Miscellaneous Tracts on some curious, and very interesting Subjects in Mechanics, Physical-Astronomy, and Speculative Mathematics ...

Simpson, Thomas

London, 1757

ETH-Bibliothek Zürich

Shelf Mark: Rar 5521

Persistent Link: https://doi.org/10.3931/e-rara-9162

A very exact method for finding the place of a planet in its orbit, from a correction of ward hypothsis, by means of one, or more equations, applied to the motion about the upper focus.

A very exact METHOD for finding the PLACE of a PLANET in its orbit, from a Correction of WARD's *hypothesis*, by means of One, or more Equations, applied to the motion about the upper focus.

Fig. 18. LET ABPC be the ellipses in which the planet revolves about the sun in the lower focus S; let F be the upper focus, and M and m any two places of the planet indefinitely near to each other; and let FM, Fm, SM, Sm be drawn, as likewise MN, perpendicular to the greater axis AP: from the center F, with the radius FD $= 1$, let the circumference of a circle DEe be described, and from its intersection with FM draw EH perpendicular to AP: put AO $(= \mathrm{OP}) = a$, OB $(= \mathrm{OC}) = b$, OF $(= \mathrm{OS}) = c$, FM $= u$, FH $= x$, EH $= y$, DE $= z$, and E$e = \dot{z}$: then SM being $(= \mathrm{AP} - \mathrm{FM}) = 2a - u$, by the nature of the ellipses, and FN $(= \mathrm{FH} \times \frac{\mathrm{FM}}{\mathrm{FE}}) = xu$, we have, by a known property of triangles, $\overline{\mathrm{SM} + \mathrm{FM}} \times \overline{\mathrm{SM} - \mathrm{FM}}$ $(2a \times \overline{2a - 2u}) = 2\mathrm{SF} \times \mathrm{ON}$ $(4c \times \overline{c + xu})$: from which equation u (FM) is found $= \frac{a^2 - c^2}{a + cx} = \frac{bb}{a + cx}$ (because $bb = aa - cc$): whence SM $(= 2a - u)$ is also had in terms of x, being $= \frac{aa + cc + 2acx}{a + cx}$.

Now the area EFe being expressed by $\frac{1}{2}\dot{z}$ $(= \mathrm{FE} \times \frac{1}{2}\mathrm{E}e)$, we have FE^2 (1) : FM^2 :: $\frac{1}{2}\dot{z}$: the area MF$m = \frac{1}{2}\dot{z} \times \overline{\mathrm{FM}}|^2$. Therefore, the angle SMm being equal to FMA, or FmA (by the nature of the curve) we also have FM$\times$Mm : S$m\times m$M (or FM : SM) : : the area $(\frac{1}{2}\dot{z} \times \overline{\mathrm{FM}}|^2)$ of the triangle FMm : the area of the triangle SM$m = \frac{1}{2}\dot{z} \times \mathrm{FM} \times \mathrm{SM}$ (*Elem.* 23. 6.) $= \frac{\frac{1}{2}b^2\dot{z} \times \overline{aa + cc + 2acx}}{\overline{a + cx}|^2} = \frac{1}{2}b^2\dot{z} + \frac{\frac{1}{2}b^2c^2\dot{z} \times \overline{1 - xx}}{\overline{a + cx}|^2} = \frac{1}{2}b^2\dot{z} + \frac{\frac{1}{2}b^2c^2y^2\dot{z}}{\overline{a + cz}|^2} =$ the fluxion of the area ASM. In order to find the

