A treatise of spherical geometry, containing its fundemental properties; the doctrine of its loci, the maxima and minima of spherical lines and areas: with an application of these elements to a ...

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The constructions of spherical problems. Book II.

THE

CONSTRUCTION

OF

SPHERICAL PROBLEMS.

BOOK II.

PROBLEM I.

GIVEN one ſide A C, an angle adjacent A C B, and the ſum of the remaining angles A and B, to determine the great circle ſpherical triangle A B C. — (Pl. 7. fig. 7.)

ANALYSIS. Suppoſe the thing done, and compleat the parallelogram A D B C, and produce D A, B C, to meet in I and V; then, becauſe angle D = C and B = A, alſo their ſum S = D + B + C + A being given, we have angle $A = B = \frac{S - D - C}{2} = \frac{S - 2C}{2}$; conſequently, angle C A V = 90° — D A C, and

A C V = 90° − A C B, are alſo given; and we have in the triangle A C V two angles, and the included ſide A C, to conſtruct the triangle.

THEOREM I.

If the angular points A B C, of the great circle ſpherical triangle A B C, be the poles of three great circles; theſe great circles, by their interſections, will form another triangle F D E, which is called ſupplemental to the former, that is, the ſides F D, D E, E F, are the ſupplements of the meaſures of the oppoſite angles C, B, A, of the triangle A B C, and the meaſures of the angles F, D, E, of the triangle F D E, will be the ſupplements of the ſides A C, B C, B A, in the triangle A B C. — (Pl. 7. fig. 14.)

Let A B, produced, meet D E, E F, in G, M, and A C meet F D, and F E in K L, and B C meet F D, D E, in N, H.

Since A is the pole of F E, and the circle A C paſſes through A, E F will paſs through the pole of A C; and ſince A C paſſes through C, the pole of F D; F D will paſs through the pole of A C, therefore, the pole of A C is in the point F, in which the arches F D, E F, interſect each other. In the ſame manner D is the pole of B C, and E the pole of A B.

And ſince F, E, are the poles of A L, A M, F L, and E M are quadrants, and E L, E M, together, that is, F E and M L together are equal to a ſemicircle. But ſince A is the pole of M L, M L is the meaſure of the angle B A C, conſequently F E is the ſupplement of the meaſure of the angle B A C. In the ſame manner E D, D F, are the ſupplements of the meaſures of the angles A B C, B C A.

Since, likewiſe, C N, B H, are quadrants, C N, B H, together, that is, N H, B C, together, are equal to a ſemicircle; and ſince D is the pole of N H, N H is the meaſure of the angle F D E; therefore, the meaſure of the angle F D E is the ſupplement of the ſide B C. In the ſame manner it is ſhown, that the meaſures of the angles D E F, E F D, are the ſupplements of the ſides A B, A C, in the triangle A B C.

PROBLEM II.

Given the ſides A B, B C, and the ſum of the angles at the baſe A + C, to determine the great circle ſpherical triangle A B C. — (Pl. 7. fig. 8.)

ANALYSIS. Suppoſe the thing done, and about A, B, C, as poles, deſcribe the great circles F E, D E, D F, and produce F D, F E, till they meet in I; then, ſince F E is the ſupplement of the meaſure of angle A, (T. I.) the ſupplement of F E, I E, muſt be the meaſure of angle A. And in like manner we prove that D I is the meaſure of angle C.

Again, angle D E F being the meaſure of the ſupplement of A B, angle D E I muſt be the meaſure of A B; and for the ſame reaſon angle I D E will be found the meaſure of B C. Hence in the triangle A B C, we have given

the angle I E D meaſure of A B,
——— I D E ——— B C,
——— D I + E I ——— A + C.

Compleat the parallelogram D E G H, and join D, G: Becauſe, in the triangle D E G, the angle E is given, the ſum of the angles $EDG+DGE=FI-IDE$, and $GE=IF-DI-IE$; the problem is reduced to the laſt.

PROBLEM III.

Given one ſide A C, the vertical angle A, and the ſum of the remaining ſides A B and B C, to determine the ſpherical triangle A B C, of uniform curvature, &c.—(Pl. 7. fig. 9.)

