Opuscula statico-mechanica principiis analyseos finitorum superstructa

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Dissertatio VII. De investigatione centri gravitatis in corporibus, eorumque superficiebus.

DISSERTATIO VII.

DE INVESTIGATIONE CENTRI GRAVITATIS IN CORPORIBUS, EORUMQUE SUPERFICIEBUS.

216. Theorema.

Si ABDC (48. Fig.) *ſit corpus quodcumque, in quo ſingulae ſectiones* mn *baſibus* AB, CD *inter ſe parallelis et aequalibus pariter parallelae et aequales ſua centra gravitatis habeant in recta* PQ *intercepta inter centra gravitatis* P, Q *baſium* AB, CD; *erit punctum medium* a *rectae* PQ *centrum gravitatis corporis* ABDC. Fig. 48.

Demonſtratio.

Si per punctum medium a rectae PQ tranſeat planum MN perpendiculare ad PQ; erit momentum cujusvis ſectionis mn reſpectu plani MN aequale producto ex ipſa ſectione mn in diſtantiam ba puncti b rectae PQ, per quod tranſit ſectio mn, a puncto medio a rectae PQ (150. §.): cum ergo omnes ejusmodi ſectiones ſint inter ſe aequales, et pro qualibet ſectione mn cadente ad unam partem plani MN detur una ſectio mn jacens ad partem oppoſitam plani MN in pari diſtantia ba; neceſſe eſt, ut ſumma momentorum omnium ſectionum ex una parte plani MN cogitabilium aequetur ſummae momentorum omnium ſectionum cogitabilium ex parte oppoſita plani MN. Planum MN dividit itaque totum corpus ABDC in duas partes, quarum momenta reſpectu ejusdem plani ſunt inter ſe aequalia (147. §.); et ideo eſt MN planum gravitatis corporis ABDC (163. §.): eapropter, quia PQ eſt diameter gravitatis ejusdem corporis (183. §.), debet punctum a eſſe illius centrum gravitatis (162. §.).

217. Corollarium 1.

Recta PQ intercepta inter centra gravitatis P, Q baſium ABCDE, abcde dati cujuscunque priſmatis Bd per (190. §) determinabilia tranſit per centra gravitatis omnium ſectionum priſmatis illius baſibus parallela- Fig. 49.

rum (214. §.): medium ejus punctum α est ergo centrum gravitatis prismatis Bd (216. §.).

218. Corollarium 2.

Bases prismatis regularis sunt polygona regularia, inter quorum centra jacet axis ejusdem prismatis: centrum gravitatis prismatis regularis coincidit ergo cum puncto medio illius axeos (191. 217. §.).

219. Corollarium 3.

Fig. 50. Et ob (196. §.) per (216. §.) erit medium punctum α axeos PQ dati cujuscunque cylindri ABCD (50. Fig.) centrum gravitatis ejusdem cylindri.

220. Corollarium 4.

Fig. 51. 52. 53. 54. Sic quoque, si datum corpus sit segmentum cylindricum ABCcab (51. Fig.) abscissum plano ABba axi cylindri parallelo, vel frustum cylindricum ABCDdabc (52. Fig.) inter plana Ad, Bc axi cylindri parallela interceptum, aut sector cylindricus ABCDdabc (53. Fig.), vel dimidius annulus cylindricus (54. Fig.); coincidet ob (216. §.) centrum gravitatis ejusmodi corporis cum medio puncto α rectae PQ interceptae inter centra gravitatis P, Q illius basium per (194. 195. 197. 198. 199. 202. §.) determinabilia.

221. Problema.

Fig. 55. 56. *Invenire centrum gravitatis datae pyramidis, datique coni* ABD (55. 56. Fig.).

Solutio.

1. In cono ABD jacet centrum gravitatis cujuslibet sectionis pq parallelae illius basi BD in axe AC (196. §.): in pyramide autem (55. Fig.), si ex centro gravitatis C baseos BD ducatur recta AC ad verticem A, transibit haec recta per unum punctum c cujuslibet sectionis pq parallelae basi BD, quod, quia rectae pc, ce, BC, CE ductae ad puncta extima laterum homologorum pe, BE comprehendunt triangula pce, BCE similia, debet esse centrum gravitatis sectionis pq (214. §.); adeoque etiam centrum gravitatis cujuslibet sectionis pq parallelae basi BD pyramidis jacet in recta AC inter ejus verticem A et centrum gravitatis C baseos intercepta. In cono et pyramide (56. 55. Fig.) debet ergo ejus centrum gravitatis jacere in recta AC (216. §.).