fluent

fluent hereof, let $\frac{1}{\overline{a+cx}|^2}$ be reſolved into the ſeries $\frac{1}{a^2} - \frac{2cx}{a^3}$, *&c.* (whereof the two firſt terms will, here, be ſufficient) by which means our fluxion will become $\frac{1}{2}b^2\dot{z} + \frac{\frac{1}{2}b^2c^2y^2\dot{z}}{a^2} - \frac{b^2c^3x\dot{z}y^2}{a^3}$; which, becauſe $y\dot{z} = -\dot{x}$, and $x\dot{z} = \dot{y}$, will be reduced to $\frac{1}{2}b^2\dot{z} + \frac{b^2c^2}{2a^2} \times -\dot{x}y - \frac{b^2c^3y^2\dot{y}}{a^3}$; whoſe fluent will therefore be truly expreſſed by $\frac{1}{2}b^2z + \frac{b^2c^2}{2a^2} \times$ area DHE $- \frac{b^2c^3y^3}{3a^3}$, or $\frac{1}{2}b^2z + \frac{b^2c^2}{2a^2} \times \overline{\frac{1}{2}z - \frac{1}{2}xy} - \frac{b^2c^3y^3}{3a^3}$ (becauſe the area DHE $= \frac{1}{2}$DE $\times$ DF $- \frac{1}{2}$HF $\times$ EH $= \frac{1}{2}z - \frac{1}{2}xy$). This expreſſion, when M coincides with P, and z is = the ſemi-circumference DEK $(= p)$, will become $\frac{b^2p}{2} + \frac{b^2c^2p}{4aa} = \frac{b^2dp}{2}$ (ſuppoſing $d = 1 + \frac{cc}{2aa}$) = the area of the ſemi-ellipſis ABP. Therefore we have, as $\frac{b^2dp}{2} : \frac{b^2dz}{2} - \frac{b^2c^2xy}{4aa} - \frac{b^2c^3y^3}{3a^3}$ (= the area ASM) :: p (= the length of an arch of 180°) : $z - \frac{c^2xy}{2a^2d} - \frac{2c^3y^3}{3a^3d}$ = the length of an arch (A) expreſſing the planet's mean anomaly: from which equation, $z = A + \frac{c^2xy}{2a^2d} + \frac{2c^3y^3}{3a^3d} = A + \frac{c^2}{4a^2d} \times \overline{\text{ſin. } 2z} + \frac{2c^3}{3a^3d} \times \overline{\text{ſin. } z}|^3$ (becauſe xy, or co-ſin. $z \times$ ſin. z = $\frac{1}{2}$ ſin. $2z$): where the two laſt terms being very ſmall in compariſon of the others (and, therefore, z nearly = A), we may, inſtead of ſin. z and ſin. $2z$, ſubſtitute ſin. A and ſin. $2A$; by which means we have $z = A + \frac{cc}{4a^2d} \times \text{ſin. } 2A + \frac{2c^3}{3a^3d} \times \overline{\text{ſin. A}}|^3$. From whence it appears, that, in order to have the angle AFM at the upper focus, the mean anomaly (A) of the planet at the time given, muſt be increaſed by the quantity, or correction $\frac{cc}{4a^2d} \times \text{ſin. } 2A + \frac{2c^3}{3a^3d} \times \overline{\text{ſin. A}}|^3$.

But to expreſs the value of this correction in ſeconds of a degree (which in practice is the moſt commodious) it will

will be, as 3,1416 (the length of an arch of 180 degrees) is to 648000 (the number of ſeconds in that arch) ſo is $\frac{cc}{4a^2d} \times$ ſin. $2A + \frac{2c^3}{3a^3d} \times \overline{\text{ſin. A}}|^3$ to $51567 \times \frac{c^2}{a^2d} \times$ ſin. $2A + 137513 \times \frac{c^3}{a^3d} \times \overline{\text{ſin. A}}|^3$ = the number of ſeconds in the ſaid correction: the logarithm of the latter term of which will, therefore, be $= 5,1383 -$ log. $d + 3$ log. $\frac{c}{a} + 3$ log. ſin. A; and that of the former $= 4,7123 -$ log. $d + 2$ log. $\frac{c}{a} +$ log. ſin. 2A. But, to render theſe expreſſions ſtill more convenient for practice, the log. of d, by reaſon of its ſmallneſs, may be, either, intirely neglected, or elſe ſo aſſumed, to be nearly a mean of what it is known to be in the planetary orbits. Accordingly, by aſſuming the excentricity $c = \frac{8}{100}$ of the mean diſtance (which is a ſmall matter leſs than the excentricity of *Mars*, but ſomething greater than *thoſe* of the *Moon*, *Saturn*, and *Jupiter)*, the value of d $\left(= 1 + \frac{cc}{2aa}\right)$ will be $= 1,0032$, and its logarithm $= 0,0014$. Whence the log. of the former part of our correction, by ſubſtituting this value, will be $5,1369 + 3$ log. $\frac{c}{a} + 3$ log. ſin. A $= 3 \times 1,7123 +$ log. $\frac{c}{a} +$ log. ſ. A; and *that* of the latter $= 4,7109 + 2$ log. $\frac{c}{a} +$ log. ſin. 2A: which, expreſſed in words at length, give the following

Practical Rules.

