}{dt} + \frac{dx}{ds}\frac{dx}{dt}\right)$$

$$= -g\frac{dy}{dt} + \frac{R}{M}\frac{ds}{dt}, \quad \because \frac{dy^2}{ds^2} + \frac{dx^2}{ds^2} = 1.$$

Now if ρ be the radius of curvature of the curve on which the body moves at the point (xy), then by the Differential Calculus

$$\frac{1}{\rho} = \frac{\frac{dy}{dt}\frac{d^2x}{dt^2} - \frac{dx}{dt}\frac{d^2y}{dt^2}}{\frac{ds^3}{dt^3}}$$

t being a function of x and y, as is the case here:

$$\therefore \frac{R}{M} = g\frac{dy}{ds} + \frac{v^2}{\rho}; \quad v = \text{velocity}.$$

This expression shews that the pressure consists of two parts, one the part of the forces which act upon the body resolved along the normal, and the other the centrifugal force arising from the motion. (Art. 254.)

Prop. *A body moves on a cycloid, the axis of the cycloid being vertical: required to find the time of an oscillation and to shew that it is independent of the extent of the vibration.*

395. We have shewn that $\frac{ds^2}{dt^2} = 2g(h - x)$;

$$\therefore -\frac{dt}{ds} = \frac{1}{\sqrt{2g}}\frac{1}{\sqrt{h-x}},$$

the negative sign being taken because the arc decreases as the time increases.

Now the equation to the cycloid is

$$y = \sqrt{2ax - x^2} + a \text{ vers}^{-1}\frac{x}{a},$$

the lowest point being the origin;

$$\therefore \frac{dy}{dx} = \frac{a - x}{\sqrt{2ax - x^2}} + \frac{a}{\sqrt{2ax - x^2}} = \sqrt{\frac{2a - x}{x}};$$

$$\therefore \frac{ds}{dx} = \sqrt{1 + \frac{dy^2}{dx^2}} = \sqrt{\frac{2a}{x}}.$$

Hence $\dfrac{dt}{dx} = \dfrac{dt}{ds}\,\dfrac{ds}{dx} = -\sqrt{\dfrac{a}{g}}\,\dfrac{1}{\sqrt{hx - x^2}}$;

$$\therefore t = C - \sqrt{\frac{a}{g}} \text{ vers}^{-1}\frac{2x}{h}$$

when $t = 0$, $x = h$; $\therefore 0 = C - \sqrt{\dfrac{a}{g}}\,\pi$

$$t = \sqrt{\frac{a}{g}}\left\{\pi - \text{vers}^{-1}\frac{2x}{h}\right\}$$

and, whenever the body stops, the velocity, or $\dfrac{ds}{dt} = 0$; and therefore $x = h$, and the values of $\text{vers}^{-1}\dfrac{2x}{h}$ when $x = h$ are

$$\pm \pi, \ \pm 3\pi, \ \pm 5\pi, \ldots\ldots$$

and therefore the values of t are

$$2\pi\sqrt{\frac{a}{g}}, \quad 4\pi\sqrt{\frac{a}{g}}, \quad 6\pi\sqrt{\frac{a}{g}}, \ldots\ldots$$

which shew that the body will oscillate backwards and forwards, the interval of time in which each oscillation is performed being $2\pi\sqrt{\dfrac{a}{g}}$.

This expression is independent of h and therefore points out the remarkable fact that however large the arc of vibration be the time of oscillation is the same in all.

For this reason the cycloid is called a Tautochronous Curve. *

Prop. *A particle moves on a circular arc acted upon by gravity: required the time of oscillating through a given portion of the arc.*

396. As before $\frac{ds^2}{dt^2} = 2g(h - x)$ and the equation to the circle from the lowest point is $y^2 = 2ax - x^2$;

$$\therefore \frac{dy}{dx} = \frac{a - x}{\sqrt{2ax - x^2}}, \quad \frac{ds}{dx} = \sqrt{1 + \frac{dy^2}{dx^2}} = \frac{a}{\sqrt{2ax - x^2}};$$

* It may be interesting to ascertain whether there are any other tautochronous curves when gravity is the force acting.

We have $\frac{dt}{dx} = -\frac{1}{\sqrt{2g}} \frac{1}{\sqrt{h-x}} \frac{ds}{dx}$

$$= -\frac{1}{\sqrt{2g}} \frac{ds}{dx} \left\{ \frac{1}{h^{\frac{1}{2}}} + \frac{1}{2} \frac{x}{h^{\frac{3}{2}}} + \frac{1.3}{2.4} \frac{x^2}{h^{\frac{5}{2}}} + \ldots\ldots + \frac{1.3\ldots(2n-1)}{2.4\ldots2n} \frac{x^n}{h^{\frac{2n+1}{2}}} + \ldots\ldots \right\}.$$

Now $\frac{ds}{dx}$ is independent of h: and consequently the integral of the general term

$$-\frac{1}{\sqrt{2g}} \cdot \frac{1.3\ldots(2n-1)}{2.4\ldots2n} \frac{x^n}{h^{\frac{2n+1}{2}}} \frac{ds}{dx}$$

must be of the form $c \cdot \left(\frac{x}{h}\right)^{\frac{2n+1}{2}}$, c being a constant, in order that when taken between the limits $x = 0$ and $x = h$ the result may be independent of h: then

$$\int x^n \frac{ds}{dx} dx = \frac{A}{2n+1} x^{\frac{2n+1}{2}}, \quad A \text{ a constant};$$

$$\therefore \frac{ds}{dx} = \frac{A}{2} x^{-\frac{1}{2}},$$

$$\therefore s = A x^{\frac{1}{2}}$$

$$s^2 = A^2 x,$$

and this is the equation to the cycloid and therefore this is the only tautochronous curve for gravity.