2. Quam-

2. Quamobrem sumta parte indeterminata $z = Ac$ in recta $AC = a$ cogitetur per punctum c transire sectio pq parallela basi BD, quae abscindat ex ABD partem $Apq = m$; erit, si ponatur tota massa $ABD = M$

$$m : M = z^3 : a^3, \text{ hinc } m = \frac{M}{a^3} z^3.$$

3. Sit jam MN planum perpendiculare ad AC, et μ momentum massae $m = pAq$ respectu ejusdem plani; erit per (171. §.)

$$\varepsilon\mu = z\varepsilon m; \text{ et in (2) } \varepsilon m = \frac{3M}{a^3} z^2 \varepsilon z:$$

$$\text{igitur } \varepsilon\mu = \frac{3M}{a^3} z^3 \varepsilon z, \text{ hinc } \mu = \frac{3Mz^4}{4a^3}.$$

4. Pro $z = AC = a$ in (3) invenietur ergo $\mu = \frac{3}{4}Ma$ momentum totius pyramidis vel coni ABD respectu plani MN: cum igitur illius centrum gravitatis debeat coincidere cum uno puncto γ rectae AC ob (1); erit ejus distantia perpendicularis ab eodem plano ob (168. §.)

$$A\gamma = \frac{\mu}{M} = \frac{3}{4}Ma : M = \frac{3}{4}a = \frac{3}{4}AC.$$

222. Corollarium 1.

Data quacunque pyramide ABD (55. Fig.), cujus centrum gravitatis desideretur, quaeratur centrum gravitatis C illius baseos BD (190. §.), et recta AC inter illud verticemque A intercepta dividatur in quatuor partes aequales, quo fiat $A\gamma = \frac{3}{4}AC$; erit γ centrum gravitatis pyramidis (221. §.). Fig. 55.

223. Corollarium 2.

Cum basis pyramidis regularis sit polygonum regulare, inter cujus centrum et verticem pyramidis jacet axis pyramidis; patet, centrum gravitatis tam pyramidis regularis (55. Fig.) quam cujuslibet coni (56. Fig.) coincidere cum puncto γ illius axeos AC, cujus distantia $A\gamma$ a vertice A aequatur tribus quartis totius axeos AC (191. 221. §.).

224. Problema.

Invenire centrum gravitatis dati coni truncati (57. Fig.) *vel datae* Fig. 57. 58.
pyramidis truncatae mdnbBMDN (58. Fig.).

Solutio.

Solutio.

1. Determinatis centris gravitatis C, c basium BMDN per (191. 196. §.), ductaque recta Cc, jacebit in hac centrum gravitatis γ totius massae mdnbBMDN, quod idcirco perfecte determinabitur, si definiatur illius distantia $C\gamma$ vel $c\gamma$ ab alterutro centro C, c.

2. Quamobrem cogitetur compleri tam conus quam pyramis truncata, quo recta Cc producta occurrat vertici A, et sit in hac recta punctum a centrum gravitatis massae Abndm, et e centrum gravitatis massae ABNDM; adeoque per (221. §.)

$$Aa = \tfrac{3}{4}Ac; \quad Ae = \tfrac{3}{4}AC.$$

3. Massa ABNDM composita est ex massis Abndm, mdnbBNDM: ob (1) (2) per (155. §.) debet ergo esse

$$A\gamma = \frac{Ae.ABNDM - Aa.Abndm}{ABNDM - Abndm};$$

seu ob (2)

$$A\gamma = \tfrac{3}{4}\cdot\frac{AC.ABNDM - Ac.Abndm}{ABNDM - Abndm}.$$

4. Sed massae ABNDM, Abndm sunt inter se ut cubi dimensionum homologarum, adeoque

$$AC^3 : Ac^3 = ABNDM : abndm = \frac{ABNDM.Ac^3}{AC^3}:$$

quare fiet, facta substitutione et reductione in (3)

$$A\gamma = \tfrac{3}{4}\cdot\frac{AC^4 - Ac^4}{AC^3 - Ac^3}.$$

5. Ut autem formula complectatur dimensiones immediate in dato cono, vel data pyramide truncata determinabiles, ducantur ex centris gravitatis c, C basium mbnd, MBND ad puncta homologa d, D rectae $cd = r$, $CD = R$ (in cono truncato radios basium exprimentes), et $d\delta = cC = a$ sit parallela rectae AC: erit enim $D\delta : DC = d\delta : AC$, et $D\delta : dc = d\delta : Ac$. Quare habebimus

$$R - r : R = a : AC = \frac{aR}{R - r};$$

$$R - r : r = a : Ac = \frac{ar}{R - r}.$$

In

In (4) debebit itaque esse

$$A\gamma = \tfrac{3}{4} a \cdot \frac{R^4 - r^4}{(R^3 - r^3)(R - r)} \,.$$

6. Hoc modo determinatur distantia centri gravitatis γ coni et pyramidis truncatae a vertice A coni et pyramidis integrae: cum autem non detur punctum A, et puncta C, c semper debeant determinari (1); derivetur ex (5) distantia ejusdem centri γ ab alterutro puncto C, c. Hinc, quia est $C\gamma = AC - A\gamma$, et $c\gamma = A\gamma - Ac$, sumtis pro AC, Ac, $A\gamma$ valoribus ex (5), fiet

$$C\gamma = \tfrac{1}{4} a \cdot \frac{R^4 + 3r^4 - 4Rr^3}{(R^3 - r^3)(R - r)} \,;$$

$$c\gamma = \tfrac{1}{4} a \cdot \frac{3R^4 + r^4 - 4rR^3}{(R^3 - r^3)(R - r)} \,.$$

225. Theorema.

Si revolutione plani QXC (59. Fig.), *ad* C *rectanguli inter duas rectas* QC, XC *et quamcunque lineam curvam* QX *intercepti, circa rectam* QC *generetur corpus rotundum* XQZ; *debebunt centra gravitatis cujuslibet segmenti* pQq *plano* prqs *ad* QC *perpendiculari abscissi, et cujuslibet frusti* ABqp *intercepti inter sectiones* ADBE, prqs *ad* QC *perpendiculares jacere in axe* QC. Fig. 59.