1°. To the ſum of the conſtant logarithm 1,7123 and the log. of the excentricity in parts of the mean diſtance, add the log. ſine of the mean anomaly; the ſum (rejecting the radius) being tripled, will give the log. of the firſt equation (in ſeconds) to be added to the mean anomaly.

2°. To the ſum of the conſtant log. 4,7109 and twice the log. of the excentricity, add the log. ſine of twice the mean anomaly; the ſum (rejecting the radius) will be the log. of the ſecond equation; to be added or ſubtracted, according as the mean anomaly is leſs, or greater than 90 degrees. Here

Here, and in what follows, the anomaly is to be always reckoned from, or to the aphelion, the neareſt way; in which the ſeconds may be omitted, in computing the propoſed corrections. Which corrections being made, the true anomaly, or angle at the lower focus, will be had from the common proportion, by ſaying, as the aphelion-diſtance is to the perihelion-diſtance, ſo is the tangent of half the mean anomaly, thus corrected, to the tangent of half the true anomaly ſought.

As an example of the uſe of the method here laid down, let the excentricity be ſuppoſed = 0,048219 (being that aſſigned, by Dr. HALLEY, to the orbit of Jupiter) and let the mean anomaly be 45°.

Then, log. excent. . .	$\bar{2},6832$	2 log. excent. . . .	3,3664
+ conſt. log. . .	1,7123	+ conſt. log. . . .	4,7109
= log. for 1ſt equ.	0,3955	= log. for 2d equ.	2,0773
log. ſin. anom. . .	9,8494	log. ſin. 2 anom.	10,0000
	0,2449	2d equ. = 119½″	2,0773
firſt equ. = 5½″	0,7347		

From $\bar{1},978537$ = log. of perihelion-diſt. 0,951781,
ſubtr. 0,020452 = log. of the aphelion-diſt. 1,048219;
the rem. $\bar{1},958085$, will be a (3d) conſt. log. for this orbit: to which
add 9,617596 = log. tang. ½ cor. anom. 22° 31′ 2½″,
ſo ſhall 9,575681 = log. tang. ½ true anomaly 20° 37′ 40″:
whoſe double, 41° 15′ 20″, is therefore the true anomaly required.

The ſame excentricity being retained; let the mean anomaly be, now, ſuppoſed 120 degrees.

Then, log. ſin. anom.	9,9375	log. ſin. 2 anom. . .	9,9375
log. for firſt equat.	0,3955	log. for 2d equation	2,0773
	0,3330	2d equat. = 103½″	2,0148
firſt equat. = 10″	0,9990		

Hence 120° + 10″ — 1′ 43½″ = 119° 58′ 26½″ = the cor. anomaly.

Therf. log. tang. ½ cor. anom. 59° 59′ 13½″ . . .	10,238331
+ third conſt. log. for this orbit	$\bar{1},958085$
= log. tang. ½ true anom. 57° 32′ 10″	10,196416

 Whence

Whence the anomaly itſelf is given $= 3^s\, 25^\circ\, 4'\, 20''$.

If the excentricity be aſſumed $= 0{,}066777$ (being the greateſt that Dr. HALLEY gives to the lunar orbit) the two conſtant logarithms, to be added to the ſines of the mean anomaly and of its double, will be $0{,}5369$ and $2{,}3601$: whence if the ſaid anomaly be taken $= 50^\circ$ (at which, according to the *Doctor's Table, pro expediendo calculo æquationis centri lunæ*, the whole correction is a *maximum)*, the former part of the ſaid correction will be found $= 18\frac{1}{2}''$, and the latter part $= 225\frac{1}{2}''$: therefore the ſum of both is $244''$, or $4'\, 4''$; agreeing, exactly, with the quantity given in the *Table*. And in the very ſame manner, the proper corrections correſponding to other anomalies and excentricities may be computed; the error never amounting to above a ſingle ſecond in any of the planets, except *Mars* and *Mercury* : in the place of *Mars*, the greateſt error will be two or three ſeconds; and in that of *Mercury*, about as many minutes.—As to the *Earth* and *Venus*, the ſecond equation, alone, will be ſufficient to give their places to leſs than a ſecond.

