The mathematical principles of mechanical philosophy, and their application to the theory of universal gravitation

Pratt, John Henry

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Chapter X.

CHAPTER X.

MOTION OF A RIGID BODY ACTED ON BY FORCES OF FINITE INTENSITY.

427. In considering the equilibrium of a rigid body (Art. 27) we stated, that, in consequence of our ignorance of the nature and laws of the forces by which the molecules are held together, we are unable to deduce the conditions of equilibrium of a body from those of a single particle. By the aid, however, of the principle of the transmission of force through a body (Art. 28) we deduced certain relations which the impressed forces, that act upon the body when in equilibrium must satisfy independently of the molecular forces. It is evident that the system of molecular forces are themselves in equilibrium independently of the other forces which act upon the body.

In considering the motion of a rigid body we fall upon the same difficulty. We know nothing of the laws of the molecular forces, and consequently cannot calculate the motion of the body by calculating the motion of its molecules separately. But we may surmount this in the manner we overcame the difficulty just mentioned.

Let mX, mY, mZ be the *impressed* moving forces which act upon the particle m, not including the molecular forces which act upon this particle. Let xyz be the co-ordinates to m at the time t: then $m\dfrac{d^2x}{dt^2}$, $m\dfrac{d^2y}{dt^2}$, $m\dfrac{d^2z}{dt^2}$ are the *effective* moving forces of m (Art. 211).

Now by the first of the general principles enunciated in Art. 226, the forces

$$m\left(X-\frac{d^2x}{dt^2}\right),\quad m\left(Y-\frac{d^2y}{dt^2}\right),\quad m\left(Z-\frac{d^2z}{dt^2}\right)$$

acting on m parallel to the axes of x, y, z respectively, and similar forces acting on all the other particles ought, together with the molecular forces by which the particles of the body act upon each other, to satisfy the equations of equilibrium of forces acting on a rigid body.

But the molecular forces are of themselves in equilibrium, since the molecules retain their relative situations during the motion.

Hence the forces $m\left(X-\frac{d^2x}{dt^2}\right)$, $m\left(Y-\frac{d^2y}{dt^2}\right)$, $m\left(Z-\frac{d^2z}{dt^2}\right)$ acting on m and similar forces acting on the other particles of the body ought to satisfy the six equations of equilibrium of forces acting on a rigid body, given in Art. 65. Wherefore we have the six equations of motion

$$\Sigma . m\left(X-\frac{d^2x}{dt^2}\right)=0, \quad \Sigma . m\left(Y-\frac{d^2y}{dt^2}\right)=0, \quad \Sigma . m\left(Z-\frac{d^2z}{dt^2}\right)=0,$$

$$\Sigma . m\left\{y\left(Z-\frac{d^2z}{dt^2}\right)-z\left(Y-\frac{d^2y}{dt^2}\right)\right\}=0,$$

$$\Sigma . m\left\{z\left(X-\frac{d^2x}{dt^2}\right)-x\left(Z-\frac{d^2z}{dt^2}\right)\right\}=0,$$

$$\Sigma . m\left\{x\left(Y-\frac{d^2y}{dt^2}\right)-y\left(X-\frac{d^2x}{dt^2}\right)\right\}=0.$$

By these six equations we shall be able to calculate the motion of a rigid body acted on by any forces of finite intensity. They lead immediately to two Principles, one of which enables us to calculate the motion of *translation* of the body in space; and the other the motion of *rotation*.

PROP. *The motion of the centre of gravity of a body moving free in space and acted on by any forces is the same as if all the forces were applied at the centre of gravity parallel to their former directions.*

428. By the first three equations of Art. 427,

$$\Sigma . m\left(X-\frac{d^2x}{dt^2}\right)=0, \quad \Sigma . m\left(Y-\frac{d^2y}{dt^2}\right)=0, \quad \Sigma . m\left(Z-\frac{d^2z}{dt^2}\right)=0.$$

Let $\bar{x}\bar{y}\bar{z}$ be the co-ordinates to the centre of gravity,

$x'y'z'$ .... m from the centre of gravity;

$$\therefore\ x = \bar{x} + x',\ y = \bar{y} + y',\ z = \bar{z} + z'.$$

Now $\Sigma . mx' = 0$, $\Sigma . my' = 0$, $\Sigma . mz' = 0$ (Art. 413).

Hence, substituting for xyz, the above equations give, M being the whole mass of the body,

$$\frac{d^2\bar{x}}{dt^2} = \frac{\Sigma . mX}{M},\ \frac{d^2\bar{y}}{dt^2} = \frac{\Sigma . mY}{M},\ \frac{d^2\bar{z}}{dt^2} = \frac{\Sigma . mZ}{M}:$$

and these are the equations we should obtain for the motion of the centre of gravity supposing the forces all applied at that point. Hence the Proposition is proved.

Prop. *The motion of rotation of a body acted on by any forces and moving freely is the same as if the centre of gravity were fixed and the same forces acted.*

429. The last three of the equations of Art. 427 are

$$\Sigma . m\left\{y\left(Z - \frac{d^2z}{dt^2}\right) - z\left(Y - \frac{d^2y}{dt^2}\right)\right\} = 0,$$

$$\Sigma . m\left\{z\left(X - \frac{d^2x}{dt^2}\right) - x\left(Z - \frac{d^2z}{dt^2}\right)\right\} = 0,$$

$$\Sigma . m\left\{x\left(Y - \frac{d^2y}{dt^2}\right) - y\left(X - \frac{d^2x}{dt^2}\right)\right\} = 0.$$

Now let $\bar{x}$, $\bar{y}$, $\bar{z}$ be the co-ordinates to the centre of gravity, and let (as before) $x = \bar{x} + x'$, $y = \bar{y} + y'$, $z = \bar{z} + z'$.

Let these be put in the above equations, observing that $\Sigma . mx' = 0$, $\Sigma . my' = 0$, $\Sigma . mz' = 0$ (Art. 413), and that therefore the differential coefficients of these with respect to t vanish; also bearing in mind the equations of last Article we have after all reductions,

$$\Sigma . m \left\{ y' \left(Z - \frac{d^2 z'}{dt^2} \right) - z' \left(Y - \frac{d^2 y'}{dt^2} \right) \right\} = 0,$$

$$\Sigma . m \left\{ z' \left(X - \frac{d^2 x'}{dt^2} \right) - x' \left(Z - \frac{d^2 z'}{dt^2} \right) \right\} = 0,$$

$$\Sigma . m \left\{ x' \left(Y - \frac{d^2 y'}{dt^2} \right) - y' \left(X - \frac{d^2 x'}{dt^2} \right) \right\} = 0.$$

But these are precisely the equations we should have obtained on the supposition that the centre of gravity were fixed, and that point taken as the origin of moments. Hence the Proposition is true.

430. From the first of the Principles demonstrated in the last two Articles we gather, that all the calculations we have made of the motion of a material particle will be true also of the centre of gravity of a rigid body. It remains then to ascertain the motion of the other parts of the body relative to the centre of gravity: and this the latter Principle enables us to accomplish, as we shall shew in the following Chapters. We shall consider the motion of rotation of a body first about any fixed axis, either passing through the centre of gravity or not, and lastly about a fixed point.