Demonstratio.

Omnes sectiones, ut e. gr. ADBE, ad QC perpendiculares, proinde basi XRZS corporis XQZ parallelae sunt totidem circuli, quorum centra jacent in axe rotationis QC, unde per (196. 216. §) necessario sequitur, quod debebat demonstrari.

226. Problema.

Data aequatione ad curvam QX *inter coordinatas orthogonas* $z = Qa$, $y = pa$, *investigare centra gravitatis partium* pQq, ABqp *corporis* XQZ *revolutione plani* QXC *ad* C *rectanguli circa axem abscissarum* QC *geniti, pro sectionibus* ADBE, prqs *basi* XRZS *parallelis et ad* QC *perpendicularibus.*

Solutio.

1. Centrum gravitatis partis pQq sit punctum γ, et centrum gravitatis partis ABqp sit punctum α in axe rotationis QC (225. §.):

 osten-

oſtendendum eſt, qua ratione diſtantias centrorum gravitatis γ, α, a puncto Q liceat inveſtigare.

2. Sit MN planum ad QC perpendiculare, M maſſa pQq, et μ ejus momentum reſpectu plani MN, adeoque $\varepsilon M = \pi y^2 \varepsilon z$ (Vol. I. §. 490.): habebimus per (171. 168. §.)

$$\varepsilon M = \pi y^2 \varepsilon z; \quad \varepsilon \mu = \pi y^2 z \varepsilon z; \quad Q\gamma = \frac{\mu}{M}.$$

Quamobrem, ut data aequatione ad curvam QX inter coordinatas $z = Qa$, $y = pa$ inveniatur diſtantia $Q\gamma$ centri gravitatis γ maſſae pQq, integrari debent exponentes differentiales εM, $\varepsilon \mu$, ſic ut illorum integralia pro $z = o$ pariter aequentur nihilo, cum evaneſcente $z = Qa$ tam maſſa $M = pQq$, quam ejus momentum μ reſpectu plani MN debeat aequari nihilo.

3. Porro, quia ſectio ADBE in determinata diſtantia Qb a vertice Q dari ſupponitur, ſectione altera prqs jacente in diſtantia variabili $Qa = z$; habebit tam ſegmentum AQB quam illius momentum reſpectu plani MN valorem determinatum, pars vero ABqp ſegmenti pQq ejusdemque momentum reſpectu plani MN erit valoris variabilis: exponentes differentiales maſſae ABqp ejusque momenti aequabuntur ergo exponentibus differentialibus maſſae $pQq = M$, ejusque momenti μ (Vol. I. §. 91.). Ob (2) poteſt ergo poni

$$\varepsilon M = \pi y^2 \varepsilon z; \quad \varepsilon \mu = \pi y^2 z \varepsilon z;$$

intelligendo per M maſſam ABqp, et per μ illius momentum reſpectu plani MN. Verum, quia maſſa ABqp decreſcente $Qa = z$ decreſcit, et pro $z = Qb$ fit aequalis nihilo, ut ope harum formularum tam maſſa $M = ABqp$, quam ejus momentum μ rite determinetur, ſic erunt integrandi exponentes differentiales εM, $\varepsilon \mu$, ut illorum integralia pro $z = Qb$ aequentur nihilo. Inventis vero completis valoribus pro M, μ, innoteſcet eo ipſo diſtantia centri gravitatis α maſſae ABqp a puncto Q, nimirum

$$Q\alpha = \frac{\mu}{M} \text{ per (168. §.)}.$$

227. Corollarium I.

Aequatio ad curvam QX inter coordinatas $z = Qa$, $y = pa$ ſit $y^2 = Az + Bz^2$, et $e = Qb$ data diſtantia ſectionis ADBE a puncto Q; habebimus pro centro gravitatis α maſſae ABqp in (226. §. 3. n.)

εM

$$\varepsilon M = \pi(Az + Bz^2)\varepsilon z; \quad \varepsilon\mu = \pi(Az^2 + Bz^3)\varepsilon z;$$
$$M = \pi(\tfrac{1}{2}Az^2 + \tfrac{1}{3}Bz^3) + C; \quad \mu = \pi(\tfrac{1}{3}Az^3 + \tfrac{1}{4}Bz^4) + C;$$
$$\pi(\tfrac{1}{2}Ae^2 + \tfrac{1}{3}Be^3) + C = 0; \quad C = -\pi(\tfrac{1}{2}Ae^2 + \tfrac{1}{3}Be^3);$$
$$\pi(\tfrac{1}{3}Ae^3 + \tfrac{1}{4}Be^4) + C = 0; \quad C = -\pi(\tfrac{1}{3}Ae^3 + \tfrac{1}{4}Be^4);$$
$$Q\alpha = \frac{\mu}{M} = \frac{4A(z^3 - e^3) + 3B(z^4 - e^4)}{6A(z^2 - e^2) + 4B(z^3 - e^3)}.$$