To obtain a farther correction, which will be neceſſary when the orbit is very eccentrical, we may (inſtead of the two firſt only) make uſe of a greater number of terms of the ſeries $\frac{1}{a^2} - \frac{2cx}{a^3} + \frac{3c^2x^2}{a^4} - \frac{4c^3x^3}{a^5}$ &c. $\left(= \frac{1}{\overline{a+cx}|^2}\right)$; by which means the fluxion $\left(\frac{1}{2}b^2\dot{z} + \frac{\frac{1}{2}b^2c^2y^2\dot{z}}{\overline{a+cx}|^2}\right)$ of the area MSA deſcribed about the focus S, which is proportional to the time, will be here repreſented by $\frac{1}{2}b^2\dot{z} + \frac{1}{2}b^2 \times \overline{e^2y^2\dot{z} - 2e^3y^2x\dot{z} + 3e^4y^2x^2\dot{z} - 4e^5y^2x^3\dot{z}}$ &c. (ſuppoſing $e = \frac{c}{a}$). But it is well known that y^2 $(\overline{\text{ſin.}\ z}|^2)$ $= \frac{1}{2} - \frac{1}{2}$ co-ſin. $2z$; whence y^2x ($= y^2 \times$ co-ſin. z) $= \frac{1}{2}$ co-ſin. $z - \frac{1}{2}$ co-ſin. $2z \times$ co-ſin. $z = \frac{1}{2}$ co-ſin. $z - \frac{1}{4}$ co-ſin. $z - \frac{1}{4}$ co-ſin. $3z$* $= \frac{1}{4}$ co-ſin. $z - \frac{1}{4}$ co-ſin. $3z$; and therefore y^2x^2

* This, and all that follows to the ſame purpoſe, is nothing more than the application of the THEOREM, *That the rectangle of the co-ſines of any two angles (the radius being unity) is equal to half the ſum of the co-ſines of the ſum and difference of thoſe angles.*

=

$= \frac{1}{4}\,\text{co-ſin.}\,z \times \text{co-ſin.}\,z - \frac{1}{4}\,\text{co-ſin.}\,3z \times \text{co-ſin.}\,z = \frac{1}{8} + \frac{1}{8}\,\text{co-ſin.}\,2z - \frac{1}{8}\,\text{co-ſin.}\,2z - \frac{1}{8}\,\text{co-ſin.}\,4z = \frac{1}{8} - \frac{1}{8}\,\text{co-ſin.}\,4z$: whence alſo $y^2x^3 = \frac{1}{8}\,\text{co-ſin.}\,z - \frac{1}{8}\,\text{co-ſin.}\,4z \times \text{co-ſin.}\,z = \frac{1}{8}\,\text{co-ſin.}\,z - \frac{1}{16}\,\text{co-ſin.}\,3z - \frac{1}{16}\,\text{co-ſin.}\,5z$. Which values being now ſubſtituted for their equals, our fluxion, above given, will become $\frac{1}{2}b^2\dot{z} + \frac{1}{2}b^2$ into $e^2 \times \overline{\frac{1}{2}\dot{z} - \frac{1}{4} \times 2\dot{z}\,\text{co-ſin.}\,2z} - 2e^3 \times \overline{\frac{1}{4}\dot{z}\,\text{co-ſ.}\,z - \frac{1}{12} \times 3\dot{z}\,\text{co-ſ.}\,3z} + 3e^4 \times \overline{\frac{1}{8}\dot{z} - \frac{1}{32} \times 4\dot{z}\,\text{co-ſ.}\,4z} - 4e^5 \times \overline{\frac{1}{8}\dot{z}\,\text{co-ſ.}\,z - \frac{1}{48} \times 3\dot{z}\,\text{co-ſ.}\,3z - \frac{1}{80} \times 5\dot{z}\,\text{co-ſ.}\,5z} + \&c.$ and, conſequently, the fluent thereof $= \frac{1}{2}b^2z + \frac{1}{2}b^2$ into $e^2 \times \overline{\frac{1}{2}z - \frac{1}{4}\,\text{ſin.}\,2z} - 2e^3 \times \overline{\frac{1}{4}\,\text{ſ.}\,z - \frac{1}{12}\,\text{ſ.}\,3z} + 3e^4 \times \overline{\frac{1}{8}z - \frac{1}{32}\,\text{ſin.}\,4z} - 4e^5 \times \overline{\frac{1}{8}\,\text{ſine}\,z - \frac{1}{48}\,\text{ſine}\,3z - \frac{1}{80}\,\text{ſine}\,5z}\ \&c. = \frac{1}{2}b^2$ into $\overline{1 + \frac{1}{2}e^2 + \frac{3}{8}e^4} \times z - \overline{\frac{1}{2}e^3 + \frac{1}{2}e^5} \times \text{ſin.}\,z - \frac{1}{4}e^2\,\text{ſin.}\,2z + \overline{\frac{1}{6}e^3 + \frac{1}{12}e^5} \times \text{ſin.}\,3z - \frac{3e^4}{32}\,\text{ſin.}\,4z + \frac{e^5}{20}\,\text{ſin.}\,5z$ (ſuppoſing all ſuch terms wherein e riſes to more than five dimenſions, to be diſregarded). This expreſſion, when $z = p$ (= the ſemi-circumference AEK) becomes $\frac{1}{2}b^2 \times \overline{1 + \frac{1}{2}e^2 + \frac{3}{8}e^4\ \&c.} \times p = \frac{1}{2}b^2p \times \overline{1 - ee}|^{-\frac{1}{2}} = \frac{\frac{1}{2}b^2p}{\sqrt{1 - \frac{cc}{aa}}} = \frac{\frac{1}{2}b^2p \times a}{\sqrt{a^2 - c^2}} = \frac{1}{2}abp =$ the area of the ſemi-ellipſe ABP. Hence it will be, as $\frac{1}{2}abp$ (area ABP) : $\frac{1}{2}abz - \frac{1}{2}b^2 \times \overline{\frac{1}{2}e^3 + \frac{1}{2}e^5} \times \text{ſin.}\,z - \frac{1}{2}b^2 \times \frac{1}{4}e^2\,\text{ſin.}\,2z\ \&c.$ (area ASM) : : p (the length of an arch of 180°) : $z - \frac{b}{a} \times \overline{\frac{1}{2}e^3 + \frac{1}{2}e^5} \times \text{ſin.}\,z - \frac{b}{a} \times \frac{e^2}{4}\,\text{ſin.}\,2z\ \&c. =$ the length of the arch A, expreſſing the planet's mean anomaly: whence, by ſubſtituting $f = \frac{b}{a}$, we have $A = z - \overline{1 + e^2} \times \frac{1}{2}fe^3\,\text{ſin.}\,z - \frac{1}{4}fe^2\,\text{ſin.}\,2z + \overline{1 + \frac{1}{2}e^2} \times \frac{1}{6}fe^3\,\text{ſin.}\,3z - \frac{3fe^4}{32} \times \text{ſin.}\,4z + \frac{fe^5}{20} \times \text{ſine}\,5z$.