$$\therefore \frac{dt}{dx} = -\frac{a}{\sqrt{2g}} \frac{1}{\sqrt{(h-x)(2ax-x^2)}}.$$

We are not able to integrate this function of x: it is reducible to one of the class called Elliptic Transcendents, the properties of which Legendre has discussed in his *Traité des Fonctions Elliptiques:* tables are given of the approximate values of the integral for given values of x*.

By means of series, however, the integral can be obtained approximately.

$$\frac{dt}{dx} = -\tfrac{1}{2}\sqrt{\frac{a}{g}} \frac{1}{\sqrt{hx-x^2}} \left(1-\frac{x}{2a}\right)^{-\frac{1}{2}}$$

$$= -\tfrac{1}{2}\sqrt{\frac{a}{g}} \frac{1}{\sqrt{hx-x^2}}$$

$$\times \left\{1+\tfrac{1}{2}\frac{x}{2a}+\frac{1.3}{2.4}\left(\frac{x}{2a}\right)^2+\ldots+\frac{1.3\ldots(2n-1)}{2.4\ldots 2n}\left(\frac{x}{2a}\right)^n+\ldots\right\}.$$

Now $\displaystyle\int \frac{x^n dx}{\sqrt{hx-x^2}} = \frac{2n-1}{2n} h \int \frac{x^{n-1}dx}{\sqrt{hx-x^2}} - \frac{x^{n-1}\sqrt{hx-x^2}}{n}$,

and between the limits $x = h$ and $x = 0$, we have

$$\int_h^0 \frac{x^n dx}{\sqrt{hx-x^2}} = \frac{2n-1}{2n} h \int_h^0 \frac{x^{n-1}dx}{\sqrt{hx-x^2}};$$

$$\therefore \int_h^0 \frac{x\,dx}{\sqrt{hx-x^2}} = \frac{h}{2}\,\text{vers}^{-1}\frac{2x}{h} + \text{constant} = -\frac{\pi h}{2};$$

* Let $x = h\sin^2\theta$: then $\theta = \frac{\pi}{2}$ when $x = h$ or $t = 0$;

$$\therefore \int \frac{dx}{\sqrt{(h-x)(2ax-x^2)}} = \int \frac{2\sin\theta\cos\theta\, d\theta}{\sqrt{\cos^2\theta\,(2a-h\sin^2\theta)\sin^2\theta}}$$

$$= \frac{2}{\sqrt{2a}} \int \frac{d\theta}{\sqrt{1-\frac{h}{2a}\sin^2\theta}};$$

which is an elliptic function of the first order.

$$\int_h^0 \frac{x^2\,dx}{\sqrt{hx-x^2}} = -\frac{1.3}{2.4}\pi h^2, \quad \int_h^0 \frac{x^3}{\sqrt{hx-x^2}} = -\frac{1.3.5}{2.4.6}\pi h^3$$

and so on;

$$\therefore\ T = \frac{\pi}{2}\sqrt{\frac{a}{g}}$$

$$\times\left\{1+\left(\frac{1}{2}\right)^2\frac{h}{2a}+\left(\frac{1.3}{2.4}\right)^2\left(\frac{h}{2a}\right)^2+\ldots+\left(\frac{1.3\ldots(2n-1)}{2.4\ldots2n}\right)^2\left(\frac{h}{2a}\right)^n+\ldots\right\}.$$

When the arc of vibration is very small, then

$$T = \frac{\pi}{2}\sqrt{\frac{a}{g}}$$

and the time of an oscillation $= \pi\sqrt{\dfrac{a}{g}}$, which coincides with that in a cycloid, observing that the a in this case is four times the a in that.

The next approximation gives a correction of the time $= \dfrac{\pi}{2}\sqrt{\dfrac{a}{g}}\,\dfrac{h}{8a}$; and the ratio this bears to the time of oscillation $= \dfrac{h}{8a} = (\frac{1}{4}$ chord of $\frac{1}{2}$ angle of oscillation$)^2$.

Thus if the body oscillate on each side of the vertical through an angle of which the chord is $\frac{1}{10}$, the time of oscillation will be greater by a $\frac{1}{1600}$th part than that calculated by the formula $\pi\sqrt{\dfrac{a}{g}}$.

397. Instead of supposing the body to move on a curve, we may imagine it suspended by a string of invariable length, or a thin wire considered of no weight. In this case the instrument is called a *Pendulum*, and is of great importance in physical researches. For if l be the length of a pendulum oscillating in a second (or unit of time) then $\pi\sqrt{\dfrac{l}{g}} = 1$,

and $g = \pi^2 l$,

By this formula we may estimate the relative intensity of the Earth's attraction at different stations on the surface, above, or below it.