228. Corollarium 2.

Si fiat $e = Qb = 0$, abibit maſſa $ABqp$ in ſegmentum pQq: quodſi ergo ſit γ centrum gravitatis ſegmenti pQq, obtinebitur illius diſtantia $Q\gamma$ a puncto Q pro $e = 0$ ex (227. §.), nimirum

$$Q\gamma = \frac{4A + 3Bz}{6A + 4Bz} \cdot z.$$

229. Corollarium 3.

Si QXC ſit quadrans circuli, cujus revolutione circa radium $QC = r$ generetur dimidius globus XQZ, continebit aequatio generalis in (227. §.) valores $A = 2r$, $B = -1$, pro quibus per (227. 228. §.) invenientur ſequentes expreſſiones diſtantiae $Q\alpha$ centri gravitatis α fruſti ſphaerici $ABqp$ intercepti inter ſectiones $ADBE$, $prqs$ perpendiculares ad radium QC, et diſtantiae $Q\gamma$ centri gravitatis γ ſegmenti ſphaerici pQq a puncto Q.

$$Q\alpha = \frac{8r(z^3 - e^3) - 3(z^4 - e^4)}{12r(z^2 - e^2) - 4(z^3 - e^3)};$$

$$Q\gamma = \frac{8r - 3z}{12r - 4z} \cdot z.$$

230. Corollarium 4.

Erat $z = Qa$: ſi igitur ponas $z = QC = r$ in (229. §.) obtinebis diſtantiam $Q\alpha$ centri gravitatis α fruſti ſphaerici inter baſim XZ dimidii globi XQZ et ſectionem AB ei parallelam intercepti a puncto Q, nimirum

$$Q\alpha = \frac{8r(r^3 - e^3) - 3(r^4 - e^4)}{12r(r^2 - e^2) - 4(r^3 - e^3)}.$$

231. Corollarium 5.

Quodſi autem in (230. §.) fiat $e = Qb = o$, obtinebitur diſtantia $Q\alpha$ centri gravitatis α dimidii globi XQZ a puncto Q, atque ex hac reperietur ejus diſtantia $C\alpha = QC - Q\alpha = r - Q\alpha$ a centro C, nimirum

$$Q\alpha = \tfrac{5}{8}r; \quad C\alpha = \tfrac{3}{8}r.$$

Verum pro $z = 2r$ fiet in (229. §.) $Q\gamma = r$, id eſt: centrum gravitatis integri globi coincidit cum illius centro magnitudinis.

232. Corollarium 6.

Pari ratione applicari poſſunt praecedentes formulae (227. 228. §.) ad arcum QX parabolae, ellipſeos, et hyperbolae, qui ſua revolutione circa axem QC generet *paraboloidem*, *ſphaeroidem ellipticum*, vel *conoidem hyperbolicum* XQZ: aequatio enim generalis $y^2 = Az + Bz^2$ in (227. §.) complectetur $A = p$ et $B = o$ pro parabola; $A = p$ vero et $B = \frac{p}{a}$ vel $B = -\frac{p}{a}$ pro hyperbola vel ellipſi (Vol. I. §. 508. 540. 563.).

233. Corollarium 7.

In ſpecie itaque diſtantia centri gravitatis γ cujuslibet ſegmenti pQq paraboloidis XQZ ſubtenſi a ſectione $prqs$ perpendiculari ad axem QC a vertice Q erit per (232. 228. §.) $Q\gamma = \frac{2}{3}z = \frac{2}{3}Qa$.

234. Corollarium 8.

Diſtantia autem centri gravitatis γ ſegmenti pQq conoidis hyperbolici, revolutione arcus hyperbolae Qp circa axem transverſum QC geniti, ab ejus vertice Q erit pro $z = Qa$ per (228. 232. §.).

$$Q\gamma = \frac{4az + 3z^2}{6a + 4z}.$$

235. Corollarium 9.

Et diſtantia centri gravitatis γ ſegmenti pQq ſphaeroidis elliptici, revolutione arcus ellipſeos Qp circa axem transverſum geniti, ab ipſius vertice Q, erit per (228. 232. §.)

$$Q\gamma = \frac{4az - 3z^2}{6a - 4z}.$$

236. Co-

236. Corollarium 10.

Pro $z = QC = \frac{1}{2}a$ erit $Q\gamma = \frac{5}{16}a = \frac{5}{8}QC$ diſtantia centri gravitatis γ dimidii ſphaeroidis elliptici XQZ ab ejus vertice Q (235. §.): et pro $z = a$ fiet $Q\gamma = \frac{1}{2}a$, id eſt, centrum gravitatis integri ſphaeroidis elliptici coincidit cum ejus centro magnitudinis.

Scholion.