CON. Having drawn A T c, A B b, including the given angle A, lay off A T, A t, each equal $\frac{AC+AB+BC}{2}$; from T, t, erect the perpendiculars T O, O t to meet in O; with radius $TO=Ot$ and center O, deſcribe the circle T P t, and from A lay off A C, the given ſide. Laſtly, lay off $CP=CT$, with radius C P and center C, which will be a tangent to T P at P, and produced, will cut A B in B.

DEM. By conſtruction, and T. V. B. III. A is the vertical angle, A C B the ſum of the ſides, and A C is the given ſide.

COR. If A c and angle A had been given,

and A b – c b, the difference of the other ſide and baſe, the conſtruction and demonſtration would have been exactly the ſame.

SCHOLIUM. The ſolutions of ſpherical problems may often be facilitated by the method uſed in P. II. Thus, if we ſuppoſe A C B a great circle triangle, (Pl. 7. fig. 8.) A B + B C, A C and angle A given as in the problem, and circles drawn about the poles C, A, B, we have given

D F E = the meaſure of ſupplement A C,
E D F + D E F ———— ſupplement C B + B A,
and F E ———— ſupplement of A,

to conſtruct the triangle D E F, being the ſame data as in P. I.

Or the problem may be conſtructed thus: Suppoſe C B, B A, Pl. 7. fig. 17. produced till they meet in I, and compleat the parallelogram C A P D; join C P, then in the triangle C A P are given C A, A P, $(= BI - \overline{CB + BA})$ and angle A (= ſupplement angle C A B) to determine the reſt.

PROBLEM IV.

Given the vertical angle A, and the

ſum of each ſide (A C + C Q and A B + B Q) added to its adjacent ſegment, made by an arch A Q O biſecting the vertical angle, to determine the ſpherical triangle. — (Pl. 7. fig. 9.)

ANALYSIS. Since the ſum of each ſide, added to its adjacent ſegment, is given, the perimeter is given, whence making the ſame preparation, as in P. III, the radius T O = O P of the circle T P is alſo given, and ſince T C = C P, if we lay off A V = A C + C Q, we have V T = Q P, which laſt will conſequently be given. Again, ſince A T, and the angles T A O, A T O, are all given, T O A is given, and, hence, the following

CON. Having deſcribed the ſmall circle T P, deſcribe a circle C Q P B of equal curvature and convexity with A C and A B, to touch it in any point P. Lay off P Q = A T − A V, and through O, Q, draw the great circle O Q A. From O lay off T O, forming the known angle A O T; and from T draw the circle T A a tangent to T P in T, and cutting C A in A, and C B in C. Draw A B t a tangent to T P in t, (P. IX. B. I.) and cutting C B in B; ſo is C A B the triangle required; the demonſtration of which is plain from the analyſis.

PROBLEM V.

Given the baſe C B, the ſum of the ſides A B + A C, and the vertical angle A, to conſtruct the great circle ſpherical triangle, &c. — (Pl. 7. fig. 16.)

CON. Having laid off the vertical angle A, and drawn the circle T t, as in P. IV, from A lay off L A, V A, each $= \frac{1}{2}. \overline{AB + AC - CB}$. Erect the perpendicular arches L Q, V Q, meeting in Q; and with radius L Q = V Q, (C. IV. T. XXII. B. I.) deſcribe the circle L V. Laſtly, draw the circle C I P B, touching the circles L V, T t, in I and P reſpectively, (P. VIII. B. I. of conſtruction) and the thing is done.

DEM. By conſtruction, C A + A B + C B (T. V. B. III.) = the given perimeter, and T A t = the given vertical angle. Alſo T A − L A = T L = B C. (T. VI. B. III. *)

* This propoſition (and its corollaries) muſt be conſidered as relative to great circle triangles only, for though it involves no abſurdity as there propoſed, it has been diſcovered (ſince it was put to the preſs) that it can only apply in that particular caſe.