Prop. *A seconds pendulum is carried to the top of a mountain; required to find the height of the mountain by observing the change in the time of oscillation.*

398. Let r be the radius of the Earth, considered spherical; h the height of the mountain; l the length of the pendulum: the force of gravity on bodies outside of the Earth varies inversely as the square of the distance from the centre: hence $\dfrac{gr^2}{(r+h)^2}$ is gravity at the top of the mountain. Let n be the number of oscillations the pendulum makes in a day, or in $24 \times 60 \times 60$ seconds: then time of oscillation $= \dfrac{24 \times 60 \times 60}{n}$:

$$\therefore\ 1 = \pi\sqrt{\frac{l}{g}} \text{ and } \frac{24\times60\times60}{n} = \pi\sqrt{\frac{l\,(r+h)^2}{gr^2}} = \frac{\pi\,(r+h)}{r}\sqrt{\frac{l}{g}};$$

$$\therefore\ \frac{h}{r} = \frac{24 \times 60 \times 60}{n} - 1,$$

which gives the height of the mountain. For the sake of example suppose the pendulum loses $5''$ a day:

$$\text{then } n = 24 \times 60 \times 60 - 5,$$

$$\frac{h}{r} = \left(1 - \frac{1}{24 \times 12 \times 60}\right)^{-1} - 1 = \frac{1}{24 \times 12 \times 60} \text{ nearly};$$

$$\therefore\ h = \frac{4000}{24 \times 12 \times 60} = \tfrac{1}{4} \text{ mile nearly.}$$

Prop. *To find the depth of a mine by observing the change of oscillation in a seconds pendulum.*

399. The gravity in the interior of the Earth varies directly as the distance from the centre: if, then, h be the depth, $\dfrac{g\,(r-h)}{r}$ is gravity at the bottom of the mine:

$$\therefore 1 = \pi\sqrt{\frac{l}{g}}, \quad \frac{24 \times 60 \times 60}{n} = \pi\sqrt{\frac{lr}{g(r-h)}};$$

$$\therefore 1 - \frac{h}{r} = \left(\frac{n}{24 \times 60 \times 60}\right)^2;$$

from which h can be found. If, as before, the pendulum lose $5''$ a day

$$\frac{h}{r} = 1 - \left(1 - \frac{1}{24 \times 60 \times 12}\right)^2 = \frac{1}{12 \times 60 \times 12} \text{ nearly};$$

$$\therefore h = \tfrac{1}{2} \text{ mile nearly}.$$

400. The results deduced by the pendulum, as far as we have at present explained its construction, would lead to erroneous conclusions; since we have supposed the rod supporting the *bob*, as the lower extremity is termed, to have no weight. We must leave the correction of this to a future part of the work, in which we shall shew that l must not be taken equal to the length of the pendulum; but some other expression which it is unnecessary to give here.

401. Owing to the remarkable property of the cycloid, that its evolute is an equal cycloid, we can easily make the bob of a flexible pendulum move in a cycloidal arc.

For let CA (fig. 95) be the pendulum when remaining at rest: PAP' the cycloid in which the bob is to move, the length of the axis being half that of the pendulum: CQ, CQ' the evolutes of PAP'. Now move the bob to the right, and let the upper portion of the pendulum bend round CQ and the other portion remain straight, touching CQ in Q. Then since CQ is the evolute of AP, the extremity of the pendulum will be in the curve AP: and by this contrivance the bob will be made to describe the cycloid PAP'.

This suggests the following means of correcting a common pendulum which makes small oscillations. Let a small portion of the upper extremity be flexible: (consisting of watch spring, &c.) and let it be suspended between two cycloidal cheeks, as in fig. 96. Then the small oscillations of the bob will be in a cycloid, and in the expression for the time of oscillation the correction depending on $\frac{h}{2a}$ is avoided: see Art. 396.

402. The following Table contains the results of experiments with a seconds pendulum on various parts of the Earth. It is extracted from the *Mécanique Céleste.*

Places.	Latitudes.	Lengths of a Seconds Pendulum.
Peru	0°.00	0.99669
Porto Bello	10.61	0.99689
Pondicherry	13.25	0.99710
Jamaica	20.00	0.99745
Petit-Goave	20.50	0.99728
Cape of Good Hope	37.69	0.99877
Toulouse	48.44	0.99950
Vienna	53.57	0.99987
Paris	54.26	1.00000
Gotha	56.63	1.00006
London	57.22	1.00018
Petersburgh	64.72	1.00074
Arensgberg	66.60	1.00101
Ponoi	74.22	1.00137
Lapland	74.53	1.00148

403. Mr Airy, in a Paper which was read before the Philosophical Society of Cambridge in the year 1826, has reduced the usual theorems for the alteration in the time and extent of vibration produced by the difference between cycloidal and circular arcs, by the resistance of the air, by the friction at the point of suspension, and by other disturbing causes, to

a very general investigation which leads to results remarkable for their simplicity. Since the principle of the pendulum is of vast importance in physical researches we shall not scruple to introduce large extracts from this valuable communication.

Prop. *A pendulum is acted upon by a small disturbing force: required the alteration in the time and extent of its oscillations.*

404. We shall suppose that the undisturbed pendulum moves with its extremity in a cycloidal arc, since in this case the calculation is not approximate.