Paucas adhuc applicationes praecedentis problematis ſubjungam, prius quam ad inveſtigationem centri gravitatis ſuperficierum corporum rotundorum transeam: commendo autem tyronibus, ut ipſimet ſibi plura his ſimilia problemata proponant, eademque propria diligentia reſolvere ſtudeant, atque iſtud conſilium, quoties opportuna occaſio in aliis quoque diſquiſitionibus ſe obtulerit, ſequantur: plurimum ſane lucrabuntur, qui continua ejusmodi exercitatione uſum calculi differentio-integralis in primo horum opusculorum volumine pertractati ſibi familiarem reddiderint.

237. Problema.

Invenire tam maſſam quam centrum gravitatis corporis pQq *geniti revolutione arcus parabolae* Qp (60. Fig.) *circa tangentem* QC *in ejus vertice* Q. Fig. 60.

Solutio.

1. Pro coordinatis orthogonis $z = Qa$, $y = pa$, maſſa $M = pQq$, ejusque momento μ reſpectu plani in Q ad QC perpendicularis, debebit eſſe per (226. §.)

$$\varepsilon M = \pi y^2 \varepsilon z; \quad \varepsilon\mu = \pi y^2 z \varepsilon z.$$

2. Centrum vero gravitatis γ corporis pQq jacebit in axe rotationis QC (225. §.) in diſtantia $Q\gamma = \frac{\mu}{M}$ a puncto Q (168. §.).

3. Sed, ſi QB ſit axis parabolae Qp, ad quem tangens QC eſt perpendicularis (Vol. I. §. 513.); erit $u^2 = pv$ pro ordinata orthogona $u = pb$, et abſciſſa $v = Qb$ (Vol. I. §. 508): igitur erit $v^2 = \frac{u^4}{p^2}$. Quamobrem, ob $pb = Qa$, $Qb = ap$, habebimus etiam $y^2 = \frac{z^4}{p^2}$, et ob (1)(2).

$$\varepsilon M = \frac{\pi}{p^2} z^4 \varepsilon z; \quad \varepsilon\mu = \frac{\pi}{p^2} z^5 \varepsilon z;$$

$$M = \frac{\pi}{5p^2} z^5; \qquad \mu = \frac{\pi}{6p^2} z^6; \quad Q\gamma = \frac{5}{6}z = \frac{5}{6}Qa.$$

238. Problema.

Fig. 45. *Invenire massam et centrum gravitatis corporis* pQq (45. Fig.) *revolutione plani* pQa *ad* a *rectanguli circa axem* QC *cissoidis* QS *geniti.*

Solutio.

Aequatio ad cissoidem QS inter coordinatas $y = pa$, $z = Qa$ est $y^2 = \frac{z^3}{2r - z}$ (Vol. I. §. 582.) pro $2r = QC$; centrum vero gravitatis γ corporis pQq revolutione plani pQa circa axem cissoidis QC geniti jacet in ipso axe QC (225. §.): quodsi ergo M sit massa, et μ momentum ejusdem corporis respectu plani in Q ad QC perpendicularis, habebimus per (226. §)

$$\varepsilon M = \frac{\pi z^3 \varepsilon z}{2r - z}; \quad \varepsilon \mu = \frac{\pi z^4 \varepsilon z}{2r - z}; \quad Q\gamma = \frac{\mu}{M}.$$

Quamobrem erit

$$\varepsilon M = -\pi z^2 \varepsilon z - 2r\pi z \varepsilon z - 4\pi r^2 \varepsilon z + \frac{8\pi r^3 \varepsilon z}{2r - z};$$

$$\varepsilon \mu = -\pi z^3 \varepsilon z - 2r\pi z^2 \varepsilon z - 4r^2 \pi z \varepsilon z - 8r^3 \pi \varepsilon z + \frac{16\pi r^4 \varepsilon z}{2r - z}.$$

Adeoque integrando per (226. §. 2. n.) juxta (Vol. I. §. 242. 249. 252. §.) obtinebimus.

$$M = 8\pi r^3 \log \frac{2r}{2r - z} - \frac{\pi z^3}{3} - r\pi z^2 - 4\pi r^2 z;$$

$$\mu = 16\pi r^4 \log \frac{2r}{2r - z} - \frac{\pi z^4}{4} - \frac{2r\pi z^3}{3} - 2\pi r^2 z^2 - 8\pi r^3 z.$$

Hoc modo determinatur tam massa M corporis pQq, quam distantia $Q\gamma$ illius centri gravitatis γ a vertice Q, pro qualibet abscissa $z = Qa$.

239. Problema.

Fig. 60. *Invenire massam et centrum gravitatis corporis* pQq *geniti revolutione arcus* Qp (60. Fig.) *ellipseos ad axem transversum* QB *relatae circa tangentem* QC *in vertice* Q.

Solutio.