Now, to find from hence the value of z, in terms of A and the ſines of its multiples, we may, for a firſt approximation, aſſume $\overline{1 + e^2} \times \frac{1}{2}fe^3\,\text{ſin.}\,z + \frac{1}{4}fe^2\,\text{ſin.}\,2z\ \&c.$ as equal to $\overline{1 + e^2} \times \frac{1}{2}fe^3\,\text{ſin.}\,A + \frac{1}{4}fe^2\,\text{ſin.}\,2A - \overline{1 + \frac{1}{2}e^2} \times \frac{1}{6}fe^3\,\text{ſin.}\,3A\ \&c.$ which laſt quantity being denoted by Q, we ſhall have $A = z - Q$, and conſequently $z = A + Q$.

But

But ſin. z (= ſin. $\overline{A+Q}$) = ſin. A × co-ſin. Q + ſin. Q × co-ſin. A = ſin. A + Q × co-ſin. A, nearly (becauſe Q being very ſmall, its co-ſine will differ inſenſibly from the radius, and the ſine very little from the arch itſelf). In the ſame manner, ſ. $2z$ (= ſ. $\overline{2A+2Q}$) = ſ. 2A + 2Q × co-ſ. 2A, ſin. $3z$ = ſ. 3A + 3Q × co-ſ. 3A, &c. Which values being therefore ſubſtituted in the given equation, it will become $A = z - Q - Q \times \overline{\overline{1+e^2} \times \tfrac12 fe^3 \text{co-ſ.}A + \tfrac12 fe^2 \text{co-ſ.}2A - \overline{1+\tfrac12 e^2} \times \tfrac12 fe^3 \text{co-ſ.}3A \ \&c.}$ But the firſt term of the ſeries, $\overline{1+e^2} \times \tfrac12 fe^3$ co-ſin. A, drawn into $-$ Q (if all terms having more than five dimenſions of e be neglected) will be barely = $\tfrac12 fe^3$ × co-ſin. A × $-\tfrac14 fe^2$ ſin. 2A = $-\tfrac18 f^2 e^5$ × co-ſin. A × ſin. 2A = $-\tfrac1{16} f^2 e^5 \times \overline{\text{ſin. }3A + \text{ſin. }A}$.