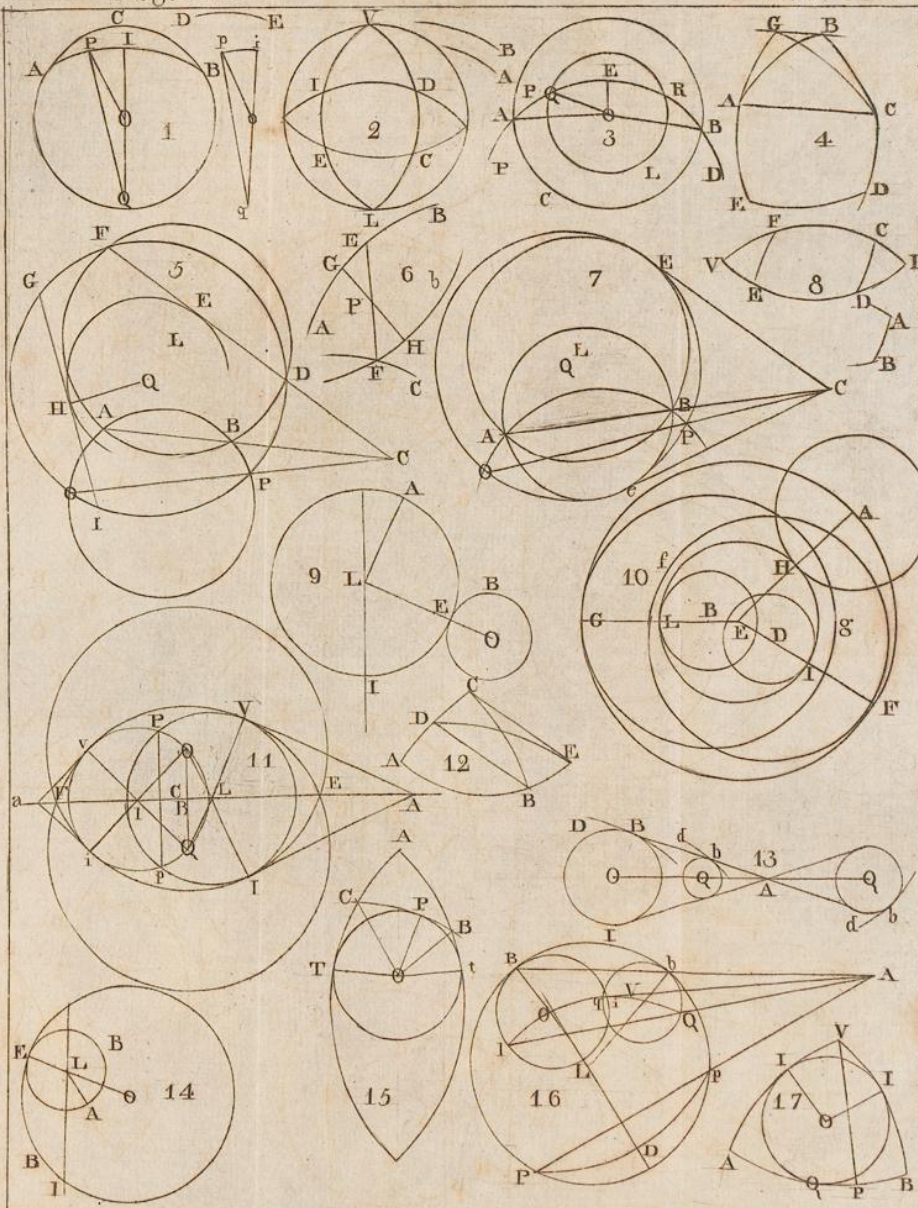
PROBLEM VI.

Given the baſe CB, the perpendicular OP, and vertical angle COB, to conſtruct the great circle triangle. — (Pl. 6. fig. 15.)

ANALYSIS. Suppoſe the thing done, and with radius OP, and center O, deſcribe the ſmall circle TPt. Through C and B, draw the great circles ACT, ABt, tangents to TPt in T, t, and meeting in A, and join C, O; B, O; T, O, and tO.

Becauſe PC = TC, and TO = OP, and angle T = P and CO, common to the two triangles TCO, COP, they are equal and ſimilar, and conſequently angle TOC = COP; and ſince in like manner it may be proved that BOt = BOP, therefore the whole angle TOt is equal twice COB, a given quantity.

CON. Hence having deſcribed TPt, and laid off from O the angle TOt, then if tangents TA, tA, be drawn to TPt at T and t, the problem is reduced to the laſt.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PROBLEM VII.

Given the vertical angle B V C, and the radii D O, Q E, of two circles inſcribed in the parts B V R, C V R, of a great circle ſpherical triangle B V C, made by a line V R, biſecting that angle to conſtruct the triangle.—(Pl. 7. fig. 13.)