1. Centrum gravitatis γ debet jacere in axe rotationis QC, ita ut, pro $pa = y$, $Qa = z$, $M = pQq$, et momento μ ejusdem massae

ſae reſpectu plani in Q ad QC perpendicularis debeat fieri (226. §. 1. 2. n.)

$$\varepsilon M = \pi y^2 \varepsilon z; \quad \varepsilon \mu = \pi y^2 z \varepsilon z; \quad Q\gamma = \frac{\mu}{M}.$$

2. Cum vero $z = Qa = pb$, et $y = ap = Qb$ ſint coordinatae orthogonae ellipſeos relatae ad axem transverſum QB ſumtum pro linea abſciſſarum; debet eſſe per (Vol. I. §. 525.)

$$z^2 = \frac{b^2 y}{a} - \frac{b^2 y^2}{a^2};$$ igitur debet eſſe

$$y^2 = \tfrac{1}{2} a^2 - \frac{a^2}{b^2} z^2 + \frac{a^2}{2b} \sqrt{(b^2 - 4z^2)}.$$

Hinc et ex (1) obtinebimus

$$\varepsilon M = \tfrac{1}{2} \pi a^2 \varepsilon z - \frac{\pi a^2}{b^2} z^2 \varepsilon z + \frac{\pi a^2}{2b} \varepsilon z \sqrt{(b^2 - 4z^2)};$$

$$\varepsilon \mu = \tfrac{1}{2} \pi a^2 z \varepsilon z - \frac{\pi a^2}{b^2} z^3 \varepsilon z + \frac{\pi a^2}{2b} z \varepsilon z \sqrt{(b^2 - 4z^2)}.$$

Quare integrando juxta (226. §. 2. n.) per (Vol. I. §. 242. 249. 248. 289. 238.) inveniemus

$$M = \tfrac{1}{2} \pi a^2 z - \frac{\pi a^2}{3b^2} z^3 + \frac{\pi a^2}{4b} z \sqrt{(b^2 - 4z^2)} + \frac{\pi b a^2}{8} \text{Arc Sin} \frac{2z}{b};$$

$$\mu = \tfrac{1}{4} \pi a^2 z^2 - \frac{\pi a^2}{4b^2} z^4 - \frac{\pi a^2}{24b} \sqrt{(b^2 - 4z^2)^3} + \frac{\pi a^2 b^2}{24}.$$

Hinc itaque determinabitur tam maſſa $M = pQq$, quam diſtantia $Q\gamma = \frac{\mu}{M}$ illius centri gravitatis γ a vertice Q pro data quacunque abſciſſa $z = Qa$.

240. Corollarium 1.

Si Qb ſit ſemiaxis transverſus $= \frac{1}{2}a$, hinc pb ſemiaxis conjugatus $= \frac{1}{2}b$ ellipſeos; exhibebit pQq corpus revolutione quadrantis elliptici Qp circa illius tangentem QC in vertice Q geniti, cujus et maſſa M, et diſtantia $Q\gamma$ centri gravitatis γ a vertice Q invenietur pro $z = \frac{1}{2}b$ ex (239. §.), ut ſequitur.

$$M = \frac{(10 + 3\pi)}{48} \pi a^2 b; \quad Q\gamma = \frac{17b}{4(10 + 3\pi)} = \frac{17}{2(10 + 3\pi)} . Qa.$$

241. Co-

241. Corollarium 2.

Ellipſis abit in circulum, ſi ejus axes a, b fiant aequales: ſi ergo in (239. §.) ponas $a = b$, obtinebis ſequentes expreſſiones pro maſſa M corporis pQq revolutione arcus circularis Qp circa illius tangentem QC geniti, ejus momento μ reſpectu plani in Q ad QC perpendicularis, et diſtantia ipſius centri gravitatis γ a puncto contactus Q, pro diametro $= a$ circuli, et quavis abſciſſa $z = Qa$.

$$M = \tfrac{1}{2}\pi a^2 z - \tfrac{1}{3}\pi z^3 + \frac{\pi a}{4} z\sqrt{(a^2 - 4z^2)} + \frac{\pi a^3}{8} \text{ Arc Sin } \frac{2z}{a};$$

$$\mu = \tfrac{1}{4}\pi a^2 z^2 - \frac{\pi z^4}{4} - \frac{\pi a}{24}\sqrt{(a^2 - 4z^2)^3} + \frac{\pi a^4}{24};$$

$$Q\gamma = \frac{\mu}{M}.$$

242. Corollarium 3.

Maſſa vero M corporis pQq revolutione quadrantis Qp circuli circa tangentem QC geniti, et diſtantia illius centri gravitatis γ a puncto contactus Q, erit pro ſemidiametro $z = Qa = \frac{1}{2}a$ (241. §.)

$$M = \frac{(10 + 3\pi)}{48}\pi a^3; \quad Q\gamma = \frac{17a}{4(10 + 3\pi)} = \frac{17}{2(10 + 3\pi)} \cdot Qa.$$

243. Corollarium 4.

Si ob (Vol. I. §. 550.) loco aequationis ad ellipſim $z^2 = \frac{b^2 y}{a} - \frac{b^2 y^2}{a^2}$ ſumas aequationem $z^2 = \frac{b^2 y}{a} + \frac{b^2 y^2}{a^2}$ ad hyperbolam in (239. §. 1. 2. n.); obtinebis ſequentes formulas pro maſſa M corporis pQq revolutione arcus pQ hyperbolae circa tangentem QC ad ejus verticem Q geniti, illius momento μ reſpectu plani in Q ad QC perpendicularis, et diſtantia $Q\gamma$ centri gravitatis γ a vertice Q.