In the ſame manner $\tfrac12 fe^2$ co-ſin. 2A × $-$Q = $-\tfrac12 fe^2$ co-ſin. 2A × $\overline{\tfrac12 fe^3 \text{ſin.}A + \tfrac14 fe^2 \text{ſin. }2A - \tfrac16 fe^3 \text{ſin. }3A}$ = $-\tfrac18 f^2 e^5 \times \overline{\text{ſin.}3A - \text{ſin. }A} - \tfrac1{16} f^2 e^4 \times \text{ſin. }4A + \tfrac1{24} f^2 e^5 \times \overline{\text{ſin. }5A + \text{ſin. }A}$: And, laſtly, $-\overline{1+\tfrac12 e^2} \times \tfrac12 fe^3$ co-ſin. 3A × $-$ Q equal to $\tfrac12 fe^3$ co-ſin. 3A × Q = $\tfrac12 fe^3$ × co-ſin. 3A × $\tfrac14 fe^2$ ſin. 2A = $\tfrac1{16} f^2 e^5 \times \overline{\text{ſin. }5A - \text{ſin. }A}$. The ſum of all which will be $\tfrac1{24} f^2 e^5$ ſin. A $- \tfrac3{16} f^2 e^5$ ſin. 3A $- \tfrac1{16} f^2 e^4$ ſin. 4A $+ \tfrac5{48} f^2 e^5$ ſin. 5A; which added to z $-$ Q (or its equal, $z - \overline{1+e^2} \times \tfrac12 fe^3$ ſin. A $- \tfrac14 fe^2$ ſin. 2A, &c.) gives $A = z - \overline{1+e^2-\tfrac1{12} fe^2} \times \tfrac12 e^3 f \text{ ſin. }A - \tfrac14 e^2 f \text{ ſin. }2A + \overline{1+\tfrac12 e^2 - \tfrac98 e^2 f} \times \tfrac16 e^3 f \text{ ſin. }3A - \overline{3f+2f^2} \times \frac{e^4}{32} \text{ſin.}4A + \overline{\tfrac1{20} f + \tfrac5{48} f^2}\, e^5 \times \text{ſin.}5A$. From whence we have $z = A + \overline{1+\frac{11ee}{12}} \times \tfrac12 e^3 f \text{ ſin. }A + \tfrac14 e^2 f \text{ ſin. }2A - \overline{1-\frac{5ee}{8}} \times \tfrac16 e^3 f \text{ ſin.}3A + \frac{5e^4 f^2}{32} \text{ſin.}4A - \frac{37 e^5 f}{240} \text{ſin.}5A$, very near; becauſe in all terms having more than three dimenſions of e, the quantities 1 and f may be uſed indifferently, for each other, without producing any errors but ſuch as conſiſt of more than five dimenſions of the converging quantity e.

But, ſince it is known that ſin. 3A = 3S $-$ S^3, and ſin. 5A = 5S $-$ 20S^3 + 16S^5 (S being the ſine of A) we ſhall, by ſubſtituting theſe values in the 2d, 4th, and 6th terms (after proper

proper reduction) have $z = A + \overline{1 + 4ee} \times \frac{2}{3}e^3fS^3 - \frac{37e^5f}{15}S^5 + \frac{1}{4}e^2f \times \text{fin. } 2A + \frac{5e^4f^2}{32} \times \text{fin. } 4A$. Hence it appears that

$$\frac{180 \times 60 \times 60}{3{,}1416} \times \overline{\overline{1+4ee} \times \frac{2}{3}e^3fS^3 - \frac{37e^5f}{15}S^5 + \frac{1}{4}e^2f\text{f. } 2A + \frac{5e^4f^2}{32}\text{f. } 4A}$$

will exprefs the number of feconds, to be added to the mean anomaly, in order to have the angle AFM at the upper focus of the ellipfis, correfponding to that anomaly. From whence is deduced the following method of calculation.