CON. On D O, deſcribe the triangle D V O right angled at D, and having its vertical angle $DVO = \frac{BVC}{4}$, and from V draw V I R a tangent to D I at I parallel to which, (through the center O of D I) draw P O a ſmall circle; join O, I, and upon O I, produced, lay off I L = Q E; and through L draw the ſmall circle L Q parallel to V A R. Again, upon the arch q a = Q E, erect the triangle a q v right angled at a, and having the angle q v a = D V O; and from V apply V A upon V R equal v a. Alſo, at right angles to V A, through A, draw the great circle P A Q, cutting L Q in Q and P O in

P; and with radius A Q and center Q, describe the circle A E. Draw V E C touching A E in E. Laftly, draw B F G C a tangent to D I and A E in F, G, refpectively, and meeting V D and V E, produced in B and C refpectively, and the thing is done.

DEM. By the nature of the circle and conftruction, angle D V I is half the given angle: Alfo, becaufe P O, V I, Q L, are parallels, and the angles at I and A both right angles, we have I O = P A and I L = A Q = Q E, the required radii. And, fince, V A = v a and angle A V Q = a v q, the point A falls at the required diftance upon V R from V, the reft is evident.

PROBLEM VIII.

Given two fides A C, C B, and the area, to conftruct the great circle fpherical triangle.—(Pl. 7. fig. 18.)

When the area is given, the fum of the angles are alfo given, (T. VIII. B. II.) in which cafe we fhall have the following

CON. Bifect one fide C B in Q, and erect the perpendicular Q V: Alfo, (by P. I.) draw

C V, meeting QV in V ſuch, that the ſum of the angles QCV + CVQ ſhall be equal $\frac{ACB + CAB + ABC}{2}$. Again, (by P. I. B. II.) draw the equal and parallel circles a B V and C I A. Laſtly, with radius C B, the other given ſide and centre C, deſcribe the ſmall circle a B, interſecting a B V in a and B; ſo will C a B or C A B be the required triangle.

DEM. Join V, A, then is the triangle C A B = C V A (C. II. T. V. B. II.) = 2. CQV, the given area, the reſt is evident.

COR. When C B neither cuts nor touches a V, the problem is impoſſible; and, hence, a and B will coincide when the triangle is a maximum, and C B be perpendicular to both a V and CQA;

And here occurs an elegant inſtance of the obſervation contained in SCHOLIUM P. III. For having deſcribed about the points A, B, C, as poles, the great circles F E, D E, and D F, (Pl. 7. fig. 8.) then, ſince the ſum of the angles C, A, and B, are given, the ſum of the ſupplement of their meaſures E F + D E + D F, is alſo given; and ſince C A and C B are both given, the meaſures of their reſpective ſupplements, angle D F E, and E D F, are alſo given; Whence the problem may be eaſily and elegantly conſtructed.

PROBLEM IX.

Given the baſe C B, the perpendicular A P, and the area, to conſtruct the ſpherical triangle. — (Pl. 7. fig. 11.)

Here the ſum of the angles A, C, B, being conſequently given, we have this

CON. Having made the triangle C V Q, and drawn the equal parallels A V a, C L B (as in the laſt problem) about the pole I of the great circle C Q B, deſcribe the parallel A E a at the diſtance A I = the complement of the given perpendicular A P to a quadrant; through A, where this interſects A V a in A, draw the great circle I A P, meeting C B (produced if neceſſary) in P; and join A, C, and A, B, and the thing is done.

DEM. It is proved (ſee the laſt problem) that triangle C A B = C V B = the given area; and A P being by conſtruction the complement of A I, is evidently equal to the given perpendicular.

PROBLEM X.

Given the perimeter, the vertical angle V, and the perpendiculars O v, O n, and O I, falling on each ſide B V, V A, and B A, from a point O within the triangle B V A to conſtruct it. — (Pl. 7. fig. 12.)

CON. Having laid off the given vertical angle V, draw b O and O a reſpectively, parallel circles to B V, V A, at the given diſtances of the perpendiculars V O, O n, and from their interſection in O, with radius equal perpendicular O I, deſcribe the circle I E. Laſtly, lay off the circle T P t, of equal perimeters as in P. IV. and draw P I a tangent to both circles at P and I reſpectively (P. VIII. B. I. of conſtruction) and the thing is done. The demonſtration is evident from the conſtruction.

PROBLEM XI.