$$\varepsilon M = \tfrac{1}{2}\pi a^2 \varepsilon z + \frac{\pi a^2}{b^2} z^2 \varepsilon z - \frac{\pi a^2}{2b} \varepsilon z\sqrt{(b^2 + 4z^2)};$$

$$\varepsilon \mu = \tfrac{1}{2}\pi a^2 z\,\varepsilon z + \frac{\pi a^2}{b^2} z^3\, \varepsilon z - \frac{\pi a^2}{2b} z\varepsilon z\sqrt{(b^2 + 4z^2)};$$

$$Q\gamma = \frac{\mu}{M}.$$

Inte-

Integrando autem exponentes differentiales εM, $\varepsilon\mu$ juxta (226. §. 2. n.) per (Vol. I. §. 242. 249. 248. 289. 238. §.) inveniemus

$$M = \tfrac{1}{2}\pi a^2 z + \frac{\pi a^2}{3 b^2} z^3 - \frac{\pi a^2}{4 b} z\sqrt{(b^2+4z^2)}$$
$$+ \frac{\pi a^2 b}{8} \log \frac{b}{2z+\sqrt{(b^2+4z)}};$$
$$\mu = \tfrac{1}{4}\pi a^2 z^2 + \frac{\pi a^2}{4 b^2} z^4 - \frac{\pi a^2}{24 b}\sqrt{(b^2+4z^2)^3} + \frac{\pi a^2 b^2}{24}.$$

244. Problema.

Data aequatione ad lineam QX (59. Fig.) *inter coordinatas orthogonas* $z = Qa$, $y = pa$, *investigare centrum gravitatis superficiei rotundae* pQq *revolutione lineae* Qp *circa axem abscissarum* QC *genitae.* Fig. 59.

Solutio.

1. Cum centrum gravitatis superficiei rotundae pQq in axe rotationis QC debeat jacere (179. 183. §.); sit illud in γ, et quaeratur ipsius distantia $Q\gamma$ a vertice Q: erit denotante M massam, et μ momentum superficiei pQq respectu plani MN in Q ad QC perpendicularis, per (168. §.)

$$Q\gamma \frac{\mu}{M}.$$

2. Cum porro debeat esse $\varepsilon\mu = z\varepsilon M$ (171. §.), si valor pro εM determinetur per (Vol. I. §. 491.), habebimus

$$\varepsilon M = 2\pi y\sqrt{(\varepsilon y^2+\varepsilon z^2)};\quad \varepsilon\mu = 2\pi z y\sqrt{(\varepsilon y^2+\varepsilon z^2)}.$$

3. Quare, data aequatione ad lineam QX inter coordinatas $z = Qa$, $y = pa$, quaerantur ope calculi integralis massa M et momentum μ ex (2), tum definiatur distantia $Q\gamma$ per (1), quae generatim hoc modo exprimi poterit

$$Q\gamma = \frac{\int y z\sqrt{(\varepsilon y^2+\varepsilon z^2)}}{\int y\sqrt{(\varepsilon y^2+\varepsilon z^2)}}.$$

4. De reliquo, quia evanescente abscissa $z = Qa$, et ordinata $y = pa$, tam superficies $M = pQq$, quam illius momentum μ respectu plani MN debet evanescere, perspicuum est, integralia M, μ in (2) (3) nihilo debere

bere aequari, tam pro $z=o$, quam $y=o$, unde conſtantium determinatio pendet (Vol. I. §. 238.).

245. Corollarium 1.

Sit QX linea recta, adeoque XQZ ſuperficies conica, ejus axis $QC=a$, et radius baſeos $XC=r$; erit $pa=y=\frac{r}{a}z$: quare erit in (244. §. 2. n.)

$$\varepsilon y=\frac{r}{a}\varepsilon z;\ \surd(\varepsilon y^2+\varepsilon z^2)=\frac{\varepsilon z}{a}\surd(a^2+r^2);$$

$$\varepsilon M=\frac{2\pi r}{a^2}z\varepsilon z\surd(a^2+r^2);\ \varepsilon\mu=\frac{2\pi r}{a^2}z^2\varepsilon z\surd(a^2+r^2);$$

$$M=\frac{\pi r}{a^2}z^2\surd(a^2+r^2);\ \mu=\frac{2\pi r}{3a^2}z^3\surd(a^2+r^2);$$

$$Q\gamma=\frac{\mu}{M}=\tfrac{2}{3}z=\tfrac{2}{3}Qa.$$

Nimirum diſtantia centri gravitatis γ ſuperficiei conicae pQq pro quavis abſciſſa $z=Qa$ ab ejus vertice Q aequatur duabus tertiis ejusdem abſciſſae: et ſic diſtantia centri gravitatis γ ſuperficiei convexae cujuslibet coni XQZ ab illius vertice QC aequalis eſt duabus tertiis axeos QC.