Let F denote the logarithm of half the leffer axis divided by half the greater, E the log. of the eccentricity divided by half the greater axis, G the log. of the fum of the fquares of half the greater axis and twice the eccentricity divided by the fquare of half the greater axis.

Take $P = 1{,}71277 + E + \frac{1}{3}F + \frac{1}{3}G$,
$Q = 1{,}14130 + E + \frac{1}{5}F$,
$R = 4{,}71236 + 2E + F$,
$S = 4{,}50824 + 4E + 2F$;

then the logarithms of four equations (in feconds), to be applied to the mean anomaly (A), will be

$$3 \times \overline{P + \text{log. fin. } A - \text{log. rad.}}$$
$$5 \times \overline{Q + \text{log. fin. } A - \text{log. rad.}}$$
$$R + \text{log. fin. } 2A - \text{log. rad.}$$
$$S + \text{log. fin. } 4A - \text{log. rad.}$$

Of which equations the firft is always to be added, and the fecond always fubtracted; the other two being to be added, or fubtracted, according as the fines of their refpective arguments, 2A and 4A, are pofitive, or negative.

The two principal of thefe equations agree with thofe before given; and are the fame, in effect, with the two equations laid down (without demonftration) by Sir ISAAC NEWTON, in the Scholium to the 31ft Propofition of the firft book of his *Principia*. The latter of which, in the *Laws of the Moon's Motion*, prefixed to that Work, feems to be reprefented, as *defective*; it being there afferted, that, *the inequality in the motion of a planet about the upper focus, confifts of three parts*; as if the nature of the fubject admitted of juft that number, and no

more

more; whereas the parts, or equations ariſing in the conſideration, are without number, being the terms of an infinite ſeries, wherein the eccentricity is the converging quantity. Sir ISAAC NEWTON has given two terms of this ſeries, which are right; but the new equation added by his *Commentator*, is not ſo; the ſign thereof, the coefficient, and the law by which it increaſes and decreaſes, being all different from what they ought to be.

This equation (expreſſed according to the above notation, where 1 repreſents half the greater axis, f half the leſſer axis, e the eccentricity, and A the mean anomaly) he makes to be $-\frac{1}{2}e^4 \times \overline{\text{ſin. A}}|^3 \times \text{co-ſin. A}$. But it ought to be $+\frac{3f+2f^2}{32} \times e^4 \text{ ſin.} 4A$ ($= \frac{5e^4}{32} \times \text{ſin.}4A$, nearly), as is ſhewn above; this being the 3d term of the general ſeries, and the next in order after thoſe given by Sir ISAAC NEWTON; who appears, more than once*, to have been diſadvantageouſly (I might ſay, unfairly) repreſented, and that, under the covers of his own book: a circumſtance that cannot be attended to without ſome concern and diſlike, by thoſe who entertain a due regard for the merit of an Author to whom the mathematical world is ſo much indebted.

I ſhall now put down one ſingle example of the uſe of the equations above derived; wherein I ſhall ſuppoſe the eccentricity to be $\frac{20589}{100000}$ parts of the ſemi-tranſverſe axis (the ſame as is aſſigned by Dr. HALLEY to the orbit of Mercury). Here, then, we have $E = -\bar{1},313635$; F ($= \log.\sqrt{1-ee} = \frac{1}{2}\log.\overline{1-ee}$) $= -0,009406$; G ($=\log.1+4ee$) $= 0,068024$; whence $P = 1,046$; $Q = 0,453$; $R = 3,3302$; $S = 1,744$:

* In the 28th Propoſition of his third book, it is found that the moon's diſtance from the earth in the ſyzigies is to the diſtance in the quadratures (ſetting aſide the conſideration of eccentricity) as 69 to 70; which is confirmed by what is demonſtrated in a ſubſequent part of this our Work, as well as by the calculations of others; nevertheleſs the truth of this proportion is called in queſtion, and a new one is laid down, which makes the ſaid diſtances to be in the proportion of 59 to 60. See *Laws of the Moon's Motion*, p. 11 and 12.

which

which values will ſerve in all caſes belonging to this orbit. Whence, ſuppoſing the given anomaly to be 50°, we have

1,0460	0,4530	3,3302	1,744
9,8842	9,8842	9,9933	9,534
0,9302	0,3372	3,3235	1,278.
3	5		
2,7906	1,6860		

Of which reſulting logarithms, the numbers correſponding are $617\frac{1}{2}$, $48\frac{1}{2}$, $2106\frac{1}{4}$, and 19; whereof the firſt and third being added, and the other two ſubtracted, we have 50° 44′ 16″ for the corrected anomaly, or the angle at the upper focus; whereof the half is 25° 22′ 8″:

Therefore log. tang. 25° 22′ 8″ 9,6759338
$+$ log. of the ratio of the gr. and leaſt diſt. 1,8185730
$=$ log. tang. $\frac{1}{2}$ true anomaly 17° 20′ 28″ 9,4945068.