Let s be the distance of the pendulum at the time t from the lowest point of the cycloid, s being measured along the arc described, l the length of the pendulum. Then the resolved part of gravity along the tangent is $g\frac{dx}{ds}$, x being measured vertically upwards: and $s^2 = 2lx$ is the equation to the cycloid;

$$\therefore\ g\frac{dx}{ds} = \frac{g}{l}s.$$

Wherefore the equation of motion of the bob of the pendulum is

$$\frac{d^2s}{dt^2} = -\frac{g}{l}s, \text{ or if } n^2 = \frac{g}{l},$$

$$\frac{d^2s}{dt^2} + n^2 s = 0.$$

The solution of this equation is

$$s = a\sin(nt + b),$$

where a and b are arbitrary constant quantities depending on the length of the arc of vibration and the time of passing the lowest point.

The velocity at time $t = \frac{ds}{dt} = na\cos(nt + b)$.

We shall now suppose that f is a small disturbing accelerating force resolved along the tangent: the equation of motion then is

$$\frac{d^2 s}{dt^2} + n^2 s = f.$$

The solution of this equation we shall assume to be

$$s = a \sin (nt + b)$$

(conformably to the principle of the variation of parameters) a and b being considered unknown functions of t, which it is our business now to determine.

Since there are two functions a and b we may assume any relation between them that we please, since we have but one quantity (s) to determine. Let this assumption be that the velocity is still expressed by $na \cos (nt + b)$: the convenience of this we shall soon discover.

$$\text{Now } s = a \sin (nt + b);$$

$$\therefore \frac{ds}{dt} = na \cos (nt + b) + \frac{da}{dt} \sin (nt + b) + a \cos (nt + b) \frac{db}{dt},$$

$$\text{and } \therefore \frac{da}{dt} \sin (nt + b) + a \cos (nt + b) \frac{db}{dt} = 0,$$

this is the assumed relation between a and b.

$$\text{Again since } \frac{ds}{dt} = na \cos (nt + b);$$

$$\therefore \frac{d^2 s}{dt^2} = - n^2 a \sin (nt+b) + n \frac{da}{dt} \cos (nt+b) - na \sin (nt+b) \frac{db}{dt},$$

in this substitute for $\frac{d^2 s}{dt^2}$ its value;

$$\therefore n \frac{da}{dt} \cos (nt + b) - na \sin (nt + b) \frac{db}{dt} = f,$$

this is the second equation between a and b.

Eliminating successively $\frac{db}{dt}$ and $\frac{da}{dt}$ from these, we have

$$\frac{da}{dt} = \frac{f}{n}\cos(nt+b), \quad \frac{db}{dt} = -\frac{f}{na}\sin(nt+b).$$

If we could solve these equations we should have the complete determination of the motion. In few cases is this practicable: in all to which we shall have to apply the investigation an approximation is sufficient.

We suppose f to be a very small force. Hence the variable parts of a and b are of the same order of magnitude as f and consequently may be neglected on the right-hand side of the above equations if we agree to neglect the square and higher powers of f.

In order to find the alteration in the extent of vibration which takes place in one oscillation we must integrate $\frac{f}{n}\cos(nt+b)$ through the limits of t corresponding to one oscillation: that is from a value of t which gives $nt+b=\alpha$ to the value of t which gives $nt+b=\pi+\alpha$. Here α may be any quantity: in different cases we shall find it convenient to integrate between different limits.

$$\therefore \textit{increase of arc of semi-vibration} = \frac{1}{n}\int f\cos(nt+b)\,dt$$

between the above-mentioned limits.

To find the alteration in the time of oscillation, let T, T' be the values of t at two successive arrivals of the pendulum at the lowest point; B, B' the values of b at these times. Then

$$nT + B = m\,.\,\pi, \quad nT' + B' = (m+1)\,.\,\pi;$$

$$\therefore n(T'-T) + B' - B = \pi,$$

$$T' - T = \frac{\pi}{n} - \frac{1}{n}(B'-B).$$

$$\text{Now } B' - B = \int_T^{T'} \frac{db}{dt}\,dt = -\frac{1}{na}\int_T^{T'} f\sin(nt+b)\,dt$$

between the proper limits;

$\therefore$ the increase of time of oscillation $= \dfrac{1}{n^2 a} \displaystyle\int_T^{T'} f \sin (nt + b)\, dt,$

and *the proportionate increase of time of oscillation*

$$= \frac{1}{\pi n a} \int_T^{T'} f \sin (nt + b)\, dt.$$

If the circumstances are such that we must integrate through two vibrations, then

proportionate increase of time of osc. $= \dfrac{1}{2\pi n a} \displaystyle\int f \sin (nt + b)\, dt.$

These formulæ are convenient when f can be expressed in terms of t. If however f be expressed in terms of s, as is the case particularly in clock escapements, we must modify the formulæ

$$\frac{da}{ds} = \frac{da}{dt}\,\frac{dt}{ds} = \frac{1}{na \cos (nt + b)}\,\frac{da}{dt} = \frac{f}{n^2 a},$$

$$\text{and } \frac{db}{ds} = \frac{1}{na \cos (nt + b)}\,\frac{db}{dt}$$

$$= -\frac{f}{n^2 a^2} \tan (nt + b) = -\frac{f}{n^2 a^2}\,\frac{s}{\sqrt{a^2 - s^2}};$$

$\therefore$ *increase of arc of semi-vibration* $= \dfrac{1}{n^2 a} \displaystyle\int_0^s f ds,$

proportionate increase of the time of vib^n. $= \dfrac{1}{\pi n^2 a^2} \displaystyle\int_0^s \frac{f ds}{\sqrt{a^2 - s^2}}.$

We shall subjoin a variety of examples.