246. Corollarium 2.

Sit Qp arcus circuli $r=QC=XC$ deſcripti; erit aequatio ad QX

$$y^2=2rz-z^2:\ \text{igitur}\ \varepsilon y=\frac{(r-z)\varepsilon z}{\surd(2rz-z^2)}.$$

Pro maſſa M ſuperficiei convexae pQq ſegmenti ſphaerici, ejus momento μ reſpectu plani MN, et diſtantia $Q\gamma$ ipſius centri gravitatis γ a vertice Q habebimus itaque juxta (244. §.)

$$\varepsilon M=2r\pi\varepsilon z;\ \varepsilon\mu=2r\pi z\varepsilon z;$$

$$M=2r\pi z;\ \mu=r\pi z^2;\ Q\gamma=\tfrac{1}{2}z.$$

Medium ergo punctum altitudinis Qa ſegmenti ſphaerici pQq eſt centrum gravitatis illius ſuperficiei convexae; et ideo etiam medium punctum

punctum radii QC perpendicularis ad basim XRZS dimidiae sphaerae est centrum gravitatis illius superficiei convexae.

247. Corollarium 3.

Si Qp sit arcus parabolae, proinde $M = pQq$ superficies convexa segmenti conoidis XQZ parabolici revolutione parabolae QX circa illius axem QC geniti, et γ centrum gravitatis ejusdem superficiei; erit per (Vol. I. §. 508.) et (244. §.)

$$y^2 = pz; \quad \varepsilon y = \frac{p\varepsilon z}{2y};$$

$$\varepsilon M = \pi \varepsilon z \sqrt{(p^2 + 4pz)}; \quad \varepsilon \mu = \pi z \varepsilon z \sqrt{(p^2 + 4pz)};$$

$$Q\gamma = \frac{\mu}{M}.$$

Integralia vero M, μ determinabis per (Vol. I. §. 248. 282. 238.), sic ut singula pro $z = 0$ aequentur nihilo.

248. Corollarium 4.

Quodsi autem Qp sit arcus ellipseos, et QC ejus semiaxis transversus, hinc $M = pQq$ superficies convexa segmenti sphaeroidis elliptici revolutione ellipseos circa axem transversum geniti, γ vero centrum gravitatis ejusdem superficiei; erit per (Vol. I. §. 540.) et (244. §.)

$$y = \frac{b}{a}\sqrt{(az - z^2)}; \quad \varepsilon y = \frac{b}{2a} \cdot \frac{(a - 2z)\varepsilon z}{\sqrt{(az - z^2)}};$$

$$\left.\begin{aligned} \varepsilon M &= \frac{\pi b c}{a^2}\, \varepsilon z \sqrt{\left(\frac{a^2 b^2}{c^2} + az - z^2\right)}; \\ \varepsilon \mu &= \frac{\pi b c}{a^2}\, z \varepsilon z \sqrt{\left(\frac{a^2 b^2}{c^2} + az - z^2\right)}; \end{aligned}\right\} \text{pro } c = 2\sqrt{(a^2 - b^2)}.$$

Quamobrem, cum exponentes hi differentiales per (Vol. I. §. 288. 294.) facile integrentur, si singula integralia per (Vol. I. §. 238.) ita definiantur, ut ea pro $z = 0$ aequentur nihilo; determinabitur eo ipso distantia centri gravitatis γ a vertice Q, nimirum $Q\gamma = \frac{\mu}{M}$, pro qualibet altitudine $z = Qa$ segmenti pQq: et si fiat $z = \frac{1}{2}b = QC$, reperietur distantia centri gravitatis superficiei convexae dimidii sphaeroidis elliptici XQZ ab ejus vertice Q.

249. Corollarium 5.

Et ſi Qp ſit arcus hyperbolae, cujus revolutione circa axem transverſum QC generetur ſuperficies convexa pQq conoidis hyperbolici; habebimus ob (Vol. I. §. 563.) per (244. §.) ſequentes expreſſiones pro maſſa M ſuperficiei pQq, ejus momento μ, et centro gravitatis γ: cum igitur exponentes differentiales εM, $\varepsilon \mu$ per (Vol. I. §. 288. 294.) ſint perfecte integrabiles, ſi illorum integralia per (Vol. I. §. 238.) ſic determinentur, ut ſingula pro $z=0$ aequentur nihilo, invenietur eo ipſo diſtantia $Q\gamma$ centri gravitatis γ a vertice Q.

$$y = \frac{b}{2a}\sqrt{(az+z^2)};\quad \varepsilon y = \frac{b}{2a}\cdot\frac{(a+2z)\varepsilon z}{\sqrt{(az+z^2)}}$$

$$\left.\begin{array}{l} \varepsilon M = \dfrac{\pi b c}{a^2}\,\varepsilon z\sqrt{\left(\dfrac{a^2 b^2}{c^2}+az+z^2\right)}; \\ \varepsilon \mu = \dfrac{\pi b c}{a^2}\,z\varepsilon z\sqrt{\left(\dfrac{a^2 b^2}{c^2}+az+z^2\right)} \end{array}\right\}\text{ pro } c = 2\sqrt{(a^2+b^2)}.$$

$$Q\gamma = \frac{\mu}{M}.$$

DISSER-