Which concluſion is true to a ſecond. Nor will the error, in any part of this orbit, amount to more than about two or three ſeconds.—If you would have the reſult depended on to a ſingle ſecond, or if the orbit be ſuppoſed ſtill more eccentrical than that of Mercury, then the following method may be of uſe.

Say, as half the greater axis of the ellipſis is to the eccentricity, ſo is the ſine of the mean anomaly to the ſine of an angle; which ſubtract from the mean anomaly, and to the log. ſine of the remainder (which I call the eccentric anomaly) add the ſum of the log. of the eccentricity and the conſtant log. 1,758123: the aggregate (rejecting the radius) will be the logarithm of an angle, in degrees and decimal parts; which, ſubtracted from the angle firſt found, leaves a correction to be added (under its proper ſign) to the mean anomaly: with which corrected anomaly, let the whole operation be repeated, if needful, by always adding the laſt correction to the mean anomaly. Then it will be, as the greater ſemi-axis of the ellipſis is to the leſſer, ſo is the tangent of the corrected anomaly to the tangent of the angle at the upper focus of the ellipſis: whence the angle at the lower focus, or the true anomaly, may alſo

alſo be known by the common proportion, and that to any aſſigned degree of exactneſs.

This method, in all the planets, except *Mars* and *Mercury*, anſwers to a ſecond, at one operation. In the former of theſe two, the error, when greateſt, will amount to about three or four ſeconds; and in the latter, to nearly as many minutes; in which caſe, three operations will be neceſſary: but in order to avoid that trouble, the following calculation may be uſed; which is ſo exact as to anſwer, even in the orbit of Mercury, to leſs than half a ſecond, without repeating the operation.

Add together twice the log. of the eccentricity, the log. ſecant of the angle firſt found (as above), and the log. co-ſine of the ſum of the eccentric anomaly and mean anomaly once corrected (in all of which angles the ſeconds may be neglected). The aggregate (ſubtracting twice the radius) will be the log. of a fraction to be added to unity, when the ſaid ſum of anomalies is between 90 and 270°; but otherwiſe, ſubtracted therefrom: then the log. of this ſum, or remainder being ſubtracted from the log. of the firſt correction, you will have the log. of the true correction to be added (under its proper ſign) to the mean anomaly given.

Thus, for example, let the mean anomaly be 70; and let the eccentricity be = 0,20589 (as in the preceding example).

Here, ſin. mean anom. 70°	9,9729858
log. eccent.	1,3136353
∴ ang. firſt found = 11° 9′,33	9,2866211;

whence 58° 50′,67 = the eccentric anomaly:

whoſe ſine	9,9323552
+ the log. proper for the orbit . .	1,0717583
= log. of 10°,0952	1,0041135:

which angle, or its equal, 10° 5′,71, being ſubtracted from 11° 9′,33, leaves the firſt correction 1° 3′,62 = 63′,62.

Moreover,

Moreover, by adding together 70°, 1° 3′, and 58° 50′, we have

129° 53′, whose co-sine	9,8070
+ sec. 11° 9′ ang. first found . . .	10,0083
+ twice log. eccent.	$\bar{2}$,6272
= log. of 0,0272	$\bar{2}$,4425

Log. first cor. 63′,62	1,80359
— log. 1,0272	0,01187
= log. true cor. 61′,9 . . .	1,79172

Now the mean anom. + true cor. = 71° 1′,9, whose log. tang.	10,4638087
+ log. of the semi-conjugate axis	$\bar{1}$,9905940
= log. tang. of 70° 38′,82	10,4544027

And the log. tang. of the half hereof, 35° 19′,41	$\underline{9}$,8504350
+ log. of the ratio of the gr. and least distances	1,8185730
= log. tang. of 25° 1′,02	9,6690080

whose double 50° 2′,04, or 50° 2′ 2″, is the true anomaly required.