Ex. 1. *Instead of vibrating in a cycloid let the pendulum vibrate in a circle.*

Here the force $= g \sin \dfrac{s}{l} = \dfrac{gs}{l} - \dfrac{gs^3}{6l^3}$ nearly;

$$\therefore f = \frac{g}{6l^3} s^3 = \frac{ga^3}{6l^3} \sin^3 (nt + b);$$

therefore proportionate increase in time of vibration

$$= \frac{g a^2}{6 \pi n l^3} \int \sin^4 (nt + b)\, dt.$$

Now $\int \sin^4 (nt+b)\, dt = \frac{1}{8} \int \{3 - 4 \cos 2 (nt+b) + \cos 4 (nt+b)\} dt$

$$= \frac{1}{8} \{3t - \frac{2}{n} \sin 2 (nt + b) + \frac{1}{4n} \sin 4 (nt + b)\} + C$$

$$= \frac{3}{8} \frac{\pi}{n}, \text{ from } nt + b = 0 \text{ to } \pi;$$

$\therefore$ proportionate increase of time $= \frac{g a^2}{16 n^2 l^3} = \frac{a^2}{16 l^2}$ since $n^2 = \frac{g}{l}$.

The increase of arc of vib. $= \frac{g a^3}{6 n l^3} \int \cos (nt + b) \sin^3 (nt + b)\, dt$

$$= \frac{g a^3}{24 n^2 l^3} \sin^4 (nt + b) + C = 0 \text{ between the limits,}$$

as we might easily have foreseen.

Ex. 2. *Suppose the friction at the point of suspension to be constant.*

Here $f = -c$, since the friction *retards* the motion; and the motion is considered *from* the lowest point. It will be convenient to take the integrals during that time in which the friction acts in the same direction: that is, from the beginning of a vibration to its end, or from $nt + b = -\frac{\pi}{2}$ to $nt + b = \frac{\pi}{2}$;

$\therefore$ increase of arc $= -\frac{c}{n} \int \cos (nt + b)\, dt$

$$= -\frac{c}{n^2} \sin (nt + b) + C = -\frac{2c}{n^2},$$

proportionate increase of time $= -\frac{c}{\pi n a} \int \sin (nt + b)\, dt$

$$= \frac{c}{\pi n^2 a} \cos (nt + b) + C = 0,$$

between the limits $nt + b = -\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Ex. 3. *Suppose the resistance of the air to produce a force varying as the* m[th] *power of the velocity or* = k v[m], m *being any whole number.*

The velocity in moving from the lowest point

$$= \frac{ds}{dt} = na \cos (nt + b);$$

$$\therefore f = -kn^m a^m \cos^m (nt + b);$$

therefore increase of arc

$$= -kn^{m-1} a^m \int \cos^{m+1} (nt + b)\, dt \text{ from } nt + b = -\frac{\pi}{2} \text{ to } \frac{\pi}{2}$$

$$= -k\pi n^{m-2} a^m \frac{m(m-2) \ldots\ldots\ldots\ldots 1}{(m+1)(m-1) \ldots\ldots 2} \quad (m \text{ odd})$$

$$= -2kn^{m-2} a^m \frac{m(m-2) \ldots\ldots\ldots\ldots 2}{(m+1)(m-1) \ldots\ldots 3} \quad (m \text{ even}).$$

When $m = 2$ (the law usually taken) the decrease of the arc $= \frac{4ka^2}{3}$.

The proportionate increase of time of oscillation

$$= -\frac{k}{\pi} n^{m-1} a^{m-1} \int \cos^m (nt + b) \sin (nt + b)\, dt$$

$$= -\frac{kn^{m-2} a^{m-1}}{\pi (m+1)} \cos^{m+1} (nt + b) + C$$

$$= 0 \text{ between } nt + b = -\frac{\pi}{2} \text{ and } \frac{\pi}{2}$$

whether m be a positive integer or fraction.

Ex. 4. *Suppose the resistance of the air is expressed by any function of the velocity.*

Here $f = \phi(v)$ for the descent and $-\phi(v)$ for the ascent, and the increase of the arc of vibration

$$= \frac{1}{n^3 a} \int \phi(v) \frac{\cos(nt+b)}{\sin(nt+b)} dv = \frac{1}{n^3 a} \int \frac{v\phi(v)\,dv}{\sqrt{n^2a^2 - v^2}}$$

from $v = 0$ to $v = 0$ again. But it must be observed that from $v = 0$ to $v = na$ (that is, from $s = -a$ to $s = 0$) the radical must be taken with a negative sign, because $\sin(nt+b)$ is then negative. The increase of the arc is consequently

$$-\frac{1}{n^3 a} \int_0^{na} \frac{v\phi(v)\,dv}{\sqrt{n^2a^2 - v^2}} + \frac{1}{n^3 a} \int_{na}^{0} \frac{v\phi(v)\,dv}{\sqrt{n^2a^2 - v^2}},$$

$$\text{and therefore decrease} = \frac{2}{n^3 a} \int_0^{na} \frac{v\phi(v)\,dv}{\sqrt{n^3a^2 - v^2}}.$$

The proportionate increase of time of vibration

$$= -\frac{1}{\pi n a} \int \phi(v) \sin(nt+b)\,dt = \frac{1}{\pi n^3 a^2} \int \phi(v)\,dv$$

$$= \frac{1}{\pi n^3 a^2} \psi(v) = 0, \text{ from } v = 0 \text{ to } v = 0.$$

Hence a resistance which is constant, or which depends on the velocity, does not alter the time of vibration.