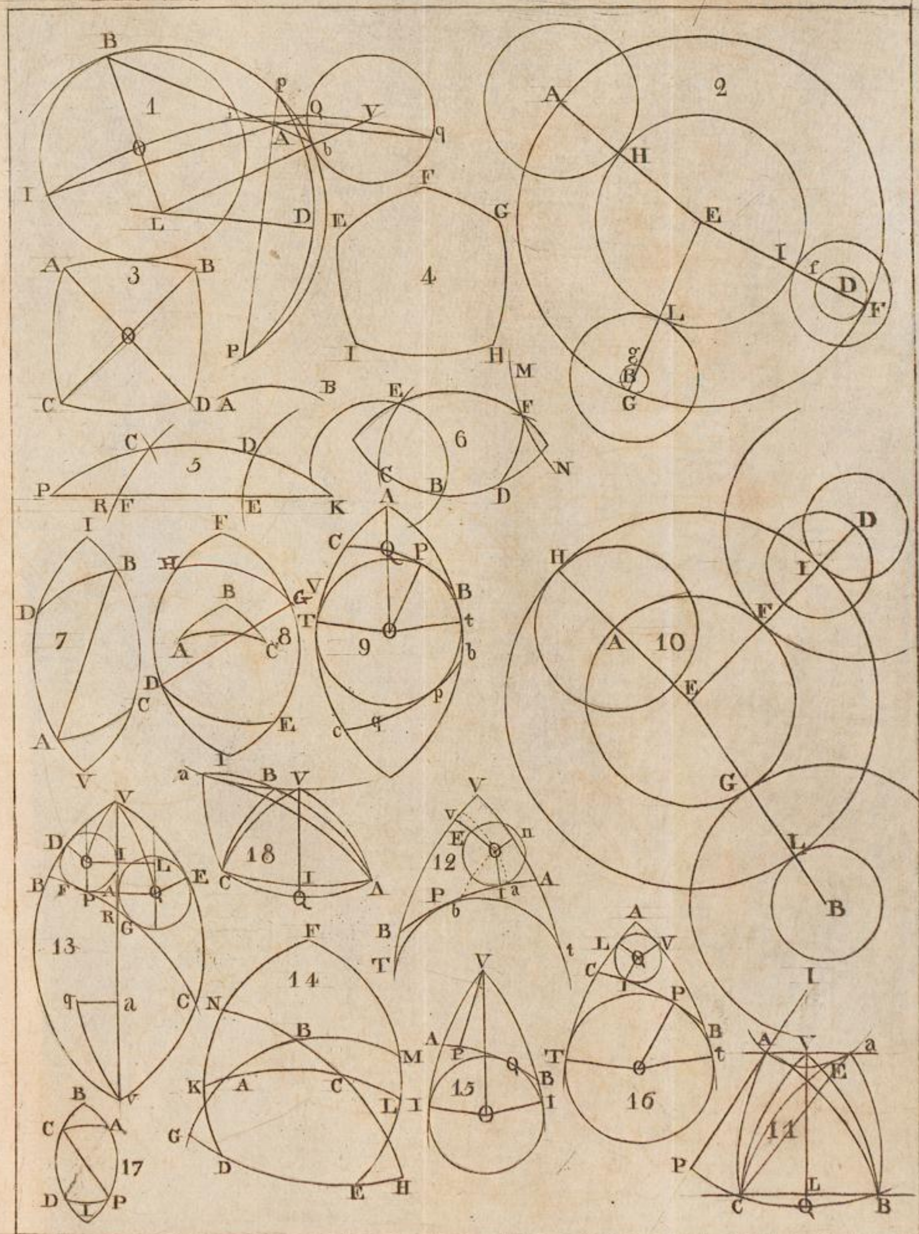
Given the vertical angle V, and the difference between each ſide and its adjacent ſegment of the baſe made by a perpendicular let fall thereon, from the vertical angle AV−AP and VB−PB, to conſtruct the great circle ſpherical triangle AVB.—(Pl. 6. fig. 17.)

PROBLEM XII.

Given the vertical angle V, and the ſum of each ſide added to its adjacent ſegment of the baſe made by a perpendicular VP let fall thereon, from the vertex V; AV+AP and VB+BP, to conſtruct the great circle ſpherical triangle AVB.—(Pl. 7. fig. 15.)

☞ The ſolutions of theſe two problems are left to the ingenious.

FINIS.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