Ex. 5. *Let the resistance be that produced by a current of air moving in the plane of vibration with a velocity* V *greater than the greatest velocity of the pendulum: and varying as the square of their relative velocity.*

Here $\phi(v) = -k(V-v)^2$ when the pendulum moves in the direction of the current

$$\phi(v) = k(V+v)^2 \text{ when it moves in the opposite direction.}$$

By the formula in the last Example, when the pendulum moves in the direction of the current, the arc is increased by

$k\left(\frac{2V^2}{n^2} - \frac{Va\pi}{n} + \frac{4a^2}{3}\right)$ and when it returns the arc is diminished by $k\left(\frac{2V^2}{n^2} + \frac{Va\pi}{n} + \frac{4a^2}{3}\right)$.

The diminution in two vibrations $= \frac{2kVa\pi}{n}$. The time is unaffected.

Ex. 6. *Let a force* F *act through a very small space* x *at the distance* c *from the lowest point.*

The increase of the arc $= \frac{1}{n^2 a} \int_c^{c+x} F ds = \frac{Fx}{n^2 a}$ nearly.

The proportionate increase of the time of vibration

$$= \frac{1}{\pi n^2 a^2} \int_c^{c+x} \frac{F s ds}{\sqrt{a^2 - s^2}},$$

if the general value of the integral be $\phi(s)$, then the proportionate increase of time $= \phi(c+x) - \phi(c) = \phi'(c)x$

$$= \frac{Fx}{\pi n^2 a^2} \frac{c}{\sqrt{a^2 - c^2}}.$$

If, then, an impulse be given when the pendulum is at its lowest point, $c = 0$ and the time of vibration is unaffected.

405. Since the preceding theory is applicable to every case in which a pendulum is acted on by small forces, it can be applied to determine the effect produced on the motion of the pendulum of a clock, or the balance of a watch, by the machinery which serves to maintain that motion.

If a pendulum vibrate uninfluenced by any external forces except that of gravity, the resistance of the air and the friction of the point of suspension gradually reduce the extent of vibration. But this diminution goes on very slowly. A pendulum suspended on knife edges has been observed to vibrate more than seven hours before its arc was reduced from two degrees to $\frac{1}{5}$th of a degree. In order to maintain vibrations of the same or nearly the same length (which for clocks is indispensable) a force must act on the pendulum: this force is generally given by the action of a tooth of the seconds wheel on the inclined surfaces of small arms or pallets carried by the pendulum: and the whole apparatus is called an *escapement*.

Now it appears from Examples 2, 3, 4 and 5 of the last Article, that the friction and the resistance of the air do not affect the time of vibration. The maintaining force, therefore, must be impressed in such a manner as not to alter the time of vibration. The escapements of clocks in general use may be divided into the three following classes: recoil escapements, dead-beat escapements, and the escapements in which the action of the wheels raises a small weight which by its descent accelerates the pendulum: this last is Cumming's escapement. A full discussion of these will be found in Mr Airy's communication. He comes to the conclusion that the dead-beat escapement is far superior to any other.

406. In this the wheel acts on the pallet for a small space near the middle of the vibration, and during the remainder of the vibration it has no effect except in producing a slight friction. The impact also at the beat does not tend to accelerate or retard the pendulum. Neglecting then the consideration of the friction, we have a constant force F, which begins to act when $x = -c$ and ceases when $x = c'$. Hence by Ex. 6. of last Article, proportionate increase of time

$$= \frac{F}{\pi n^2 a^2} \int_{-c}^{c'} \frac{s\,ds}{\sqrt{a^2 - s^2}} = \frac{F}{\pi n^2 a^2} \left\{\sqrt{a^2 - c^2} - \sqrt{a^2 - c'^2}\right\}$$

$$= \frac{F}{\pi n^2 a^2} \frac{c'^2 - c^2}{\sqrt{a^2 - c^2} + \sqrt{a^2 - c'^2}} = \frac{F}{2\pi n^2 a^3} (c'+c)(c'-c) \text{ nearly};$$

an extremely small quantity, since c and c' are very small when compared with a, and $c' - c$ may be made almost as small as we please, though it cannot be made absolutely zero; for the wheel must be so adapted to the pallets, that when it is disengaged from one it may strike the other, not on the acting surface, but a little above it; that is, the instant of disengagement from a pallet must follow the instant at which the pendulum is in its middle position by a rather longer time than that by which the instant of beginning to act preceded it. Hence c' must be rather greater than c. But the difference may be made so small that the effect on the clock's rate shall be almost insensible. This escapement, then, approaches very nearly to absolute perfection: and in this respect theory and practice are in exact agreement.

Mr Airy suggests a construction (*Trans. Cam. Phil. Soc.* Vol. III. p. 125.) for a clock escapement similar in its principles to the best detached escapements of chronometers.

PROP. *To prove that the velocity of a particle moving on a smooth surface is independent of the path described, but depends solely on the co-ordinates of position.*

407. Let R be the normal pressure between the surface and particle at the time t, M the mass of the particle; $\alpha\beta\gamma$ the angles which the direction of R makes with the axes: then, X, Y, Z being the other forces acting on the particle, the equations of motion are

$$\frac{d^2x}{dt^2} = X + \frac{R}{M}\cos\alpha, \quad \frac{d^2y}{dt^2} = Y + \frac{R}{M}\cos\beta,$$

$$\frac{d^2z}{dt^2} = Z + \frac{R}{M}\cos\gamma.$$

Multiply these by $2\frac{dx}{dt}$, $2\frac{dy}{dt}$, $2\frac{dz}{dt}$ and add; then

$$\frac{d\,.v^2}{dt} = 2\left(X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt}\right)$$

$$+\frac{2R}{M}\left(\frac{dx}{dt}\cos\alpha + \frac{dy}{dt}\cos\beta + \frac{dz}{dt}\cos\gamma\right).$$

But $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$ are the cosines of the angles which the tangent line to the curve described makes with the axes; hence

$$\frac{dx}{ds}\cos\alpha + \frac{dy}{ds}\cos\beta + \frac{dz}{ds}\cos\gamma$$

equals the cosine of the angle which this tangent makes with the normal, and therefore equals zero;

$$\therefore v^2 = 2\int (Xdx + Ydy + Zdz),$$

and X, Y, Z being functions of x, y, z this expression when integrated will be a function of x, y, z, the co-ordinates of position, and does not depend on the path described.

PROP. *A particle moves in a spherical bowl acted on by gravity: required to determine the motion.*

408. The equations of motion are (z being vertical)

$$\frac{d^2x}{dt^2} = -\frac{R}{M}\cos\alpha, \quad \frac{d^2y}{dt^2} = -\frac{R}{M}\cos\beta, \quad \frac{d^2z}{dt^2} = g - \frac{R}{M}\cos\gamma,$$

also $x^2+y^2+z^2=a^2$ is the equation to the surface: in this case,

$$\cos\alpha = \frac{x}{a}, \quad \cos\beta = \frac{y}{a}, \quad \cos\gamma = \frac{z}{a},$$

then (as in last Article)

$$\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2} = C + 2gz.$$

Let V and k be the initial values of the velocity and of z: then

$$\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2} = V^2 - 2g(k-z),$$

$$\text{also } x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0;$$

$$\therefore x\frac{dy}{dt} - y\frac{dx}{dt} = \text{const.} = h,$$

$$\text{likewise } x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt} = 0,$$

By eliminating $\frac{dx}{dt}$ and $\frac{dy}{dt}$ from these, we have

$$t = \int \frac{a\,dz}{\sqrt{(a^2-z^2)\{V^2 - 2g(k-z)\} - h^2}}.$$

This is an elliptic function, Art. 396. If this could be integrated, then z (and consequently x and y) is known in terms of t, and the motion is determined.

409. We may obtain approximate results by supposing the oscillations to be very small.

In this case, let θ be the angle that the radius drawn to the particle makes with the vertical, ψ the angle which the vertical plane in which θ is measured makes with the vertical plane through the centre of the sphere and the point of projection; let the velocity of projection $(V) = \beta\sqrt{ga}$, β being a small numerical quantity, the direction of V horizontal, α the initial value of θ; then

$$k = a - \tfrac{1}{2}a\alpha^2, \quad z = a - \tfrac{1}{2}a\theta^2, \quad h^2 = a^3 g\alpha^2\beta^2,$$

$$y = x\tan\psi, \quad x^2 + y^2 + z^2 = a^2;$$

$$\therefore \frac{dt}{d\theta} = -a\theta\frac{dt}{dz} = -\sqrt{\frac{a}{g}}\frac{\theta}{\sqrt{(\alpha^2-\theta^2)(\theta^2-\beta^2)}},$$

$$\frac{d\psi}{d\theta} = \frac{d\psi}{dt}\frac{dt}{d\theta} = \frac{1}{x^2+y^2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)\frac{dt}{d\theta} = -\frac{\alpha\beta}{\theta\sqrt{(\alpha^2-\theta^2)(\theta^2-\beta^2)}}.$$

The first of these equations gives

$$2t = -\sqrt{\frac{a}{g}}\int\frac{2d\,.\,\theta^2}{\sqrt{(\alpha^2-\beta^2)^2 - \{2\theta^2 - (\alpha^2+\beta^2)\}^2}}$$

$$= \sqrt{\frac{a}{g}}\cos^{-1}\left\{\frac{2\theta^2 - (\alpha^2+\beta^2)}{\alpha^2-\beta^2}\right\}, \text{ const.} = 0;$$

$$\therefore \theta^2 = \tfrac{1}{2}(\alpha^2+\beta^2) + \tfrac{1}{2}(\alpha^2-\beta^2)\cos 2\sqrt{\frac{g}{a}}\,t:$$

this shews that the pendulum makes isochronous oscillations in the moveable vertical plane: the extreme angles being α and β, and the time of oscillation being $\frac{\pi}{2}\sqrt{\frac{a}{g}}$, or half the time of oscillation when the plane of motion is constant.

Hence also $\frac{d\psi}{dt} = \sqrt{\frac{g}{a}} \frac{\alpha\beta}{\alpha^2 \cos^2 \sqrt{\frac{g}{a}}\, t + \beta^2 \sin^2 \sqrt{\frac{g}{a}}\, t}$;

$$\therefore\ \alpha \tan \psi = \beta \tan \sqrt{\frac{g}{a}}\, t,$$

from which the azimuth of the plane of oscillation is known at any time.

By substitution we have

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = (a^2 - z^2)\left(\frac{\cos^2 \psi}{\alpha^2} + \frac{\sin^2 \psi}{\beta^2}\right) = a^2 \theta^2 \left(\frac{\cos^2 \psi}{\alpha^2} + \frac{\sin^2 \psi}{\beta^2}\right),$$

and substituting for θ and ψ their values in terms of t,

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = a^2,$$

which shews that the projection of the path on a horizontal plane is an ellipse with its centre in the vertical radius of the sphere.

Cor. If $\alpha = \beta$, then $\theta^2 = \alpha^2$, $\psi = \sqrt{\frac{g}{a}}\, t$, $x^2 + y^2 = a^2\alpha^2$, and the pendulum describes a conical surface with a uniform motion.

Prop. *A particle moves on a curve surface, required to find the pressure at any instant.*

410. The equations of motion are

$$\frac{d^2x}{dt^2} = X + \frac{R}{M}\cos\alpha,\quad \frac{d^2y}{dt^2} = Y + \frac{R}{M}\cos\beta,\quad \frac{d^2z}{dt^2} = Z + \frac{R}{M}\cos\gamma.$$

Multiply by $\cos\alpha$, $\cos\beta$, $\cos\gamma$ respectively, and add, then

$$\frac{R}{M} = \frac{d^2x}{dt^2}\cos\alpha + \frac{d^2y}{dt^2}\cos\beta + \frac{d^2z}{dt^2}\cos\gamma$$
$$- \{X\cos\alpha + Y\cos\beta + Z\cos\gamma\}.$$

To calculate the former part suppose that the co-ordinate planes are so chosen, that, at the instant under consideration, the axis of z is the normal line at the point of contact of the particle: hence $\cos\alpha = 0$, $\cos\beta = 0$, $\cos\gamma = 1$, and this part becomes $\frac{d^2z}{dt^2}$.

Now z is a function of x and y: x and y are functions of t; hence

$$\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} + \frac{dz}{dy}\frac{dy}{dt},$$

$$\frac{d^2z}{dt^2} = \frac{d^2z}{dx^2}\frac{dx^2}{dt^2} + 2\frac{d^2z}{dxdy}\frac{dx}{dt}\frac{dy}{dt} + \frac{d^2z}{dy^2}\frac{dy^2}{dt^2} + \frac{dz}{dx}\frac{d^2x}{dt^2} + \frac{dz}{dy}\frac{d^2y}{dt^2}.$$

But $\frac{dz}{dx} = 0$, $\frac{dz}{dy} = 0$ as the axes are chosen.

Hence $$\frac{d^2z}{dt^2} = \frac{ds^2}{dt^2}\left\{\frac{d^2z}{dx^2}\frac{dx^2}{ds^2} + 2\frac{d^2z}{dxdy}\frac{dx}{ds}\frac{dy}{ds} + \frac{d^2z}{dy^2}\frac{dy^2}{ds^2}\right\}$$

$$= \frac{(\text{velocity})^2}{\text{radius of curvature}} = \frac{v^2}{\rho},$$

and the magnitude of this cannot depend upon the manner of fixing the axis; therefore, in general,

$$\frac{R}{M} = \frac{v^2}{\rho} - (X\cos\alpha + Y\cos\beta + Z\cos\gamma)$$

= centrifugal force − resolved part of the forces along the normal.

PROP. *A particle moves in a groove in the form of a curve of double curvature; required the pressure.*

411. The equations of motion are the same as in the last Article: $\alpha\beta\gamma$ being the angles which the direction of the pressure makes with the axes; this coincides with the radius of absolute curvature.

Let ρ be the radius, and $x_, y_, z_,$ the co-ordinates to the centre of curvature, then

$$x_, = x + \rho^2\frac{d^2x}{ds^2}, \quad y_, = y + \rho^2\frac{d^2y}{ds^2}, \quad z_, = z + \rho^2\frac{d^2z}{ds^2};$$

$$\therefore \cos\alpha = \frac{x - x_{,}}{\rho} = \rho\frac{d^2x}{ds^2}, \quad \cos\beta = \rho\frac{d^2y}{ds^2}, \quad \cos\gamma = \rho\frac{d^2z}{ds^2};$$

$$\therefore \frac{R}{M} = \rho\left\{\frac{d^2x}{dt^2}\frac{d^2x}{ds^2} + \frac{d^2y}{dt^2}\frac{d^2y}{ds^2} + \frac{d^2z}{dt^2}\frac{d^2z}{ds^2}\right\} - \{X\cos\alpha + Y\cos\beta + Z\cos\gamma\},$$

the former part, by changing the independent variable to s (as in Art. 255), becomes

$$\frac{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2}{\frac{dt^2}{ds^2}} - \frac{\frac{d^2t}{ds^2}\frac{d}{ds}\left\{\frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} + \frac{dz^2}{ds^2}\right\}}{\frac{dt^3}{ds^3}}$$

$$= \frac{1}{\rho^2}(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)\frac{ds^2}{dt^2} = \frac{1}{\rho^2}\frac{ds^2}{dt^2};$$

$$\therefore \frac{R}{M} = \frac{v^2}{\rho} - (X\cos\alpha + Y\cos\beta + Z\cos\gamma)$$

= centrifugal force − resolved part of the forces along the radius of absolute curvature.